

# Join restriction categories and the importance of being adhesive

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# Restriction Categories

A category  $\mathbb{C}$  is a **restriction category** if it has a restriction operator:

$$\frac{X \xrightarrow{f} Y}{X \xrightarrow{\bar{f}} X}$$

$$[\text{R.1}] \quad f\bar{f} = f,$$

$$[\text{R.2}] \quad \bar{f}\bar{g} = \overline{gf},$$

$$[\text{R.3}] \quad \overline{gf} = \overline{g}\bar{f},$$

$$[\text{R.4}] \quad \overline{gf} = f\overline{gf}.$$

The domain of definition of  $f$  is expressed by  $\bar{f}$ . Restriction categories are abstract categories of partial maps.

A map is **total** if  $\bar{f} = 1$ . The total maps form a subcategory.

## More properties

- ▶ The restriction idempotents  $e = \bar{e} : A \rightarrow A$  form a semilattice written  $\mathcal{O}(A)$  (in fact  $\mathcal{O}$  is a contravariant functor to the category of semilattices with stable maps: a *corestriction* category). Think of these as the “open sets of  $A$ ”.
- ▶ Restriction categories are partial order enriched with
$$f \leq g \Leftrightarrow g\bar{f} = f$$
- ▶ A map  $f : A \rightarrow B$  is a **partial isomorphism** in case there is an  $f^{(-1)} : B \rightarrow A$  such that  $ff^{(-1)} = \overline{f^{(-1)}}$  and  $f^{(-1)}f = \bar{f}$ .
- ▶ A restriction category in which *all* maps are partial isomorphism is an inverse category. A one object inverse category is an inverse semigroup with a unit!

*Inverse categories are to restriction categories what groupoids are to categories.*

# Compatibility

- ▶ Restriction categories are **compatibility** enriched with  $f \smile g \Leftrightarrow g\bar{f} = f\bar{g}$ . This relation is preserved by composition:

$$f \smile g \Rightarrow hfk \smile hgk.$$

- ▶ A set  $S \subseteq \mathbb{C}(A, B)$  is **compatible** if for every  $s, s' \in S$ ,  $s \smile s'$ .

It is reasonable to consider a join operation restricted to compatible maps ....

# Join Restriction Categories

A restriction category  $\mathbb{C}$  is a **join restriction category** if for each compatible subset  $S \subseteq \mathbb{C}(A, B)$ , the join  $\bigvee_{s \in S} s \in \mathbb{C}(A, B)$  exists:

- ▶  $\bigvee_{s \in S} s$  is the join with respect to  $\leq$  in  $\mathbb{C}(A, B)$ ,
- ▶ The join is *stable* in the sense that:  $(\bigvee_{s \in S} s)g = \bigvee_{s \in S} (sg)$ .

Four consequences:

- ▶ The join is *universal* in the sense that  $f(\bigvee_{s \in S} s) = \bigvee_{s \in S} (fs)$ .
- ▶ The join commutes with the restriction  $\overline{\bigvee_{s \in S} s} = \bigvee_{s \in S} \bar{s}$ .
- ▶ Each  $\mathcal{O}(A)$  is a *locale*. (In fact  $\mathcal{O}$  is a covariant functor to the restriction category of locales with stable maps).
- ▶ Join restriction categories allow the manifold construction (Marco Grandis).

