Topological aspects of restriction categories

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Introduction to restriction categories

A restriction category is a category with a restriction operator

$$\frac{X \xrightarrow{f} Y}{X \xrightarrow[\overline{f}]{} X}$$

which satisfies the following four axioms:

[R.1] $f\overline{f} = f$

[R.2] $\overline{g}\,\overline{f} = \overline{f}\,\overline{g}$

[R.3] $\overline{g}\,\overline{f} = \overline{g\overline{f}}$

[R.4] $\overline{g}f = f\overline{gf}$

Restriction functors are functors, which, in addition, preserve the restriction: $F(\overline{x}) = \overline{F(x)}$.
These axioms are independent ... here is a sample equality:

$$\overline{\overline{g}f} = \overline{f\overline{gf}} = \overline{f}\;\overline{gf} = \overline{gf}\;\overline{f} = \overline{gf\overline{f}} = \overline{gf}$$

Basic properties of restriction categories

- The total maps, x such that $\overline{x} = 1$, form a subcategory.
- Parallel maps can be partially ordered by $x \leq y$ iff $x = y\overline{x}$; this is a partial order enrichment (that is if $x \leq y$ then $gxf \leq gyf$).
- The *restricted isomorphisms*, $f : X \longrightarrow Y$ with a (necessarily unique) "partial inverse" $g; Y \longrightarrow X$ such that $gf = \overline{f}$ and $fg = \overline{g}$, form a subcategory (which is an *inverse category*).
- As $\overline{f} = \overline{f}\,\overline{f} = \overline{\overline{f}}$, maps e with $e = \overline{e}$ are called restriction idempotents. The set of restriction idempotents at an object X,

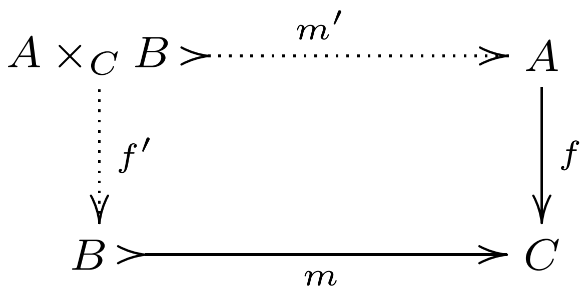
$$R(X) = \{e : X \longrightarrow X \mid \overline{e} = e\}$$

  form a commutative monoid of idempotents and therefore is a *semilattice*.
- *Restriction monics* are monic restricted isomorphisms and are splittings of restriction idempotents.

Completeness of restriction categories

An $\mathcal{M}$-stable system of monics satisfies:

- Each $m \in \mathcal{M}$ is monic
- Composites of maps in $\mathcal{M}$ are themselves in $\mathcal{M}$
- All isomorphisms are in $\mathcal{M}$
- Pullbacks along of an $\mathcal{M}$-map along any map always exists and is an $\mathcal{M}$-map.



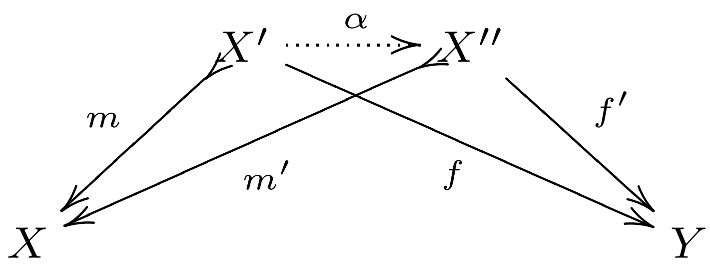
Theorem 1. *(Cockett-Lack) Every restriction category has a fully structure preserving embedded into the $\mathcal{M}$-partial map category of a category with a stable system of monics $\mathcal{M}$.*

Partial map categories

Let $(\mathbf{X}, \mathcal{M})$ be a category with a stable system of monics then $\mathsf{Par}(\mathbf{X}, \mathcal{M})$, the category of $\mathcal{M}$-partial maps, is the following:

