Elements of Category Theory

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Functors and natural transformations

Adjoints and Monads

Limits and colimits

Pullbacks
FUNCTORS
A **functor** is a map of categories $F : \mathbb{X} \rightarrow \mathbb{Y}$ which consists of a map $F_0$ of the objects and a map $F_1$ of the maps (we shall drop these subscripts) such that

- $\partial_0(F(f)) = F(\partial_0(f))$ and $\partial_1(F(f)) = F(\partial_1(f))$:

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow_{F(X)} & & \downarrow_{F(Y)} \\
F(X) & \xrightarrow{F(f)} & F(Y)
\end{array}
$$

- $F(1_A) = 1_{F(A)}$, identity maps are preserved.
- $F(fg) = F(f)F(g)$, composition is preserved.

Every category has an identity functor. Composition of functors is associative. Thus:

**Lemma**

*Categories and functors form a category Cat.*
EXAMPLES OF Set FUNCTORS

- The product (with $A$) functor

\[ \_ \times A : \text{Set} \to \text{Set}; \]

\[ \begin{array}{ccc}
X & \to & X \times A \\
f & \downarrow & f \times 1 \\
Y & \to & Y \times A \\
\end{array} \]

\[ (x, a) \]

- The exponential functor:

\[ A \Rightarrow \_ : \text{Set} \to \text{Set}; \]

\[ \begin{array}{ccc}
X & \to & A \Rightarrow X \\
f & \downarrow & A \Rightarrow f \\
Y & \to & A \Rightarrow Y \\
\end{array} \]

\[ h \]
EXAMPLES OF Set FUNCTORs

- List on $A$ (data $L(A) = \text{Nil} \mid \text{Cons } A \ L(A)$)

\[
\begin{align*}
X & \quad L(X) & \quad [x_1, x_2, \ldots] \\
\downarrow & & \downarrow L(f) \\
Y & \quad L(Y) & \quad [f(x_1), f(x_2), \ldots]
\end{align*}
\]

- Trees on $A$ (data $T(A) = Lf A \mid \text{Node } T(A) T(A)$):

\[
\begin{align*}
X & \quad T(X) & \quad \text{Node}(Lf x_1)(Lf x_2) \\
\downarrow & & \downarrow T(f) \\
Y & \quad T(Y) & \quad \text{Node}(Lf f(x_1))(Lf f(x_2))
\end{align*}
\]
EXAMPLES OF Set FUNCTORs

- The covariant powerset functor:

  \[ \mathcal{P} : \text{Set} \to \text{Set}; \quad X \to \mathcal{P}(X), \quad X' \subseteq X \]

  \[ f \downarrow \quad \mathcal{P}(f) \downarrow \quad f(X') \subseteq Y \]

- The contravariant powerset functor:

  \[ \mathcal{P} : \text{Set}^{\text{op}} \to \text{Set}; \quad X \to \mathcal{P}(X), \quad X' \subseteq X \]

  \[ f \uparrow \quad \mathcal{P}(f) \uparrow \quad f^{-1}(X') \subseteq Y \]

Note: covariant functors are functors, contravariant functors are functors BUT starting at the dual category.
Given two functors $F, G : \mathcal{X} \to \mathcal{Y}$ a \textbf{(natural) transformation} $\alpha : F \Rightarrow G$ is a family of maps in $\mathcal{Y}$ $\alpha_X : F(X) \to G(X)$, indexed by the objects $X \in \mathcal{X}$ such that for every map $f : X \to X'$ in $\mathcal{X}$ the following diagram commutes:

\[
\begin{array}{ccc}
F(X) & \xrightarrow{F(f)} & F(X') \\
\downarrow^{\alpha_X} & & \downarrow^{\alpha_{X'}} \\
G(X) & \xrightarrow{G(f)} & G(X')
\end{array}
\]

This means that $\text{Cat}(\mathcal{X}, \mathcal{Y})$ can be given the structure of a category. In fact, $\text{Cat}$ is a $\text{Cat}$-enriched category (a.k.a. a \textbf{2-category}).