# Free Join Restriction Categories

Given any restriction category  $\mathbb{X}$ , one may construct from it a free join restriction category  $\mathbb{X} \rightarrow \widehat{\mathbb{X}}$  (Marco Grandis) with

- ▶ **objects:**  $X \in \mathbb{X}$ ;
- ▶ **maps:**  $S : A \rightarrow B$  where  $S \subseteq \mathbb{X}(A, B)$  is a down-closed compatible set;
- ▶ **identities:**  $1_A = \downarrow \{1_A\} = \{e \mid e = \bar{e} : A\} = \mathcal{O}(A)$ ;
- ▶ **composition:** for maps  $S : A \rightarrow B$  and  $T : B \rightarrow C$   
 $TS = \downarrow \{ts \mid s \in S, t \in T\}$ ;
- ▶ **restriction:**  $\bar{S} = \{\bar{s} \mid s \in S\}$ ;
- ▶ **join:**  $\bigvee_{i \in \Gamma} S_i = \bigcup_{i \in \Gamma} S_i$ , where each  $S_i$  is a down closed compatible set and  $\{S_i\}_{i \in \Gamma}$  are compatible sets.

# Partial Maps Categories

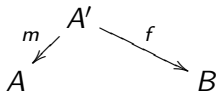
- ▶ A collection  $\mathcal{M}$  of monics is a **stable system of monics** if it includes all isomorphisms, is closed under composition and is pullback stable.
- ▶ For any stable system of monics  $\mathcal{M}$ , if  $mn \in \mathcal{M}$  and  $m$  is monic, then  $n \in \mathcal{M}$ .
- ▶ An  $\mathcal{M}$ -category is a pair  $(\mathbb{C}, \mathcal{M})$ , where  $\mathbb{C}$  is a category and  $\mathcal{M}$  is a stable system of monics in  $\mathbb{C}$ .
- ▶ Functors between  $\mathcal{M}$ -categories must preserve the selected monics *and* pullbacks of these monic. Natural transformations are “tight” (Manes) in the sense that they are cartesian over the selected monics.



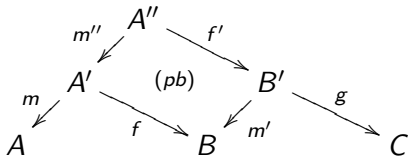
# Partial Maps Categories

The category of partial maps  $\text{Par}(\mathbb{C}, \mathcal{M})$  is:

- ▶ **objects:**  $A \in \mathbb{C}$ ;
- ▶ **maps:**  $(m, f) : A \rightarrow B$  (up to equivalence) with  $m : A' \rightarrow A$  is in  $\mathcal{M}$  and  $f : A' \rightarrow B$  is a map in  $\mathbb{C}$ :



- ▶ **identities:**  $(1_A, 1_A) : A \rightarrow A$ ;
- ▶ **composition:**  $(m', g)(m, f) = (mm'', gf')$ :



- ▶ **restriction:**  $\overline{(m, f)} = (m, m)$ .

## Completeness and representation

For a *split* restriction category,  $\mathbb{X}$ , the subcategory of total maps is an  $\mathcal{M}$ -category, where  $m \in \mathcal{M}$  if and only if it is monic and a partial isomorphism. In that case  $\text{Par}(\text{Total}(\mathbb{X}), \mathcal{M})$  is isomorphic to  $\mathbb{X}$ .

### Theorem (Completeness: Cockett and Lack)

*Every restriction category is a full subcategory of a partial map category.*

There is also a representation theorem:

### Theorem (Representation: Mulry)

*Any restriction category  $\mathbb{C}$  has a full and faithful restriction-preserving embedding into a partial map category of a presheaf category*

$$\mathbb{C} \rightarrow \text{Par}(\mathbf{Set}^{\text{Total}(\text{split}_r(\mathbb{C}))^{\text{op}}}, \widehat{\mathcal{M}})$$

# Completeness and representation with joins

When does an  $\mathcal{M}$ -category have its partial map category a join restriction category?

The answer:  $(\mathbb{X}, \mathcal{M})$  must be  $\mathcal{M}$ -adhesive ...

