Itegories and PCAs

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Motivation

In a categorical development of computability it is natural to ask:

- What basic settings support computation?
- What models are there of these settings?
- How do these settings link to practical computability in computer science?
  - Flow diagrams: they underlies modern compiler optimization techniques.
  - The λ-calculus: used in semantics and to implement functional languages.
  - Turing machines: used in the classroom!

This talk: the link to practical computability and flow diagrams .....

Motivation

What is computability?

In this talk: a category "supports computability" means "contains a PCA."

A partial combinatory algebra (in sets) consists of

- A binary partial operation $\bullet : A \times A \longrightarrow A$
- Two constants k and s such that
  - $(\mathsf{k} \bullet x) \bullet y = x$
  - $((\mathsf{s} \bullet x) \bullet y) \bullet z = (x \bullet z) \bullet (y \bullet z)$ where $(\mathsf{s} \bullet x) \bullet y$ is always defined.

To describe a partial combinator algebras one needs a theory of "partiality" so that one can say what "defined" means to start with.

Motivation

WHAT IS THE SEMANTICS OF FLOW DIAGRAMS ...

Not a new problem! (Dana Scott, Calvin Elgot, Steve Bloom, Zoltan Esik, Ernie Manes, Phil Scott, Esfan Haghverdi .... many others)

Certain ingredients are accepted:

- A notion of partiality
- Partial products
- Extensive coproducts
- A (particle style) trace on the coproduct.

Can we get by with less?

Motivation

ALGEBRAIC INGREDIENTS:

Notion of partiality = restriction category

Partial products = cartesian restriction category

Extensive coproducts = extensive restriction category

Trace on coproduct = iteration.

Idea: an itegory should be a full subcategory of a setting with all these features ...

Motivation

KEY TECHNICAL IDEA:

Macro-micro principle: large-scale properties are reflections of small-scale properties.

(works both ways!)

Extensive coproduct

$$\frac{A, B : \mathsf{Type}}{A + B : \mathsf{Type}}$$

$\longrightarrow$

Local disjunction

$$\frac{f, g : X \longrightarrow Y}{f \sqcup g : X \longrightarrow Y}$$

Trace

$$\frac{X + A \xrightarrow{f} X + B}{A \xrightarrow[\mathsf{Tr}(f)]{} B}$$

$\longrightarrow$

Kleene wand

$$\frac{A \xrightarrow{f} A \quad A \xrightarrow{g} B \quad f \perp g}{A \xrightarrow[g^{\star\!\uparrow} f]{} B}$$

What does $f \perp g$ mean?

Motivation

CLAIM: this gives a semantics of flow diagrams.

What do you have to add to get computability?

Answer: an **appending stack object** ...

Motivation

Disclaimer: This is not original!

Alex Heller (1990) set out along the same route to produce off-beat examples of recursion categories. His work was my starting point ...

(I hope there are some technical improvements ...)

Restriction categories and partial maps

A restriction category is a category with a restriction operator

$$\frac{X \xrightarrow{f} Y}{X \xrightarrow{\overline{f}} X}$$

which satisfies the following four axioms:

[R.1] $f\overline{f} = f$

[R.2] $\overline{g}\overline{f} = \overline{f}\overline{g}$

[R.3] $\overline{g}\overline{f} = \overline{g\overline{f}}$

[R.4] $\overline{g}f = f\overline{gf}$

Restriction functors are functors, which, in addition, preserve the restriction: $F(\overline{x}) = \overline{F(x)}$.
These axioms are independent ... here is a sample equality:

$$\overline{\overline{g}f} = \overline{f\overline{gf}} = \overline{f}\,\overline{gf} = \overline{gf}\,\overline{f} = \overline{gf\overline{f}} = \overline{gf}$$

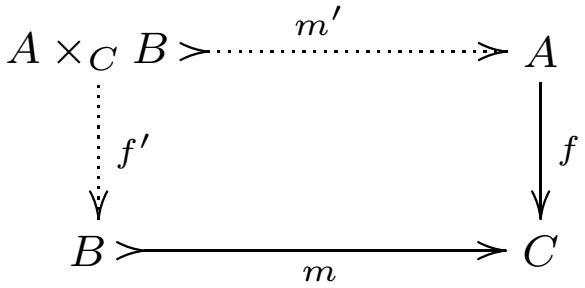
Basic properties of restriction categories

- The *total maps*, x such that $\overline{x} = 1$, form a subcategory.
- Parallel maps can be partially ordered by $x \leq y$ iff $x = y\overline{x}$; this is a partial order enrichment (that is if $x \leq y$ then $gxf \leq gyf$).
- Parallel maps are *compatible* when they agree where they are both defined $g\overline{f} = f\overline{g}$.
- The *restricted isomorphisms*, $f : X \longrightarrow Y$ with a (necessarily unique) "partial inverse" $g = f^{(-1)} : Y \longrightarrow X$ such that $gf = \overline{f}$ and $fg = \overline{g}$, form a subcategory (which is an *inverse category*).
- As $\overline{\overline{f}} = \overline{f} = \overline{f\overline{f}} = \overline{f}\,\overline{f}$, maps e with $e = \overline{e}$ are called restriction idempotents. The set of restriction idempotents at an object X, written $\mathcal{O}(X) = \{e : X \longrightarrow X \mid \overline{e} = e\}$ form a commutative monoid of idempotents and therefore is a *semilattice*.
- *Restriction monics* are monic restricted isomorphisms and are splittings of restriction idempotents.

Completeness of restriction categories

An $\mathcal{M}$-stable system of monics satisfies:

- Each $m \in \mathcal{M}$ is monic
- Composites of maps in $\mathcal{M}$ are themselves in $\mathcal{M}$
- All isomorphisms are in $\mathcal{M}$
- Pullbacks along of an $\mathcal{M}$-map along any map always exists and is an $\mathcal{M}$-map.