\textbf{Lemma}

$\text{Cat}(\mathcal{X}, \mathcal{Y})$ is a category with objects functors and maps natural transformations.
NATURAL TRANSFORMATION EXAMPLE I
Consider the category:

\[
\text{TWO} = \begin{array}{c}
E \\
\downarrow \partial_0 \\
N
\end{array}
\]

A functor \( G : \text{TWO} \rightarrow \text{Set} \) is precisely a directed graph!!

A natural transformation between two functors:

\[
\alpha : G_1 \rightarrow G_2 : \text{TWO} \rightarrow \text{Set}
\]

is precisely a morphism of the directed graphs.

\[
\alpha_N G_1(\partial_i)(f) = G_2(\partial_i)(\alpha_E(f)).
\]
NATURAL TRANSFORMATION EXAMPLE II
Consider the category $\mathbb{N}^{\text{op}}$:

\[ 0 \leftarrow \partial \leftarrow 1 \leftarrow \partial \leftarrow 2 \leftarrow \partial \leftarrow \ldots \]

A functor $F : \mathbb{N}^{\text{op}} \to \text{Set}$ is a forest. The children of a node $x \in F(n)$ in the forest is given by \( \{x' \in F(n+1) | \partial(x') = x\} \).

A natural transformation between two functors

\[ \gamma : F_1 \to F_2 : \mathbb{N}^{\text{op}} \to \text{Set} \]

is precisely a morphism of forests:

\[ \gamma_n(F_1(\partial)(x)) = F_2(\partial)(\gamma_{n+1}(x)). \]
NATURAL TRANSFORMATION ...

If functors define structure ...

Then natural transformation define the (natural) homomorphisms of that structure ...
UNIVERSAL PROPERTY
Let $G : \mathcal{Y} \to \mathcal{X}$ be a functor and $X \in \mathcal{X}$, then an object $U \in \mathcal{Y}$ together with a map $\eta : X \to G(U)$ is a universal pair for the functor $G$ (at the object $X$) if for any $f : X \to G(Y)$ there is a unique $f^\# : U \to Y$ such that

\[
\begin{array}{ccc}
X & \xrightarrow{\eta_X} & G(U) \\
\downarrow f & & \downarrow \ \\
& & G(f^\#) \\
& \downarrow & \\
& G(Y) & \\
\end{array}
\]

commutes.
UNIVERSAL PROPERTY – EXAMPLE I
let Graph be the category of directed graphs and Cat the category of categories, let the functor

\[ U : \text{Cat} \rightarrow \text{Graph} \]

be the “underlying functor” which forgets the composition structure of a category.

The map which takes a directed graph and embeds it into the graph underlying the path category as the singleton paths (paths of length one)

\[ \eta : G \rightarrow U(\text{Path}(G)); [n_1 \xrightarrow{a} n_2] \mapsto (n_1, [a], n_2) \]

has the universal property for this “underlying” functor \( U \).
UNIVERSAL PROPERTY – EXAMPLE cont.
Consider a map of directed graphs into the graph underlying a category, $h : G \rightarrow U(\mathbb{C})$, we can extend it uniquely to a functor from the path category to the category by defining

$$h^\#: \text{Path}(G) \rightarrow \mathbb{C}; (A, [a_1, \ldots, a_n], B) \mapsto h(a_1) \ldots h(a_n) : h(A) \rightarrow h(B)$$

This is uniquely determined by $h$ as where the “generating” arrows go determines where the composite arrows go.
UNIVERSAL PROPERTY – EXAMPLE ...

For those more mathematically inclined:

Consider the category of Group then there is an obvious underlying functor $U : \text{Group} \rightarrow \text{Set}$.

The pair $(\mathcal{F}(X), \eta)$ where $\eta : X \rightarrow U(\mathcal{F}(X))$ is a universal pair for this underlying functor.

The diagram expresses the property of being a “free” group (or more generally “free” algebra).
ADJOINT
Suppose $G : \mathcal{Y} \to \mathcal{X}$ has for each $X \in \mathcal{X}$ a universal pair $(F(X), \eta_X)$ so that

$$
\begin{array}{ccc}
X & \xrightarrow{\eta_X} & G(F(X)) \\
\downarrow{f} & & \downarrow{G(f^\#)} \\
G(Y) & & G(Y)
\end{array}
$$

then $G$ is said to be a **right adjoint**.

