Seely categories revisited

Robin Cockett

robin@cpsc.ucalgary.ca

University of Calgary
Department of Computer Science

Seely categories: Canada's contribution.

Robert Seely introduced "Seely Categories" as the categorical proof theory of linear logic (1989):

- *-autonomous category with products;
- $(!(\_), \delta, \epsilon)$ a comonad;
- Natural isomorphisms $s_2 :!(A \times B) \longrightarrow !(A) \otimes !(B)$ and $s_1 :!(1) \longrightarrow \top$.

Seely proved that the coKleisli category of such a comonad was Cartesian closed.

Seely categories: Cambridge contributes.

The Cambridge four (Gavin Beirman, Nick Benton, Valaria de Paiva, Martin Hyland) start working on the proof theory of MELL (multiplicative exponential linear logic). Gavin Beirman realizes that in the proof theory of MELL the comonad must be monoidal (1995).

He defined a **linear category** to be:

- Monoidal closed category (with products);
- $(!(\_), \delta, \epsilon)$ a monoidal comonad;
- Each $!(A)$ forms a comonoid $(!(\_), \Delta, e)$:
  - Δ and e are monoidal;
  - Δ and e are coalgebra morphisms;
  - Every algebra morphism is a morphism of comonoids.

Not a simple axiomatization any more!

Points out that, as an interepretation of the proof theory of MELL, Seely's original axiomatization is "unsound." Introduces *new Seely categories* as linear categories with products.

Seely categories: Manchester contributes.

Andrea Schalk (c2003) in a nice overview collected and simplified the axiomatization from the Cambridge gang of four. She removed the necessity for the category to be closed and focused on the structure of the comonads calling them *linear exponential comonads*.

- Monoidal category;
- $(!(\_), \delta, \epsilon)$ a monoidal comonad;
- In the category of coalgebras the induced tensor product is in fact a product.

Seely categories: the French contribute.

Jean-Yves Girard (1987) Linear Logic ...

Yves Lafont (1988) investigated the free cocommutative comonoid functor and showed it provided a model of linear logic ...

Paul-Andre Mellies (2005) revisiting “models of linear logic”

- Concentrates entirely on the structure of the exponential comonad in monoidal closed categories; *-autonomous categories hardly rate a mention and additives are reduced to products;
- Focuses on the linear/non-linear model acknowledging influence from Plotkin;
- Proves a converse to Kelly’s lemma: an adjunction with left adjoint iso/strict monoidal is a monoidal adjunction.
- Provided another characterization of (new) Seely categories which involved axiomatizing the Seely isomorphism more carefully.

Below we present a modification of this last idea.

Storage transformation

A **storage transformation** on a monoidal category $\mathbf{X}$ with products and a comonad $(S(\_), \delta, \epsilon)$ is a symmetric comonoidal transformation on $S : (\mathbf{X}, \times, 1) \longrightarrow (\mathbf{X}, \otimes, \top)$.

$$s_1 : S(1) \longrightarrow \top \qquad s_2 : S(A \times B) \longrightarrow S(A) \otimes S(B)$$

such that

$$\begin{tikzcd}
S(A \times 1) \arrow[r, "s_2"] \arrow[d, "\pi_0"'] & S(A) \otimes S(1) \arrow[d, "1 \otimes s_1"] \\
S(A) & S(A) \otimes \top \arrow[l, "u^R_\otimes"]
\end{tikzcd}
\qquad
\begin{tikzcd}
S(A \times B) \arrow[r, "s_2"] \arrow[d, "S(c_\times)"'] & S(A) \otimes S(B) \arrow[d, "c_\otimes"] \\
S(B \times A) \arrow[r, "s_2"'] & S(B) \otimes S(A)
\end{tikzcd}$$

$$\begin{tikzcd}
S((A \times B) \times C) \arrow[r, "s_2"] \arrow[d, "S(a_\times)"'] & S(A \times B) \otimes S(C) \arrow[r, "s_2 \otimes 1"] & (S(A) \otimes S(B)) \otimes S(C) \arrow[d, "a_\otimes"] \\
S(A \times (B \times C)) \arrow[r, "s_2"'] & S(A) \otimes S(B \times C) \arrow[r, "1 \otimes s_2"'] & S(A) \otimes (S(B) \otimes S(C))
\end{tikzcd}$$

such that δ is a comonoidal transformation.