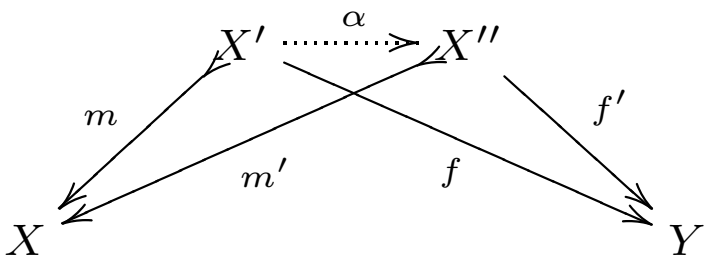
Theorem 1. *(Cockett-Lack) Every restriction category has a fully structure preserving embedded into the $\mathcal{M}$-partial map category of a category with a stable system of monics $\mathcal{M}$.*

Partial map categories

Let $(\mathbf{X}, \mathcal{M})$ be a category with a stable system of monics then $\mathsf{Par}(\mathbf{X}, \mathcal{M})$, the category of $\mathcal{M}$-partial maps, is the following:

Objects: $X \in \mathbf{X}$

Maps: Spans $(m, f) : X \to Y$ where $m \in \mathcal{M}$ up to equivalence:

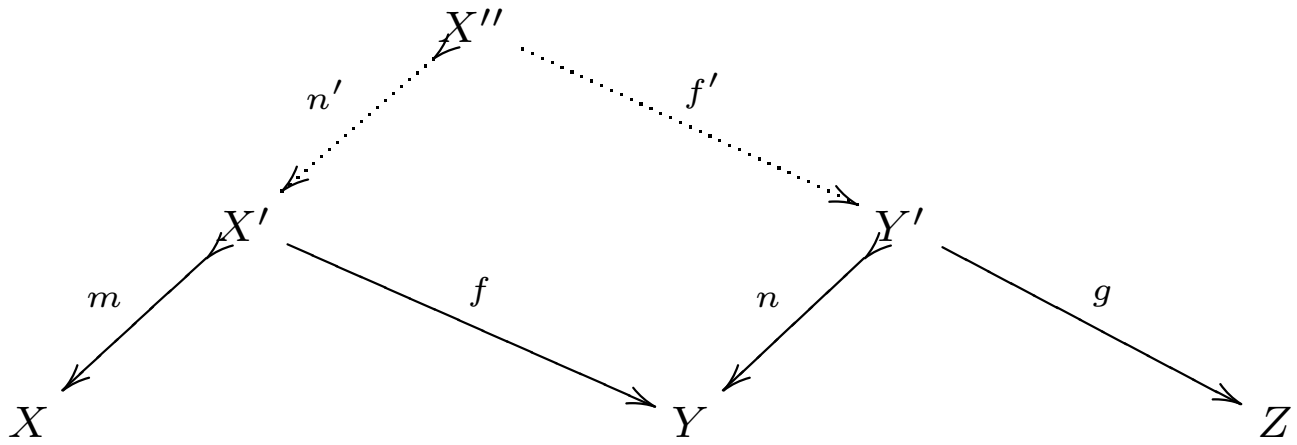


Where $(m, f) \sim (m' f')$ when there is an isomorphism α' making the diagram commute.

Identities: $(1_X, 1_X) : X \to X$

Partial map categories

Composition: By pullback:



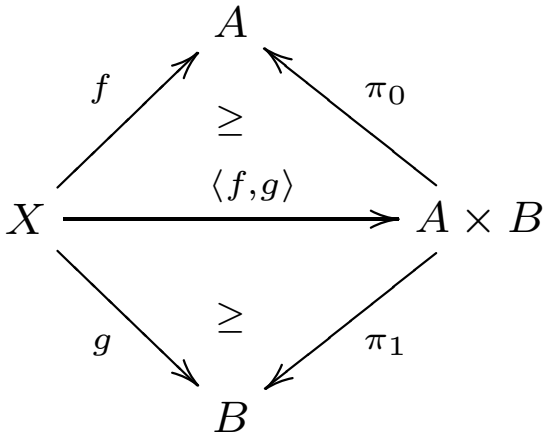
Proposition 2. *All partial map categories* $\mathsf{Par}(\mathbf{X}, \mathcal{M})$*, as above, are (split) restriction categories with* $\overline{(m, f)} = (m, m)$.

The completeness theorem is proved by spitting the idempotents of a restriction category, $\mathbf{X}$, to obtain a total map category in which the restricted monics form a system of $\mathcal{M}$-maps. The original restriction category $\mathbf{X}$ then sits elegantly inside $\mathsf{Par}(\mathsf{Total}(\mathbf{X}, \mathcal{M}))$.

Partial products

A restriction category has a *partial terminal object*, 1, in case for each object A there is a total map $!_A : A \longrightarrow 1$ such that given any other map $h : A \longrightarrow 1$ we have $h = !_A \overline{h}$.

A restriction category has *partial products* in case for each pair of object A and B there is an object $A \times B$ and total maps $\pi_0 : A \times B \longrightarrow A$ and $\pi_1 : A \times B \longrightarrow B$ such that given any pair of maps $f : X \longrightarrow A$ and $g : X \longrightarrow B$ there is a map $\langle f, g \rangle : X \longrightarrow A \times B$ with $f\overline{g} = \langle f.g \rangle \pi_0$ and $g\overline{f} = \langle f, g \rangle \pi_1$.



A restriction category has partial products and a partial terminal object if and only if its total category has products and a terminal object.

A restriction category which has partial products is a *cartesian restriction category*.

The restriction category $\mathsf{sSLat}^{\mathrm{op}}$

The category of semilattices in which the homomorphisms are *stable*, in the sense that they preserve binary products, sSLat, is defined as follows:

Objects: Semilattices $(X, \top, \wedge)$

Maps: $f : X \longrightarrow Y$ is a stable (or binary meet preserving) map. That is $f(x \wedge y) = f(x) \wedge f(y)$ (but f does *not* necessarily preserve the top, $\top$).

This category has a corestriction defined by:

coRestriction: If $f : X \longrightarrow Y$ then $\overline{f} : Y \longrightarrow Y$ has $\overline{f}(x) = f(\top) \wedge x$.

It is easy to check this is a corestriction category. Therefore, $\mathsf{sSLat}^{\mathrm{op}}$, the dual of the category of semilattices with stable maps, is a restriction category.