If $h : X \to X' \in \mathcal{X}$ then define $F(h) := (h\eta_{X'})^#$
then $F$ is a functor ...

$F$ is **left adjoint** to $G$.

$\eta : 1_X \to FG$ is a natural transformation ...
Furthermore, $\varepsilon_Y := (1_{G(Y)})^\# : GF \to 1_Y$ is a natural transformation

\[ G(Y) \xrightarrow{\eta_{G(Y)}} G(F(G(X))) \]

\[ \Downarrow \]

\[ G(Y) \]

\[ \Downarrow \]

\[ U((1_{G(Y)})^\#) \]
ADJOINT
This gives the following data (and **adjunction**):

\[(\eta, \epsilon) : F \dashv G : \mathbf{X} \to \mathbf{Y}\]

- \(F : \mathbf{X} \to \mathbf{Y}\) and \(G : \mathbf{Y} \to \mathbf{X}\) functors
- \(\eta : 1_\mathbf{X} \to FG\) and \(\epsilon : GF \to 1_\mathbf{Y}\) natural transformations
- Triangle equalities:

\[
\begin{align*}
G(Y) & \xrightarrow{\eta_{G(Y)}} G(F(G(Y))) \\
& \downarrow \epsilon_Y \\
G(Y) & \\
F(X) & \xrightarrow{F(\eta_X)} F(G(F(X))) \\
& \downarrow \epsilon_{F(X)} \\
F(X) &
\end{align*}
\]

This data is purely algebraic and is precisely to ask \(F\) be left adjoint to \(G\)!
ADJOINTE
Another important characterization:

\[
\begin{align*}
X & \xrightarrow{f = g^{\flat}} G(Y) \\
F(X) & \xrightarrow{g = f^\#} Y
\end{align*}
\]

And another important example: \textbf{cartesian closed} categories:

\[
\begin{align*}
A \times X & \xrightarrow{f} Y \\
X & \xrightarrow{\text{curry}(f)} A \Rightarrow Y
\end{align*}
\]

Semantics of the typed $\lambda$-calculus.
ADJOINT
Here is the couniversal property for $A \Rightarrow B$:

$$
\begin{align*}
A \times Y & \xrightarrow{1 \times \text{curry}(f)} A \times A \Rightarrow B \\
& \xrightarrow{\text{eval}} B
\end{align*}
$$

\[\text{curry}(f) = y \mapsto \lambda a. f(a, x)\]
MONADS (briefly)
Given an adjunction

\((\eta, \epsilon) : F \dashv G : \mathcal{X} \to \mathcal{Y}\)

consider \(T := FG\) we have two transformations:

\[\eta_X : X \to T(X) = G(F(X))\]

\[\mu_X : T(T(X)) \to T(X) = G(F(G(F(X)))) \xrightarrow{G(\epsilon_{F(X)})} G(F(X))\]

and one can check these satisfy:

\[
\begin{array}{c}
T(X) \xrightarrow{\eta_{T(X)}} T(T(X)) \xleftarrow{T(\eta_X)} T(X) \\
\mu \downarrow \quad \mu \quad \mu \\
T(X) \quad T(X) \quad T(X)
\end{array}
\]

\[
\begin{array}{c}
T(T(T(X)))) \xrightarrow{\mu} T(T(X)) \\
T(\mu) \downarrow \\
T(T(X)) \xleftarrow{\mu} T(X)
\end{array}
\]

Such a \((T, \eta, \mu)\) is called a **monad**.
ADJUNCTIONS AND MONADS

Any adjunction

\[(\eta, \epsilon) : F \dashv G : \mathcal{X} \to \mathcal{Y}\]

generates an monad on \(\mathcal{X}\) and a comonad on \(\mathcal{Y}\).

Furthermore, every monad arises through an adjunction ...