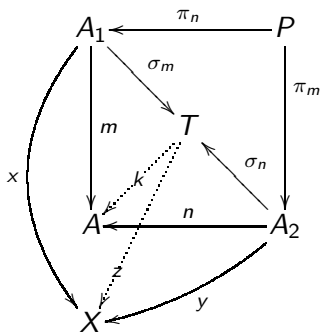
## Theorem (Cockett and Guo)

*Every join restriction category is a full subcategory of the partial map category of an adhesive  $\mathcal{M}$ -category whose gaps are in  $\mathcal{M}$ .*

The rest of the talk is about the proof of this and a few consequences ...

## First attempts ...

To form joins  $(m, x) \vee (n, y)$  in  $\text{Par}(\mathbb{C}, \mathcal{M})$ :



In order to have  $(m, x) \vee (n, y) = (k, z)$ , the **gap**  $k$  must be in  $\mathcal{M}$ , the **pushout**  $(\sigma_m, \sigma_n)$  of  $(\pi_m, \pi_n)$  must be **stable** under pulling back.

.... also need stability under composition of spans: what on earth is this????!!! ...



# Adhesive Categories

## Definition (Adhesive category, [4])

A category  $\mathbb{X}$  is said to be **adhesive** if

- (i)  $\mathbb{X}$  has pushouts along monics;
- (ii)  $\mathbb{X}$  has pullbacks;
- (iii) pushouts along monics are van Kampen squares.

**Set** and elementary toposes are adhesive  
but **Pos**, **Top**, **Grp**, and **Cat** are not [4].

We want to extend the notions of van Kampen squares and adhesive categories to **van Kampen colimits** and **adhesive  $\mathcal{M}$ -categories** ....

## van Kampen colimits in general

A colimit  $\alpha : D \Rightarrow C$ , where  $D : \mathbb{S} \rightarrow \mathbb{C}$ , is **van Kampen** if for any diagram  $D' : \mathbb{S} \rightarrow \mathbb{C}$ , any cone  $\alpha' : D' \Rightarrow X$  under  $D'$ , and any commutative diagram

$$\begin{array}{ccc} D' & \xrightarrow{\alpha'} & X \\ \Downarrow \beta & & \downarrow r \\ D & \xrightarrow{\alpha} & C \end{array}$$

in which  $\beta$  is cartesian natural transformation,  $\alpha' : D' \Rightarrow X$  is a colimit if and only if for each  $s \in \mathbb{S}$

$$\begin{array}{ccc} D'(s) & \xrightarrow{\alpha'(s)} & X \\ \beta(s) \downarrow & & \downarrow r \\ D(s) & \xrightarrow{\alpha(s)} & C \end{array}$$

is a pullback diagram.

# van Kampen colimits

Some properties:

- ▶ van Kampen colimits are pullback stable.
- ▶ Let  $D_i$  be diagrams on  $\mathbb{S}_i$ ,  $i = 1, 2$ . If both  $\alpha_1 : D_1 \Rightarrow X$  and  $\alpha_2 : D_2 \Rightarrow X$  are van Kampen colimits, then so is  $\alpha_1 \times_X \alpha_2 : D_1 \times_X D_2 \Rightarrow X$ , where  $D_1 \times_X D_2 : \mathbb{S}_1 \times \mathbb{S}_2 \rightarrow \mathbf{C}$  is given by the following pullback diagram:

$$\begin{array}{ccc} (D_1 \times_X D_2)(s_1, s_2) & \xrightarrow{\beta(s_1, s_2)} & D_2(s_2) \\ \gamma(s_1, s_2) \downarrow & & \downarrow \alpha_2(s_2) \\ D_1(s_1) & \xrightarrow{\alpha_1(s_1)} & X \end{array}$$

and  $(\alpha_1 \times_X \alpha_2)(s_1, s_2) = \alpha_1(s_1)\gamma(s_1, s_2) = \alpha_2(s_2)\beta(s_1, s_2)$ , for each  $(s_1, s_2) \in \mathbb{S}_1 \times \mathbb{S}_2$ .