Storage transformation

To say δ is comonoidal is to say explicitly that the following commutes:

$$\begin{tikzcd}
S(A\times B) \arrow[rr, "s^2"] \arrow[d, "\delta"'] & & S(A)\otimes S(B) \arrow[d, "\delta\otimes\delta"] \\
S^2(A\times B) \arrow[r, "S(\sigma_2^\times)"'] & S(S(A)\times S(B)) \arrow[r, "s_2"'] & S^2(A)\otimes S^2(B)
\end{tikzcd}$$

$$\begin{tikzcd}
S(1) \arrow[rr, "s_0"] \arrow[d, "\delta"] & & \top \arrow[d, equal] \\
S(S(1)) \arrow[r, "S(\sigma_0^\times)"'] & S(1) \arrow[r, "s_0"'] & \top
\end{tikzcd}$$

where $\sigma_2^\times = \langle S(\pi_0), S(\pi_1)\rangle : S(A\times B) \longrightarrow S(A)\times S(B)$ and $\sigma_1^\times : S(1) \longrightarrow 1$ is the unique map.

Coalgebra modalities

A **coalgebra modality** is a comonad $(S(\_), \delta, \epsilon)$ on a monoidal category such that for each A there is a natural cocommutative comonoid

$$\Delta : S(A) \longrightarrow S(A) \otimes S(A) \qquad e : S(A) \longrightarrow \top$$

such that

$$\begin{tikzcd}
S(A) \arrow[r, "\delta"] \arrow[d, "\Delta"'] & S^2(A) \arrow[d, "\Delta"] \\
S(A) \otimes S(A) \arrow[r, "\delta \otimes \delta"'] & S^2(A) \otimes S^2(A)
\end{tikzcd}$$

Coalgebra modalities

Proposition 1. *A symmetric monoidal category with products has a coalgebra modality if and only if it has a storage transformation.*

To define a coalgebra modality from a storage transformation:

$$S(A) \xrightarrow{e} \top \quad = \quad S(A) \xrightarrow{S(!)} S(1) \xrightarrow{s_1} \top$$

$$S(A) \xrightarrow{\Delta} S(A) \otimes S(A) \quad = \quad S(A) \xrightarrow{S(\Delta_\times)} S(A \times A) \xrightarrow{s_2} S(A) \otimes S(A)$$

To define a storage transformation from a coalgebra modality:

$$S(1) \xrightarrow{s_1} \top \quad = \quad S(1) \xrightarrow{e} \top$$

$$S(A \times B) \xrightarrow{s_2} S(A) \otimes S(B)$$

$$= \quad S(A \times B) \xrightarrow{\Delta} S(A \times B) \otimes S(A \times B) \xrightarrow{S(\pi_0) \otimes S(\pi_1)} S(A) \otimes S(B).$$

(Modern) linear categories

We amalgamate Bierman and Schalk's ideas and use coalgebra modalities ...

A (modern) **linear category** is a monoidal category with a coalgebra modality such that Δ and e are monoidal and coalgebra morphisms.

Note we are able to drop one of Bierman's conditions (coalgebra morphisms are morphisms of comonoids) as it is a consequence of the requirement of being a coalgebra modality.

A (modern) linear category is, thus, a model of the proof theory of MELL which is not necessarily monoidal closed.

Mellies theorem

Theorem 2. *(Mellies) A symmetric monoidal category with products and a coalgebra modality $(S(\_), \delta, \epsilon)$ is a (modern) linear category if and only if it has a storage transformation which is an isomorphism (i.e. has a Seely isomorphism).*

The surprising aspect is that having a storage transformation which is an isomorphism forces the monad to be monoidal. Here is the monoidal structure:

$$
\begin{aligned}
& \top \xrightarrow{m_\top} S(\top) \\
& \quad = \quad \top \xrightarrow{s_1^{-1}} S(1) \xrightarrow{\delta} S^2(1) \xrightarrow{S(s_1)} S(\top) \\
& S(A) \otimes S(B) \xrightarrow{m_\otimes} S(A \times B) \\
& \quad = \quad S(A) \otimes S(B) \xrightarrow{s_2} S(A \times B) \xrightarrow{\delta} S^2(A \times B) \\
& \qquad\qquad \xrightarrow{S(s_2)} S(S(A) \otimes S(B)) \xrightarrow{S(\epsilon \otimes \epsilon)} S(A \otimes B)
\end{aligned}
$$

This is what I shall call a (modern) **Seely category** ... a monoidal category with products and a coalgebra modality whose storage transformation is an isomorphism.

So Seely's original idea was nearly right!!

CoKleisli categories of Seely categories

What does the coKleisli category of a Seely category look like?

CoKleisli category of Seely categories have been the main source of applications:

- (Stable) domain theory;
- Game theoretic models of computation;
- Constructing reflexive objects (models of the λ-calculus);
- Cocommutative cofree coalgebras (Foch spaces).