$\mathsf{sSLat}^{\mathrm{op}}$ has partial products given by the coproduct in SLat (which is the same as the product).

The fundamental functor

Every restriction category has a "fundamental" restriction functor to $\mathsf{sSLat}^{\mathrm{op}}$:

$$\mathcal{O} : \mathbf{X} \longrightarrow \mathsf{sSLat}^{\mathrm{op}}; \quad \begin{array}{c} X \\ f \downarrow \\ Y \end{array} \quad \mapsto \quad \begin{array}{c} \mathcal{O}(X) \\ \uparrow \mathcal{O}(f) \\ \mathcal{O}(Y) \end{array}$$

where $\mathcal{O}(X) = \{e : X \longrightarrow X \mid e = \overline{e}\}$ and

$$\mathcal{O}(f) : \mathcal{O}(Y) \longrightarrow \mathcal{O}(X); e \mapsto \overline{ef}$$

Note that $\overline{f} = \mathcal{O}(f)(1_Y)$.

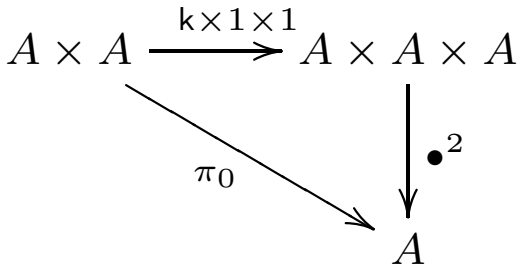
$\mathbf{X}$ has a partial terminal object if and only if this "fundamental" functor is representable

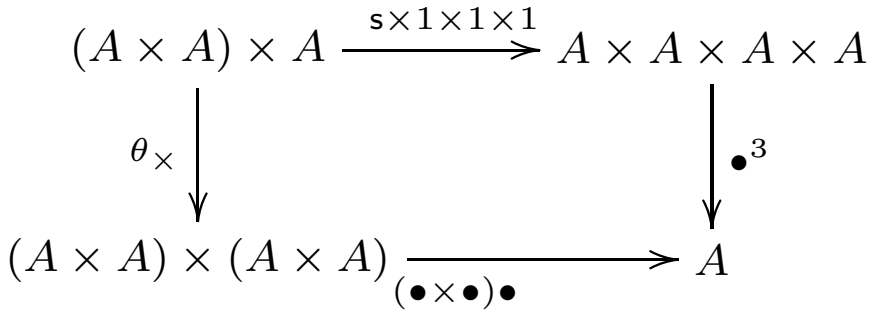
$$\mathcal{O}(X) = \mathbf{X}(X, 1)$$

PCAs revisited

Once one has a partial product one can express arbitrary partial algebras. Here is a PCA again:

$$\bullet : A \times A \longrightarrow A \qquad \mathsf{k}, \mathsf{s} : 1 \longrightarrow A \qquad \text{such that } \overline{\mathsf{k}} = 1_1 \text{ and } \overline{\mathsf{s}} = 1_1$$





$$\overline{(\mathsf{s} \times 1 \times 1)\bullet^2} = 1$$

Examples of restriction categories

1. The category of sets and partial maps $\mathsf{Par}(\mathsf{Set}, \text{monic})$.
2. Any partial map category: a favorite is $\mathsf{CRing}^{\mathrm{op}}$ with localizations $\mathcal{L}$. Then $\mathsf{Par}(\mathsf{CRing}^{\mathrm{op}}, \mathcal{L})$ is a restriction category (of relevance in algebraic geometry).
3. The category of topological spaces with partial maps defined on an open subset and a continuous map on that subset.
4. The category $\mathsf{sSLat}^{\mathrm{op}}$ with *stable maps* (i.e. binary meet preserving maps). The category of locales with stable maps (i.e. binary meet and join preserving maps).
5. Partial recursive maps on the natural numbers.
6. Given any partial algebraic theory $\mathcal{T}$ there is a classifying cartesian restriction category $C(\mathcal{T})$ with a *generic* model of the partial algebraic theory. For example, there is a generic partial combinatory algebra which lives in its own environment and gives a generic version of computability.
   One can also form the generic category with an appending stack object ...

Restriction zeros

A restriction category has a *restriction zeros* if for each pair of objects A and B there is a map $0 : A \longrightarrow B$ such that

- $f0g = 0$
- $\overline{0} = 0$

If a zero map splits then the category has a zero object.

If a restriction category has restriction zeros then it is enriched over pointed sets. Note that the category of pointed sets is isomorphic to the category of sets and partial maps.

Sets and partial maps have restriction zeros.

Coproducts and extensivity

A restriction category which has coproducts is *extensive* in case it has restriction zeros and a **decision operator**:

$$\frac{f : X \longrightarrow Y + Z}{\langle f \rangle : X \longrightarrow X + X}$$

[Dec.1] $\nabla\langle f \rangle = \overline{f}$

[Dec.2] $(f + f)\langle f \rangle = (\sigma_0 + \sigma_1)f$

A map $d : X \longrightarrow X + X$ is a **decision** in case $d = \langle d \rangle$. Binary decisions implies n-ary decisions $d : A \longrightarrow A + ... + A$ (these satisfy $\nabla d = \overline{d}$ and $(d + ... + d)d = (\sigma_1 + ... + \sigma_n)d$). An extensive restriction category is a unique decomposition category on the coproduct as the coproduct in an extensive restriction category becomes a pre-biproduct with respect to the injections and their restricted inverses:

$$\begin{aligned} f &= ((\sigma_0^{(-1)}\sigma_0) + (\sigma_1^{(-1)}\sigma_1)f = (\sigma_0^{(-1)} + \sigma_1^{(-1)})(f + f)\langle f \rangle \\ &= \nabla(\overline{\sigma_0^{(-1)}}f + \overline{\sigma_1^{(-1)}}f)\langle f \rangle = (\overline{\sigma_0^{(-1)}}f \sqcup \overline{\sigma_1^{(-1)}}f) \end{aligned}$$

where $\_ \sqcup \_$ is the join in the homset so that it is uniquely determined.