## van Kampen $\mathcal{M}$ -amalgams

A **stable poset** is a poset with *binary* meets. When  $\mathbb{S}$  is a stable poset and  $D : \mathbb{S} \rightarrow \mathcal{M}$  a diagram, an  $\mathcal{M}$ -cone  $\alpha : D \Rightarrow X$  is an  **$\mathcal{M}$ -amalgam** in case for all  $s_1, s_2 \in \mathbb{S}$  each

$$\begin{array}{ccc} D(s_1 \wedge s_2) & \xrightarrow{D(\leq)} & D(s_1) \\ D(\leq) \downarrow & & \downarrow \alpha(s_1) \\ D(s_2) & \xrightarrow{\alpha(s_2)} & X \end{array}$$

is a pullback diagram.

A stable poset  $\mathcal{M}$ -diagram  $D : \mathbb{S} \rightarrow \mathcal{M}$  is  **$\mathcal{M}$ -amalgamable** if there is an  $\mathcal{M}$ -amalgam under  $D$ .

# $\mathcal{M}$ -adhesive categories

- ▶ An  $\mathcal{M}$ -category  $\mathbb{X}$  is an  $\mathcal{M}$ -**adhesive category** if each amalgamable  $\mathcal{M}$ -diagram  $D$  has a van Kampen colimit.
- ▶ A map  $g : X \rightarrow Y$  in an  $\mathcal{M}$ -adhesive category is an  $\mathcal{M}$ -**gap** if there is a van Kampen colimit  $\nu : D \Rightarrow X$  such that each  $g\nu(s) \in \mathcal{M}$  for each  $s \in \mathbb{S}$ :

$$\begin{array}{ccc} D & \xRightarrow{\nu} & X \\ & \searrow \alpha & \downarrow g \\ & & Y \end{array}$$

**Note:**  $\mathcal{M}$ -gaps are necessarily monic so that these van Kampen colimits are  $\mathcal{M}$ -amalgams.

# Mind the gap

What is the relation to van Kampen squares? When  $\mathcal{M}$ -gaps are  $\mathcal{M}$  ...

## Theorem

*An  $\mathcal{M}$ -category is  $\mathcal{M}$ -adhesive with all  $\mathcal{M}$ -gaps in  $\mathcal{M}$  if and only if all  $\mathcal{M}$ -amalgams which are pushouts have van Kampen colimits whose gaps are in  $\mathcal{M}$ .*

The situation when the  $\mathcal{M}$ -gaps are *not* in  $\mathcal{M}$  is of interest ...

# $\mathcal{M}$ -adhesive Categories

The class  $\mathcal{M}_{\text{gap}}$  of all  $\mathcal{M}$ -gaps in an  $\mathcal{M}$ -adhesive category  $\mathbf{C}$  is a stable system of monics in  $\mathbf{C}$  with  $\mathcal{M} \subseteq \mathcal{M}_{\text{gap}}$ .

## Theorem

*If  $\mathbb{X}$  is an  $\mathcal{M}$ -adhesive category, then*

- (i)  $\mathbb{X}$  is an  $\mathcal{M}_{\text{gap}}$ -adhesive category;*
- (ii)  $(\mathcal{M}_{\text{gap}})_{\text{gap}} = \mathcal{M}_{\text{gap}}$ .*

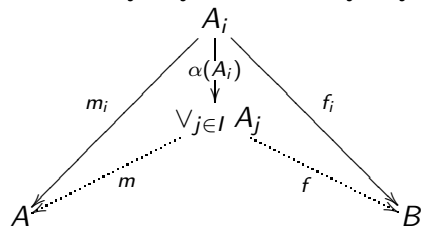
So one can always complete an  $\mathcal{M}$ -adhesive category to be closed to gaps.

# Completeness for joins

## Theorem

Let  $\mathbb{X}$  be a category with a stable system of monics  $\mathcal{M}$ . Then  $\text{Par}(\mathbb{X}, \mathcal{M})$  is a join restriction category if and only if  $\mathbb{X}$  is an  $\mathcal{M}$ -adhesive category and  $\mathcal{M}_{\text{gap}} \subseteq \mathcal{M}$ .