Given a monad \(\mathbb{T} = (T, \eta, \mu)\) on a category \(\mathcal{X}\) we may construct two categories with underlying right adjoints to \(\mathcal{X}\) which generate \(\mathbb{T}\):

the **Kleisli** category \(\mathcal{X}_T\)

and the **Eilenberg-Moore** category \(\mathcal{X}^T\)

so that any \(U : \mathcal{Y} \to \mathcal{X}\) a right adjoint which also generates \(\mathbb{T}\) sits canonically between these categories:

\[
\begin{array}{ccc}
\mathcal{X}_T & \to & \mathcal{Y} \\
\searrow & & \searrow \\
\mathcal{X} & \to & \mathcal{X}^T
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{X}_T & \to & \mathcal{Y} & \to & \mathcal{X}^T \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{X} & \to & \mathcal{X} & \to & \mathcal{X}^T \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{X} & \to & \mathcal{X} & \to & \mathcal{X}^T \\
\end{array}
\]
MONADS AND EFFECTS
Computational effects (exceptions, state, continuations, non-determinism ...) can be generated by using the composition of Kleisli categories.
Here is the definition of $X_T$ (e.g. think list monad):

**Objects:**

$X \in X$

**Maps:**

$X \xrightarrow{f} T(Y) \in X$

$X \xrightarrow{f} Y \in X_T$

**Identities:**

$X \xrightarrow{\eta_X} T(X) \in X$

$X \xrightarrow{1_X} X \in X_T$

**Composition:**

$X \xrightarrow{f} T(Y) \xrightarrow{T(f)} T^2(Z) \xrightarrow{\mu} T(Z) \in X$

$X \xrightarrow{f} Y \xrightarrow{g} Z \in X_T$
MONADS AND EFFECTS
Incomplete history of monads:

- Named by Mac Lane (Categories for Working Mathematician)
- Known first as “standard construction” (Eilenberg, Moore) also “triple” (Barr)
- Kleisli discovered the “Kleisli category”
- Ernie Manes introduced the form of a monad used in Haskell
- Moggi developed computer Science examples (rediscovered Manes form for monad) and calculi for monads (probably motivated by the partial map classifier – a very well behaved monad),
- Wadler made the connection to list comprehension and uses in programming,
- ... do syntax.

MATHEMATICS CAME FIRST ON THIS ONE ...
FUNCTORIAL CALCULUS

The functorial calculus has turned out to be a useful practical and theoretical tool in programming language semantics and implementation ...

Everyone should know it!!

Although very important this is not the focus of these talks!
INITIAL AND FINAL OBJECTS
An **initial object** in a category $\mathbb{X}$ is an object which has exactly one map to every object (including itself) in the category.

Denote the initial object by $0$ and the unique map as $?_A : 0 \rightarrow A$.

Dual to an initial object is a **final object**: a final object in a category $\mathbb{X}$ is an object to which every object has exactly one map.

Denote the final object by the numeral $1$ and the unique map by $!_A : A \rightarrow 1$.

What are these in Set, Mat($R$), and Cat?
INITIAL AND FINAL OBJECTS

- In Set the initial object is the empty set and the final object is any one element set.
- In Mat($R$) the initial object and the final object is the 0-dimensional object.
- In Cat the initial object is the empty category and the final category is any category with one object and one arrow.
INITIAL AND FINAL OBJECTS
A simple observation is:

**Lemma**

*If K and K′ are initial in C then there is a unique isomorphism α : K → K′.*

**Proof:** As K is initial there is exactly one map α : K → K′. Conversely, as K′ is initial there is a unique map α′ : K′ → K. This map is the inverse of α as αα′ : K → K is the unique endo-map on K namely the identity and similarly we obtain α′α = 1′K.

Thus initial objects (and by duality final objects) are unique up to unique isomorphism.
PRODUCTS AND COPRODUCTS

Let $A$ and $B$ be objects in a category then a **product** of $A$ and $B$ is an object, $A \times B$, equipped with two maps $\pi_0 : A \times B \to A$ and $\pi_1 : A \times B \to B$ such that given any object $W$ with two maps $f : W \to A$ and $g : W \to B$ there is a unique map $\langle f, g \rangle : W \to A \times B$, such that $\langle f, g \rangle \pi_0 = f$ and $\langle f, g \rangle \pi_1 = g$. That is:

![Diagram]

The maps $\pi_0$ and $\pi_1$ are called **projections**.

Coproducts are dual.
PRODUCTS AND COPRODUCTS

► In Set the product is the cartesian product and the coproduct is the disjoint union.

► In Mat($\mathbb{R}$) the product of $n$ and $m$ and the coproduct is $n + m$.