Extensivity cont.

Why is extensivity important?

Every map between coproducts an be written as a matrix ..

$$X_1 + ... + X_n \longrightarrow Y_1 + ... + Y_m$$

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ & \dots & \\ a_{m1} & \dots & a_{mn} \end{bmatrix}$$

where the i^{th} rows must be *separated* by a decision d_i such that $\overline{a_{ij}} = \overline{\sigma_j d_i}$

Independence spaces

An *independence space* $(X, \mathcal{X})$ is a set with a set of finite subsets $\mathcal{X} \subseteq \mathcal{P}_f(X)$ which are down-closed and contain all singleton subsets.

A morphism $f : (X, \mathcal{X}) \longrightarrow (Y, \mathcal{Y})$ between independence spaces is a partial map $f : X \longrightarrow Y$ such that restricted to each independent subset is a restricted isomorphism which preserves independence.

Each $X' \in \mathcal{X}$ determines a restriction idempotent $e_{X'} : X \longrightarrow X$, we require fe is a restricted isomorphism and $\overline{(fe)^{(-1)}}$ determines an independent subset of Y.

The idea is that independent subsets cannot be "squashed" by the maps ...

The category of independence spaces is an extensive restriction category. It does not have partial products but it has various tensors ....

Independence spaces

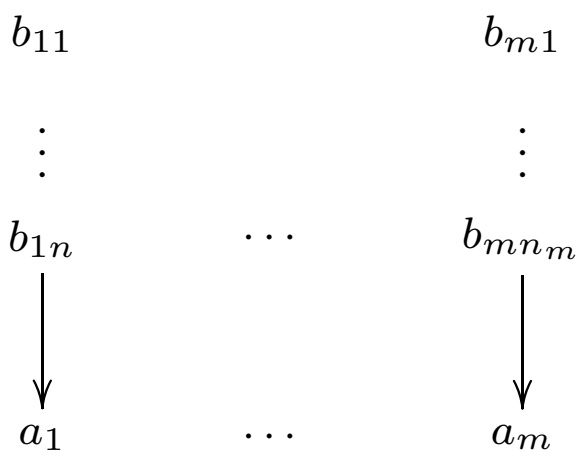
There is the following (non-symmetric) tensor product on the category:

$$(A, \mathcal{A}) \lhd (B, \mathcal{B}) = (A \times B, \mathcal{A} \lhd \mathcal{B})$$

where

$$X' \in \mathcal{A} \lhd \mathcal{B} \quad \text{if and only if} \quad \{a | \exists b.(a, b) \in X'\} \in \mathcal{A}$$

$$\& \quad X'_a = \{b | (a, b) \in X' | (a, b) \in X'\} \in \mathcal{B}$$



Every object has naturally a coassociative comultiplication ...

Independence restriction categories

Idea: enrich restriction categories in independence spaces ...

Require that $\overline{(\_)} : \mathbf{X}(A, B) \longrightarrow \mathbf{X}(A, A)$ is an independence space map (note: cannot enrich all the restriction identities). This has various consequences:

- There must be a restriction zero.
- Parallel arrows $\{f_1, ..., f_n\}$ are independent if and only if $\{\overline{f_1}, ..., \overline{f_n}\}$.
- If $\{f_1, ..., f_n\}$ are independent and $f_i \leq f_i'$ then $\{f_1', ..., f_n'\}$ are independent.
- f and g are independent the $\overline{f}g = 0$.

PROOF: (of last) If $\{f, g\}$ is independent then

$$\{(\overline{f}, \overline{f}), (\overline{f}, \overline{g}), (\overline{g}, \overline{f}), (\overline{g}, \overline{g})\}$$

is independent in $\mathbf{X}(A, A) \lhd \mathbf{X}(A, A)$ but under composition $\overline{f}\ \overline{g} = \overline{g}\ \overline{f}$ so these non-diagonal pairs are squashed under composition and so cannot be defined for the partial map of composition. However, this means that these composites must be zero. □

Examples of independences

- Given any restriction category $\mathbf{X}$ with a restriction zero declare the independent sets on $\mathbf{X}(A, B)^*$ (where we remove the zero) to be the singleton sets.
- Given any restriction category $\mathbf{X}$ with a restriction zero declare the independent sets on $\mathbf{X}(A, B)^*$ to consist of the finite sets $X' \subseteq \mathbf{X}(A, B)^*$ such that for each distinct pair $f, g \in X'$ we have $f\overline{g} = 0$.
- Given any extensive category $\mathbf{X}$ declare $\{f_1, ..., f_n\} \subseteq \mathbf{X}(A, B)^*$ independent in case there is a decision $d : A \longrightarrow A + ... + A$ such that $\overline{f_i} \leq \overline{\sigma_i^{(-1)} d}$.

Does separation in extensive categories cover all examples?

Answer: *Yes*

Disjoint restriction categories

An independence restriction category is a *disjoint* restriction category if independent pairs of parallel maps $f, g : X \longrightarrow Y$ with $\{f, g\}$ independent have a join $f \sqcup g$ which satisfies;

- It is the join:
  - $f \leq f \sqcup g$ and $g \leq f \sqcup g$
  - Given $f \leq h$ and $g \leq h$ then $f \sqcup g \leq h$.
- It is stable $(f \sqcup g)h = (fh) \sqcup (gh)$ (and universal $k(f \sqcup g) = (kf) \sqcup (kg)$).
- $\{f_{ij} | i \in I, j \in J_i\}$ is independent if and only if $\{\bigsqcup_{j \in J_i} f_{ij} | i \in I\}$ is independent.