PROOF: ( $\Leftarrow$ ) For any compatible set  $\{(m_i, f_i) \mid i \in I\}$ ,  $\nu : D \Rightarrow A$ , given by  $\nu(i) = m_i$ , is a stable  $\mathcal{M}$ -cone on  $\{A_i\}$ ,  $D$  has a VK colimit  $(\bigvee_{j \in I} A_j, \alpha)$ .  $\exists! m : \bigvee_{j \in I} A_j \rightarrow A$  and  $\exists f : \bigvee_{j \in I} A_j \rightarrow B$ :  
 $(m, f) = \bigvee \{(m_i, f_i) \mid i \in I\}$  and



$\text{Par}(\mathbb{X}, \mathcal{M})$  is a jrCat.

# Completeness for joins

( $\Rightarrow$ ) For all  $\mathcal{M}$ -diagrams  $D : \mathbb{S} \rightarrow \mathcal{M}$ , and an  $\mathcal{M}$ -amalgam  $\alpha : D \Rightarrow X$ ,

- ▶ The join  $\bigvee_{s \in \mathbb{S}} (\alpha(s), \alpha(s)) = (m, m)$  exists,  $m : C \rightarrow X \in \mathcal{M}$ ;
- ▶  $(\alpha(s), \alpha(s)) \leq (m, f)$  implies there is an  $\mathcal{M}$ -map  $\iota(s) : D(s) \rightarrow C$  implies  $\exists$  an amalgam  $\mathcal{M}$ -cone  $\iota : D \Rightarrow C$ .
- ▶  $\iota : D \Rightarrow C$  is a van Kampen colimit.

# Free joins and $\mathcal{M}$ -gaps

- ▶ Since any elementary topos is adhesive [4],  $\mathbf{Set}^{\mathbf{Total}(\mathit{split}_r(\mathbb{C}))^{\text{op}}}$  is an adhesive category.
- ▶ Since  $\mathcal{M} \subseteq \mathcal{M}_{\text{gap}}$ , there is a faithful embedding:

$$\begin{array}{c} \mathbf{Par}(\mathbf{Set}^{\mathbf{Total}(\mathit{split}_r(\mathbb{C}))^{\text{op}}}, \widehat{\mathcal{M}}) \\ \downarrow \mathcal{E} \\ \mathbf{Par}(\mathbf{Set}^{\mathbf{Total}(\mathit{split}_r(\mathbb{C}))^{\text{op}}}, \widehat{\mathcal{M}_{\text{gap}}}) \end{array}$$

## Free joins and $\mathcal{M}$ -gaps

Hence there is a unique restriction functor  $\mathcal{F} : \widehat{\mathbb{C}} \rightarrow \text{Par}(\mathbf{Set}^{\text{Total}(\text{split}_r(\mathbb{C}))^{\text{op}}}, \widehat{\mathcal{M}}_{\text{gap}})$  such that






$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\text{Par}(\mathcal{Y})J_{\mathbb{C}}} & \text{Par}(\mathbf{Set}^{\text{Total}(\text{split}_r(\mathbb{C}))^{\text{op}}}, \widehat{\mathcal{M}}) \\ \eta_{\mathbb{C}} \downarrow & & \downarrow \varepsilon \\ \widehat{\mathbb{C}} & \xrightarrow{\mathcal{F}} & \text{Par}(\mathbf{Set}^{\text{Total}(\text{split}_r(\mathbb{C}))^{\text{op}}}, \widehat{\mathcal{M}}_{\text{gap}}) \end{array}$$

commutes

The functor  $\mathcal{F}$  in the last commutative diagram is full and faithful. So constructing joins in the Grothendieck category is the same as constructing joins directly ...



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