Strong system of maps

Let $\mathbf{X}$ be any category with products then one can associate with $\mathbf{X}$ the simple fibration

$$\begin{array}{c} \mathbf{X}[\mathbf{X}] \\ \Big\downarrow{\scriptstyle \partial} \\ \mathbf{X} \end{array}$$

the simple fiber over A is $\mathbf{X}[A]$ (the simple slice) and is the coKleisli category for the comonad $A \times \_$.
Explicitly $\mathbf{X}[A]$:

Objects: $X \in \mathbf{X}$

Maps: $f : X \longrightarrow Y$ is a map $f : A \times X \longrightarrow Y$ in $\mathbf{X}$

Identities: $X \longrightarrow X$ correspond to $\pi_1 : A \times X \longrightarrow X$

Composition:

$$\frac{A \times X \xrightarrow{f} Y \quad A \times Y \xrightarrow{g} Z}{A \times X \xrightarrow[\Delta \times 1]{} A \times A \times X \xrightarrow[1 \times f]{} A \times Y \xrightarrow[g]{} Z}$$

Strong system of maps

A strong system of maps is a subfibration with some additional properties:

$$\begin{tikzcd} \mathcal{L}[\mathbf{X}] \arrow[rr] \arrow[dr] & & \mathbf{X}[\mathbf{X}] \arrow[dl, "\partial"] \\ & \mathbf{X} & \end{tikzcd}$$

where we call the maps of $\mathcal{L}[A]$ **systematic** over A.

Explicitly the systematic maps must satisfy:

- Each $\mathcal{L}[A]$ contains identities, projections and pairings of systematic maps;
- Each $\mathcal{L}[A]$ is closed to composition and if $g \in \mathcal{L}[A]$ is a retraction in $\mathbf{X}[A]$ then $gf \in \mathcal{L}[A]$ then $f \in \mathcal{L}[A]$.
- If $f \in \mathcal{L}[B]$ and $a : A \longrightarrow B$ then $(a \times 1)f \in \mathcal{L}[A]$.

Strong system of maps: examples

1. Consider the category of Abelian groups (resp. vector spaces) with arbitrary set maps: this is a left additive category that is $f(g+h) = fg + fh$ and $f0 = 0$ (but not the other distributive law). Amongst these arrows are those which are additive – that is Abelian group homomorphisms (resp. linear maps). These form a strong system of maps.

   Note that this category is the coKleisli category with respect to the comonad induced on Abelian groups (resp. vector spaces) by the free functor from sets.
2. (Girard) Coherent spaces with stable maps with the systematic maps being the linear maps (preserving union).
3. (Ehrhard) Hypercoherent spaces and hypercoherent stable maps with the systematic maps being the linear maps (preserving union).
4. (Blute, Cockett, Seely) Differential categories: the linear maps in any cartesian differential category form a stong system of maps – this was the original motivation for this work.

Classification

A strong system of maps is **classified** in case in each $\mathbf{X}[A]$ given any A there is a universal map $A \xrightarrow{\varphi} S(A)$ which makes f systematic:

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\varphi"'] & Y \\
S(X) \arrow[ur, dotted, "f^\sharp"'] &
\end{tikzcd}
$$

in $\mathbf{X}$ this means:

$$
\begin{tikzcd}
A \times X \arrow[r, "f"] \arrow[d, "1\times\varphi"'] & Y \\
A \times S(X) \arrow[ur, dotted, "f^\sharp"'] &
\end{tikzcd}
$$

Persistent classification

The systematic maps are persistent over classification in case whenever f is systematic in $\mathbf{X}[A \times X]$ then

$$\begin{tikzcd} A \times X \times W \arrow[r, "f"] \arrow[d, "1\times\varphi\times 1"'] & Y \\ A \times S(X) \times W \arrow[ur, dotted, "f^\sharp"'] & \end{tikzcd}$$

$f^\sharp$ is systematic in $\mathbf{X}[A \times S(X)]$

A **pre-storage category** is a Cartesian category with a strong system of maps which are persistently classified.

Classification

Proposition 3. *If $\mathcal{L}$ is a strong system of maps for $\mathbf{X}$ then:*

1. *If $\mathcal{L}$ is classified then S forms a strong monad which in addition satisfies:*

$$\begin{tikzcd}
S(A) \arrow[rr, "S(\Delta_\times)"] \arrow[d, "\delta"'] & & S(A \times A) \arrow[d, "\delta"] \\
S^2(A) \arrow[r, "S(\langle 1, \epsilon \rangle)"'] & S(A \times S(A)) \arrow[r, "S(\theta)"'] & S^2(A \times A)
\end{tikzcd}$$

2. *If $\mathcal{L}$ is persistent over classification then S forms a commutative monad (i.e. is monoidal on the product):*

$$\begin{tikzcd}
 & S(A \times S(B)) \arrow[r, "S(\theta)"] & S^2(A \times B) \arrow[dr, "\mu"] & \\
S(A) \times S(B) \arrow[ur, "\theta'"] \arrow[dr, "\theta"'] & & & S(A \times B) \\
 & S(S(A) \times B) \arrow[r, "S(\theta')"'] & S^2(A \times B) \arrow[ur, "\mu"'] &
\end{tikzcd}$$