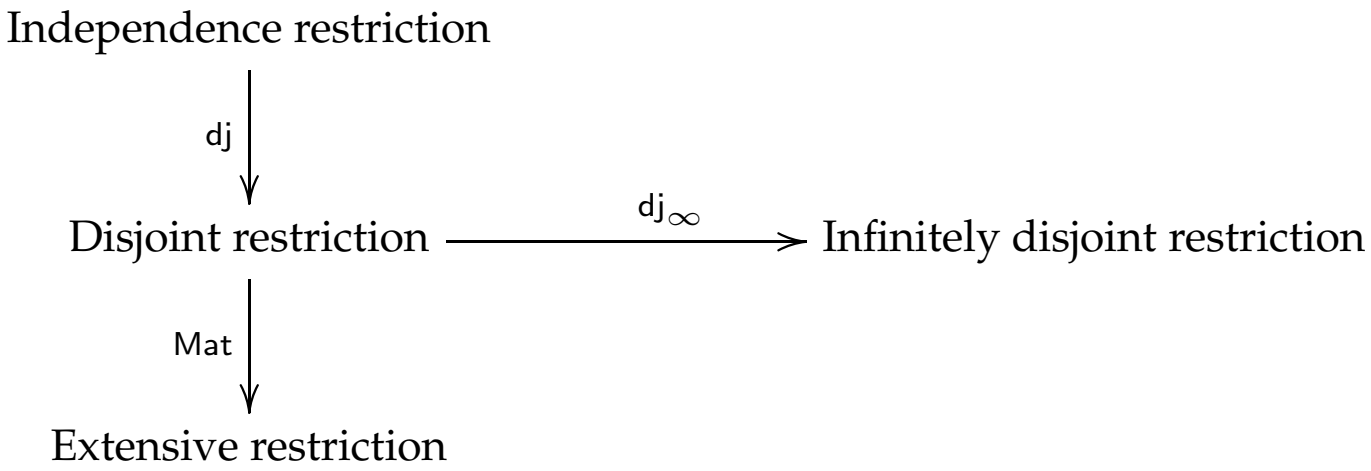
In a disjoint restriction category it suffices to give the binary independences $f \perp g$ as $\{f_1, ..., f_n\}$ is independent iff and only if

$$\begin{array}{rcl} f_{n-1} & \perp & f_n \\ f_{n-2} & \perp & f_{n-1} \sqcup f_n \\ & \ldots & \\ f_1 & \perp & f_2 \sqcup \ldots \sqcup f_n \end{array}$$

Every extensive restriction category is disjoint with respect to separation.

Constructions ...

There are a series of constructions:



Constructions ...

Given any independence restriction category there is a universal disjoint restriction category into which it embeds $\eta : \mathbf{X} \longrightarrow \mathrm{dj}(\mathbf{X})$

$\mathrm{dj}(\mathbf{X})$ is defined as follows:

Objects: The same as $\mathbf{X}$, $X \in \mathbf{X}$;

Maps: A map $F : X \longrightarrow Y$ in $\mathrm{dj}(\mathbf{X})$ is an independent subset $F \subseteq \mathbf{X}(X, Y)^*$;

Composition: $FG = \{fg | f \in F, g \in G\}$;

Identities: $1_X = \{1_X\}$;

Restriction: $\overline{F} = \{\overline{f} | f \in F\}$;

Disjointness: $\{F_i | i \in I\}$ is independent if and only if $\bigcup_i F_i$ is independent.

Join: $F \sqcup G = F \cup G$.

The embedding preserves the independence structure ...

Constructions ...

Given any disjoint restriction category there is a universal extensive restriction category into which it embeds $\eta : \mathbf{X} \longrightarrow \mathsf{Mat}(\mathbf{X})$ preserving the disjointness and independence properties:

Objects: Maps $A : I_A \longrightarrow \mathsf{obj}(\mathbf{X})$ where I is a finite set.

Maps: Matrices $M : A \longrightarrow B$ where $M : I_A \times I_B \longrightarrow \mathsf{Map}(\mathbf{X})$ with $\partial_0(M(i,j)) = A(i)$ and $\partial_1(M(i,j)) = B(j)$ and $\{M(i,j) | j \in I_B\}$ independent.

Composition: $MN(i,j) = \bigsqcup_k M(i,k)N(k,j)$

Identities: $1_A : A \longrightarrow A$ has $1_A(i,i) = 1_{A(i)}$ and $1_A(i,i') = 0$ otherwise.

Restriction: $\overline{M}(i,i) = \bigsqcup_j \overline{M(i,j)}$ and $\overline{M(i.i')} = 0$ otherwise.

Disjointness: $M \perp M'$ if and only if $M(i,j) \perp M'(i,j)$.

Join: $(M \sqcup N)(i,j) = M(i,j) \sqcup N(i,j)$.

Note $\mathsf{Mat}(\mathsf{Mat}(\mathbf{X})) \equiv \mathsf{Mat}(\mathbf{X})$ ...

Constructions ...

One last construction which will be useful:

Given any disjoint restriction category there is a universal infinitely disjoint restriction category into which it embeds $\eta : \mathbf{X} \longrightarrow \mathrm{dj}_\infty(\mathbf{X})$ preserving the disjointness and independence properties.

A set is *infinitely disjoint* if all its finite subsets are disjoint.

objects: As for $\mathbf{X}$.

Maps: $F; A \longrightarrow B$ down-closed compatible subsets $F \subseteq \mathbf{X}(A, B)$ which are closed to finite disjoint joins.

composition: $FG = \Downarrow \{\bigsqcup_{i,j} f_i g_j | f \in F, g \in G\}$ where we must add disjoint joins and down-close the set of composites.

Identity: $1_A = \{e | e \leq 1_A\}$.

Restriction: $\overline{F} = \{\overline{f} | f \in F\}$.

Disjointness: $F \perp G$ if and only if for every $f \in F$ and $g \in G$ we have $f \perp g$.

Join: The disjoint join closure of the union.

Itegories

An *itegory* is a disjoint restriction category with a combinator, which we call the **Kleene wand**:

$$\frac{A \xrightarrow{f} A \quad A \xrightarrow{g} B \quad f \perp g}{A \xrightarrow[f \dagger g]{} B}$$

which satisfies (Conway equations):

[W.1] When $f \perp h$ then $(gf)\dagger h = h \sqcup ((fg)\dagger(hg))f$;

[W.2] When $f \perp g$, $g \perp h$, and $h \perp g$ then $(f \sqcup g)\dagger h = (f\dagger g)\dagger(f\dagger h)$.