► In Cat the product puts the categories in parallel the coproduct puts them side-by-side.

Are projections epic? In Set consider $A \times 0 \xrightarrow{\pi_0} A$. 
PRODUCTS AS ADJOINTS

Given any category there is always a “diagonal” functor:

\[
\Delta : X \to X \times X;
\]

\[
\begin{array}{ccc}
X & \rightarrow & X \times X \\
\downarrow f & \mapsto & \downarrow f \times f \\
Y & \rightarrow & Y \times Y \\
\end{array}
\]

\[
(x, y) \mapsto (f(x), f(y))
\]

having products amounts to requiring that this functor is a left adjoint (namely \((Y, Z) \mapsto X \times Z)!

\[
\begin{array}{ccc}
X & \xrightarrow{\Delta} & X \times X \\
\downarrow \langle f, g \rangle & \mapsto & \downarrow f \times g \\
Y \times Z & \end{array}
\]

Here \(\Delta = \langle 1_X, 1_X \rangle\) is the diagonal map in the category ...
PRODUCTS AND COPRODUCTS
It follows \( \_ \times \_ \) is a functor \( f \times g \) is define as \( \langle \pi_0 f, \pi_1 g \rangle \):

Any binary product has a symmetry map:

Note that \( c_{AB} c_{BA} = 1_{A \times B} \) and so it is an isomorphism.
LIMITS AND COLIMITS

A diagram in $\mathbb{X}$ is a functor $D : \mathbb{G} \to \mathbb{X}$ from a small category $\mathbb{G}$. A $D$–cone over this diagram consists of an object $A$, called the apex of the cone together with for each node $N$ of $\mathbb{G}$ a map $\alpha_N : A \to D(N)$ such that for each arrow of $\mathbb{G}$, $a : N_1 \to N_2$, we have $\alpha_{N_1} G(a) = \alpha_{N_2}$.

A morphism of cones $(\alpha, h, \beta) : \alpha \to \beta$ is given by a map in $\mathbb{C}$, $h : A \to B$ between the apexes of the cones such that $\alpha_N = h \beta_N$ for all the nodes of the diagram.

Lemma

The cones over $D : \mathbb{G} \to \mathbb{C}$ form a category, $\text{Cone}_D(\mathbb{C})$, with objects the cones and maps the morphisms of cones.
A **limit** of a diagram is a final object in Cone$_D(\mathbb{C})$. The apex of this cone is written $\text{Lim}(D)$ with projections $\pi_N : \text{Lim}(D) \to G(N)$.

\[
\begin{array}{c}
\text{A} \\
\downarrow \alpha_{N_1} \downarrow \alpha_{N_2} \downarrow \alpha_{N_3} \\
D(N_1) \quad D(a) \quad D(N_2) \\
\downarrow D(b) \\
D(N_3) \\
\end{array}
\]
ADJOINTS and LIMITS
Because a limit is given by a couniversal property

RIGHT ADJOINTS PRESERVE LIMITS

Dually

LEFT ADJOINTS PRESERVE COLIMITS
EQUALIZERS An **equalizer diagram** is a parallel pair of arrows:

\[
\begin{array}{c}
A \\ f \\ g \\
\end{array} \xrightarrow{\quad} \begin{array}{c}
B
\end{array}
\]

A cone for the above equalizer diagram is determined by a map to \(q : Q \rightarrow A\). Such a map is said to equalize \(f\) and \(g\) as \(qf = qg\). A limit \((E, e)\) is called the **equalizer** (even though it is not unique) and satisfies the property

\[
Q \xrightarrow{k} E \xrightarrow{e} A \xrightarrow{f} B \xrightarrow{g}
\]

that there is a unique \(k\) such that \(ke = q\).

**Lemma**

Suppose \((E, e)\) is the equalizer of \(A \xrightarrow{f} B\) then \(e\) is monic.
COMPLETENESS AND COCOMPLETENESS

Final objects \( \mathbb{G} = 0 \), products \( \mathbb{G} = 1 + 1 \).

A category is **complete** when it has limits for all small diagrams. Dually it is **cocomplete** if it has colimits for all small diagrams.