Representation

In a category $\mathbf{X}$ with a strong system of maps a map $f : X \times Y \longrightarrow Z$ is **bisystematic** in $\mathbf{X}[A]$ in case it is systematic in $\mathbf{X}[A][X]$ and $c_{\mathbf{X}} f$ is systematic in $\mathbf{X}[A][Y]$. That is it is systematic in each argument.
We say that bisystematic maps are **represented** in case in each $\mathbf{X}[A]$ there is a universal bisystematic map $\varphi_\otimes : X \times Y \longrightarrow X \otimes Y$ so that for any bisystematic map f there is a unique systematic map $f^{\sharp\otimes}$ rendering commutative:

$$\begin{tikzcd} X \times Y \arrow[r, "f"] \arrow[d, "\varphi_\otimes"'] & Z \\ X \otimes Y \arrow[ur, dotted, "f^{\sharp\otimes}"'] & \end{tikzcd}$$

in $\mathbf{X}$ this means

$$\begin{tikzcd} A \times X \times Y \arrow[r, "f"] \arrow[d, "1 \times \varphi_\otimes"'] & Z \\ A \times X \otimes Y \arrow[ur, dotted, "f^{\sharp\otimes}"'] & \end{tikzcd}$$

Persistent representation and storage

The systematic maps are persistent over the representation of bisystematic maps in case whenever f is systematic in $\mathbf{X}[A \times X \times Y]$ (i.e. is systematic in the W argument)

$$\begin{tikzcd} A \times X \times Y \times W \arrow[r, "f"] \arrow[d, "\varphi"] & Z \\ A \times X \otimes Y \times W \arrow[ur, dotted, "f^{\sharp\otimes}"'] & \end{tikzcd}$$

then $f^{\sharp\otimes}$ is systematic in $\mathbf{X}[A \times X \otimes Y]$:

A **storage category** is a Cartesian category $\mathbf{X}$ with a strong system of maps which is retentively both classified and represented.

Theorem 4. *In any storage category* $\mathbf{X}$ *the systematic maps form a Seely category for which* $\mathbf{X}$ *is the coKleisli category.*

Strong abstract coKleisli categories

Can we describe a pre-storage category in terms of its functorial properies?

A **strong abstract coKleisli category** is a Cartesian category together with:

- A strong functor S;
- A natural transformaton $\varphi : 1_{\mathbf{X}} \longrightarrow S$ such that $\varphi_{S(\_)}$ is strong;
- An unnatural transformation such that $\epsilon_{S(\_)}$ is a strong natural transformation.

  Such that:

  - $\varphi\epsilon = 1;$
  - $S(\varphi)\epsilon = 1;$
  - $\epsilon\epsilon = S(\epsilon)\epsilon$
  - π_0 and π_1 are ϵ-natural.

Following Carsten Furhman who introduced abstract Kleisli categories to facilitate arguments about Kleisli categories.

Strong abstract coKleisli categories

Proposition 5. *In any strong abstract coKleisli category* $\mathbf{X}$ *the maps* f *in* $\mathbf{X}[A]$ *such that*

$$\begin{tikzcd}
A \times S(X) \arrow[r, "\theta"] \arrow[d, "1\times\epsilon"'] & S(A \times X) \arrow[r, "S(f)"] & S(Y) \arrow[d, "\epsilon"] \\
A \times X \arrow[rr, "f"'] & & Y
\end{tikzcd}$$

form a strong system of maps. Furthermore this system is classified if and only if φ *(itself) is strong, and the systematic maps are persistent over this classification if and only if* $(S, \varphi, \epsilon_{S(\_)})$ *is a commutative monad.*

This gives an alternative characterization of prestorage categories .... There is also a complete equational description of a comonad which gives rise to a pre-storage category (backward engineer all this!) ...

CoKleisli categories of Seely categories

SO what does the coKleisli category of a Seely category look like?

MORALLY it is a storage category!

... but it is a more little technical!

In fact, as far as I can see, at the moment, it only need be a pre-storage category ... HOWEVER there are various conditions which ensure that a pre-storage category is a storage category! This is now work in progress ... A reasonable statement of affairs, however, seems to be:

Conjecture: The coKleisli category of any *exact* Seely category which is monoidal *closed* is a storage category.
A Seely category is **exact** if the following is a coequalizer:

$$S^2(A) \underset{\epsilon}{\overset{S(\epsilon)}{\rightrightarrows}} S(A) \xrightarrow{\epsilon} A$$