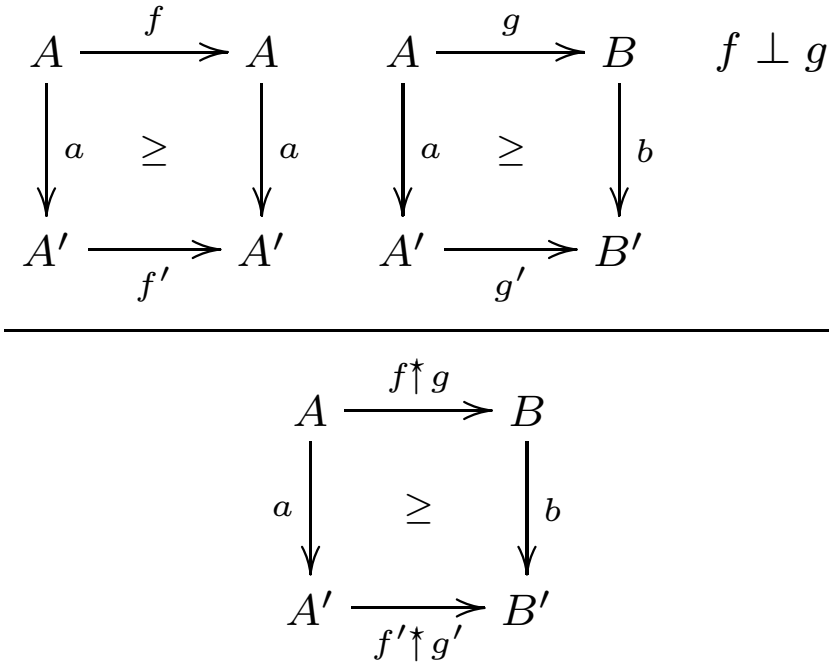
The Kleene wand $f\dagger g$ means intuitively "iterate the endomorphism f until the result lies in the definition of g and then apply g."

... it is `repeat` f `until` $\overline{g}$ `do` g.

Note f and g must be disjoint, where $f \perp g$ means $\{\overline{f}, \overline{g}\}$ is independent, so that the decision to continue or finish in this iteration is unambiguous.

Itegories

The Kleene wand is a **combinator** (uniform) and a **lax combinator** (lax uniform). This means.



where equality (resp. inequality) above the line implies equality (resp. inequality) below the line. With inequalities we call this condition **lax uniformity** while with equalities it is often called **uniformity**. Uniformity and lax uniformity are independent requirements.

Itegories

A full subcategory of an itegory is an itegory. If E is any set of idempotents then $\mathsf{Split}_E(\mathbf{X})$ is an itegory whenever $\mathbf{X}$ is.

- $f\dagger h = (f\dagger h)f \sqcup h$;
- $0\dagger g = g$;
- $f\dagger h = h(f\dagger \overline{h})$;
- If $c \perp d$ then $(bac\dagger d)ba = (acb\dagger db)a = cba\dagger dba$;
- $(f\dagger h)\overline{h} = h$;
- If $\overline{h^*}f = f$ and $h^* \perp h$ then $f\dagger h = h$.
- $f\dagger 0 = 0$;
- $f\dagger(hg) = h(f\dagger g)$;
- When $g \perp g'$ then $f\dagger g \perp f\dagger g'$ and $f\dagger(g \sqcup g') = (f\dagger g) \sqcup (f\dagger g')$;
- If $f \leq f'$ and $g \leq g'$ then $f\dagger g \leq f'\dagger g'$;

Inductive itegories

A combinator on a disjoint restriction category with the typing of a Kleene wand is an **inductive Kleene wand** in case it is a combinator - this means it is uniform - which satisfies the following conditions:

[iW.1] $(f\dagger g)f \sqcup g \leq f\dagger g$;

[iW.2] $xf \leq x$ and $hg \leq x$ implies $(f\dagger g)h \leq x$;

Proposition 3. *A disjoint restriction category with an inductive Kleene wand is an itegory.*

This has the important consequence:

Corollary 4. *Every disjoint restriction category can be embedded into an itegory.*

PROOF: Take the disjoint restriction category and infinitely complete it.
Define the Kleene wand in the completion as

$$f\dagger g = \bigsqcup_{i=0}^{\infty} gf^i$$

We must show $\{f^i g | i \leq n\}$ is independent. To establish this note that $g \perp fg$ and $g \perp f(g \sqcup fg) = fg \sqcup f^2 g$, but then $g \perp f(g \sqcup fg \sqcup f^2 g = fg \sqcup f^2 g \sqcup f^3 g$ etc. □

Traces and itegories

We shall denote the trace of a map

$$f = \begin{bmatrix} f_{00} & f_{01} \\ f_{10} & f_{11} \end{bmatrix} : A + X \longrightarrow B + X$$

$$\mathsf{Tr}(f) = \left[\begin{array}{c|c} f_{00} & f_{01} \\ \hline f_{10} & f_{11} \end{array}\right] : A \longrightarrow B$$

In a traced extensive restriction category we may define:

$$f \dagger g = \left[\begin{array}{c|c} 0 & 1 \\ \hline g & f \end{array}\right]$$

Theorem 5. *A disjoint restriction category is an itegory if and only if* $\mathsf{Mat}(\mathbf{X})$ *is a (laxly and uniformly) traced extensive category whose trace reduces to the Kleene wand as above.*

Traces and itegories

In this notation the identities required of a trace may be written:

Yanking: $1 = \left[\begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \end{array}\right]$

Tightening: $g \left[\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right] f = \left[\begin{array}{c|c} gaf & bf \\ \hline gc & d \end{array}\right]$

Superposition: $\left[\begin{array}{cc} f & 0 \\ 0 & \left[\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right] \end{array}\right] = \left[\begin{array}{cc|c} f & 0 & 0 \\ 0 & a & b \\ \hline 0 & c & d \end{array}\right]$

Compatibility: $\left[\begin{array}{c|c|c} a & b & c \\ \hline d & e & f \\ \hline h & i & j \end{array}\right] = \left[\begin{array}{c|c|c} a & c & b \\ \hline h & j & i \\ \hline d & f & e \end{array}\right]$