There is an important theorem:

**Theorem**

A category is complete if and only if it has all products and equalizers.
PULLBACKS

Another important limit is the pullback (especially to these talks). A pullback diagram is a binary fan of arrows:

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
& \downarrow & \downarrow g \\
\end{array}
\]

A cone is given by a \( Q \) together with maps \( q_A : Q \to A \) and \( q_B : Q \to B \) such that \( q_A f = q_B g \). A limit \( (E, e_A, e_B) \) is called the pullback:

\[
\begin{array}{ccc}
Q & \xrightarrow{k} & E \\
& \searrow & \downarrow e_A \\
& & A \\
& \downarrow e_B & \downarrow f \\
B & \xrightarrow{g} & C \\
\end{array}
\]

and has a unique comparison map \( k \) from any cone such that \( ke_A = q_A \) and \( ke_B = q_B \).
PULLBACKS Products and equalizers imply pullbacks:

\[
\begin{array}{c}
P \xrightarrow{g'} A \\
\downarrow f' \\
\downarrow f \\
B \xrightarrow{g} C
\end{array}
\]

is a pullback if and only if

\[
P \xrightarrow{\langle f', g' \rangle} A \times B \xrightarrow{\pi_1 g} C \xrightarrow{\pi_0 f} C
\]

is an equalizer.

In Set the pullback is a subset of the product:

\[
\{(a, b) | f(a) = g(b)\} \subseteq A \times B
\]
PULLBACKS

Lemma

*In any category the pullback of a monic along any map is a monic.*

**Proof:** Suppose \( g \) is monic and \( k_1 e_A = k_2 e_A \) then

\[
k_1 e_B g = k_1 e_A f = k_2 e_A f = k_2 e_B g
\]

so as \( g \) is monic \( k_1 e_B = k_2 e_B \).

However, this makes \( k_1 \) and \( k_2 \) comparison maps from the outer square to the pullback.

\( \square \)
Lemma

In any category $f : A \to B$ is monic iff the followings is a pullback:

$$
\begin{array}{ccc}
A & \xrightarrow{f} & A \\
\downarrow & & \downarrow \\
A & \xrightarrow{f} & B.
\end{array}
$$

**Proof:** If $xf = yf$ there is a unique comparison map

$$
\begin{array}{ccc}
X & \xrightarrow{x} & A \\
\downarrow & & \downarrow \\
A & \xrightarrow{f} & B.
\end{array}
$$

which shows $x = y$. Conversely if $f$ is monic then whenever we form the outer square $x = y$, so this gives a comparison map, whose uniqueness is forced by the fact that $f$ is monic. □
PULLBACKS
As right adjoints preserve pullbacks

RIGHT ADJOINTS PRESERVE MONICS

and dually ..
**Pullbacks**

**Lemma**

*In the following (commuting) diagram:*

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B & \xrightarrow{g} & C \\
\downarrow{a} & & \downarrow{b} & & \downarrow{c} \\
A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C'
\end{array}
\]

(i) if the two inner squares are pullbacks the outer square is a pullback;

(ii) if the rightmost square and outer square is a pullback the leftmost square is a pullback.
PULLBACKS
Products and pullbacks imply equalizers:

**Lemma**
The following square is a pullback

\[
\begin{array}{ccc}
E & \xrightarrow{e} & X \\
\downarrow{e'} & & \downarrow{\langle f,g \rangle} \\
Y & \xrightarrow{\Delta} & Y \times Y
\end{array}
\]

if and only if

\[
\begin{array}{ccc}
E & \xrightarrow{e} & X & \xrightarrow{f} & Y \\
& & \Downarrow{g}
\end{array}
\]

is the equalizer.
PULLBACKS
Pullbacks and a final object imply products:

**Lemma**

*The following square is a pullback*

\[
\begin{array}{ccc}
A \times B & \overset{\pi_0}{\longrightarrow} & A \\
\downarrow^{\pi_1} & & \downarrow^{!} \\
B & \overset{!}{\longrightarrow} & 1
\end{array}
\]

*if and only if*

\[
\begin{array}{ccc}
A & \overset{\pi_0}{\longleftarrow} & A \times B \\
 & \overset{\pi_1}{\longrightarrow} & B
\end{array}
\]

*is a product.*

And so one has all finite limits when one has pullbacks and a final object ...