Traces and itegories

Lemma 6. *In any extensive restriction category with a trace on the coproduct:*

(i) $$\left[\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right] = a \sqcup (d^{\dagger} c)b = a \sqcup \left[\begin{array}{c|c} 0 & 1 \\ \hline c & d \end{array}\right] b;$$

(ii)

$$\left[\begin{array}{ccc|c} a_{00} & \dots & a_{0n} & b_0 \\ & \dots & & \\ a_{m0} & \dots & a_{mn} & b_m \\ \hline c_0 & \dots & c_n & d \end{array}\right] = \left[\begin{array}{ccc} \left[\begin{array}{c|c} a_{00} & b_0 \\ \hline c_0 & d \end{array}\right] & \dots & \left[\begin{array}{c|c} a_{0n} & b_0 \\ \hline c_n & d \end{array}\right] \\ & \dots & \\ \left[\begin{array}{c|c} a_{m0} & b_m \\ \hline c_0 & d \end{array}\right] & \dots & \left[\begin{array}{c|c} a_{mn} & b_m \\ \hline c_n & d \end{array}\right] \end{array}\right]$$

$$= \left[\begin{array}{ccc} a_{00} \sqcup (d^{\dagger} c_0)b_0 & \dots & a_{0n} \sqcup (d^{\dagger} c_n)b_0 \\ & \dots & \\ a_{m0} \sqcup (d^{\dagger} c_0)b_m & \dots & a_{mn} \sqcup (d^{\dagger} c_n)b_m \end{array}\right]$$

Traces and itegories

Here is the proof of the second identity from being traced:

$$
\begin{aligned}
(f\dagger g)\dagger(f\dagger h) &= \left[\begin{array}{c|c} 0 & 1 \\ \hline f\dagger h & f\dagger g \end{array}\right] = \left[\begin{array}{c|c} \left[\begin{array}{c|c} 0 & 0 \\ \hline h & f \end{array}\right] & \left[\begin{array}{c|c} 1 & 0 \\ \hline g & f \end{array}\right] \\ \hline \left[\begin{array}{c|c} 0 & 1 \\ \hline h & f \end{array}\right] & \left[\begin{array}{c|c} 0 & 1 \\ \hline g & f \end{array}\right] \end{array}\right] \\
&= \left[\begin{array}{c|c|c} 0 & 1 & 0 \\ \hline 0 & 0 & 1 \\ \hline h & g & f \end{array}\right] = \left[\begin{array}{c|c|c} 0 & 0 & 1 \\ \hline h & f & g \\ \hline 0 & 1 & 0 \end{array}\right] \\
&= \left[\begin{array}{c|c} \left[\begin{array}{c|c} 0 & 1 \\ \hline 0 & 0 \end{array}\right] & \left[\begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \end{array}\right] \\ \hline \left[\begin{array}{c|c} h & g \\ \hline 0 & 0 \end{array}\right] & \left[\begin{array}{c|c} f & g \\ \hline 1 & 0 \end{array}\right] \end{array}\right] = \left[\begin{array}{c|c} 0 & 1 \\ \hline h & f \sqcup g \end{array}\right] \\
&= (f \sqcup g)\dagger h
\end{aligned}
$$

Cartesian itegories

Proposition 7. *In any extensive restriction category which has partial products necessarily the product distributes over the coproduct.*

Conversely, when one has restriction zeros, if the partial products distribute over the coproducts, one necessarily has an extensive restriction category.

SO a cartesian extensive restriction category is a **distributive restriction category**.

A **cartesian disjoint restriction category** is a disjoint restriction category with partial products in which the product functors $A \times \_$ preserve disjunction and independence.

Proposition 8. *The extensive (matrix) completion of a cartesian disjoint restriction category has partial products and, thus, is a distributive restriction category.*

The product is constructed by distributing the product structure through the coproduct structure ...

Cartesian itegories

In a **cartesian itegory** is a cartesian disjoin restriction category in which one requires that the product functors preserve iteration: that

$$(A \times f)\dagger(A \times g) = A \times (f\dagger g)$$

This ensures that context variables are not affected by iteration.

Claim that these provide the semantics of flow diagrams ...

Computability in cartesian itegories

We can work without loss in the extensive completion ...

SO assume we are working in a distributive itegory ...

Stack objects

Our aim is to build a partial combinator algebra for this we need:
A **stack object** is an object with maps:

$$\begin{aligned} \mathsf{put} &: 1 + A \times A \longrightarrow A \\ \mathsf{get} &: A \longrightarrow 1 + A \times A \end{aligned}$$

such that $\mathsf{put}\ \mathsf{get} = 1_{1+A\times A}$.

This does force put to be a total map but there is, importantly, no requirement that get need be total.

Lemma 9. *The following are equivalent conditions for an object:*

(i) $1 \prec A$, $A + A \prec A$ *and* $A \times A \prec A$;

(ii) $1 + A \prec A$ *and* $A \times A \prec A$;

(iii) $1 + A \times A \prec A$ *(it is a stack object).*

A stack object is like an old warehouse ... if you put something in you can retrieve it BUT if you forget where you put it you will never find it!

Creating a PCA

Code	Value	Stack	Code	Value	Stack
end	x	[]	exit with x		
k	x	S	end	$\mathsf{k}_0(x)$	S
$\mathsf{k}_0(x)$	y	S	end	x	S
s	x	S	end	$\mathsf{s}_0(x)$	S
$\mathsf{s}_0(x)$	y	S	end	$\mathsf{s}_1(x,y)$	S
$\mathsf{s}_1(x,y)$	z	S	x	z	$\mathsf{cons}(\mathsf{c}_0(y,z),S)$
end	v	$\mathsf{cons}(\mathsf{c}_0(y,z),S)$	y	z	$\mathsf{cons}(c_1(v),S)$
end	v'	$\mathsf{cons}(\mathsf{c}_1(v),S)$	v	v'	S

Partial combinator reduction

$$\frac{A \times A \times A \xrightarrow{(\mathsf{step}\ \ \mathsf{exit})} A \times A \times A + A}{A \times A \times A \xrightarrow[\mathsf{step}\dagger\mathsf{exit}]{} A}$$

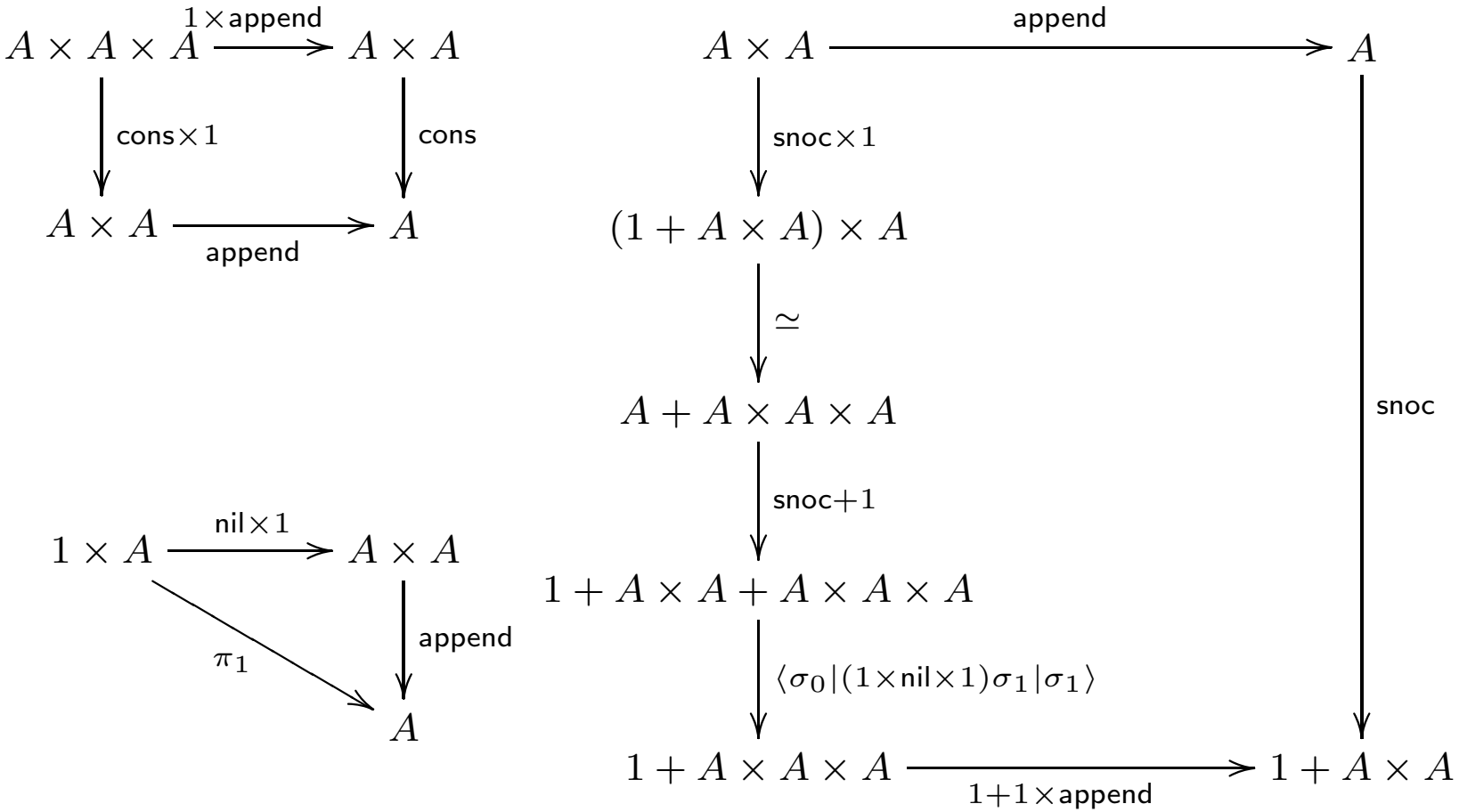
$$\_ \bullet \_ = (\mathsf{step}\dagger\mathsf{exit})(1 \times 1 \times [])$$

The identity for s is the only difficulty:

$$
\begin{array}{rcl}
((\mathsf{s} \bullet x) \bullet y) \bullet z & = & \mathsf{s}_1(x, y) \bullet z \\
& = & \mathsf{step}^\dagger\mathsf{exit}(\mathsf{s}_1(x, y), z, []) \\
& = & \left\{ \begin{array}{lcl} \sigma_0(c, v, d) & \mapsto & \mathsf{step}^\dagger\mathsf{exit}(c, v, d) \\ \sigma_1(x) & \mapsto & x \end{array} \right\} \mathsf{step}(\mathsf{s}_1(x, y), z, []) \\
& = & \mathsf{step}^\dagger\mathsf{exit}(x, z, \mathsf{cons}(\mathsf{c}_1(y, z), [])) \\
& = & \mathsf{step}^\dagger\mathsf{exit}(x \bullet z, \mathsf{end}, \mathsf{cons}(\mathsf{c}_1(y, z), [])) \\
& = & \mathsf{step}^\dagger\mathsf{exit}(y, z, \mathsf{cons}(\mathsf{c}_0(x \bullet z), [])) \\
& = & \mathsf{step}^\dagger\mathsf{exit}(y \bullet z, \mathsf{end}, \mathsf{cons}(\mathsf{c}_0(x \bullet z), [])) \\
& = & \mathsf{step}^\dagger\mathsf{exit}(y \bullet z, x \bullet z, []) \\
& = & (y \bullet z) \bullet (x \bullet z)
\end{array}
$$

Where we use repeatedly the identity: $\mathsf{step}^\dagger\mathsf{exit}(x, y, S) = \mathsf{step}^\dagger\mathsf{exit}(x \bullet y, \mathsf{end}, S)$ which, unfortunately, for a general trace and stack object may not hold!

For this identity we require that our stack object is **appending** that is has a map $\mathsf{append} : A \times A \longrightarrow A$ which satisfies



EXIT!