Objects: $X \in \mathbf{X}$

Maps: Spans $(m, f) : X \longrightarrow Y$ where $m \in \mathcal{M}$ upto eqivalence:

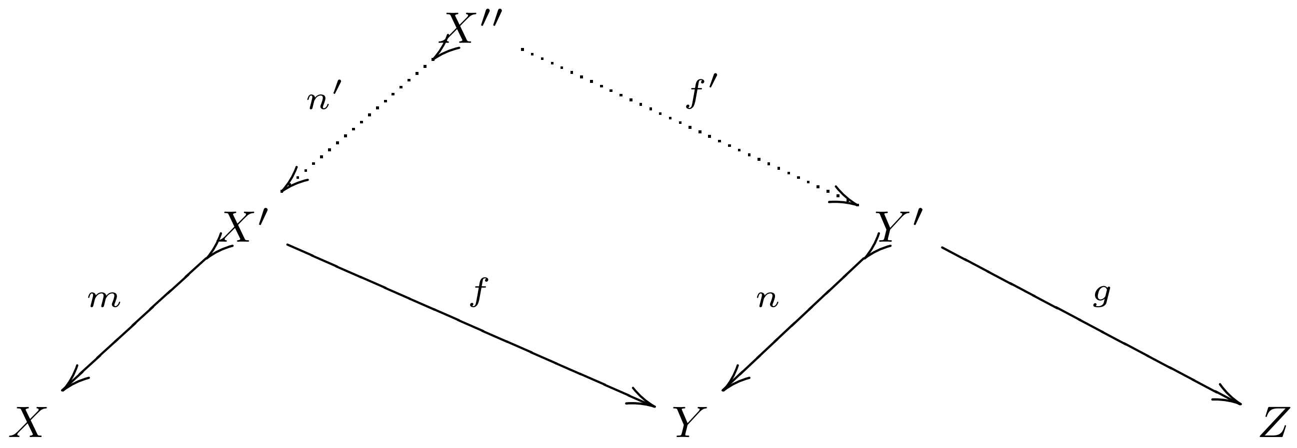


Where $(m, f) \sim (m' f')$ when there is an isomorphism α' making the diagram commute.

Identities: $(1_X, 1_X) : X \longrightarrow X$

Partial map categories

Composition: By pullback:



Proposition 2. *All partial map categories* $\mathsf{Par}(\mathbf{X}, \mathcal{M})$*, as above, are (split) restriction categories with* $\overline{(m, f)} = (m, m)$.

The completeness theorem is proven by spitting the idempotents of a restriction category, $\mathbf{X}$, to obtain a total map category in which the restricted monics form a system of $\mathcal{M}$-maps. The original restriction category $\mathbf{X}$ then sits elegantly inside $\mathsf{Par}(\mathsf{Total}(\mathbf{X}, \mathcal{M}))$.

$\mathbf{X}$ has products iff $\mathsf{Par}(\mathbf{X}, \mathcal{M})$ has partial products.

The restriction category $\mathsf{sSLat}^{\mathrm{op}}$

The category of semilattices in which the homomorphisms are *stable*, in the sense that they preserve binary products, sSLat, is defined as follows:

Objects: Semilattices $(X, \top, \wedge)$

Maps: $f : X \longrightarrow Y$ is a stable (or binary meet preserving) map. That is
$f(x \wedge y) = f(x) \wedge f(y)$ (but f does *not* necessarily preserve the top, $\top$).

This category has a corestriction defined by:

coRestriction: If $f : X \longrightarrow Y$ then $\overline{f} : Y \longrightarrow Y$ has $\overline{f}(x) = f(\top) \wedge x$.

It is easy to check this is a corestriction category. Therefore, $\mathsf{sSLat}^{\mathrm{op}}$, the dual of the category of semilattices with stable maps, is a restriction category.

$\mathsf{sSLat}^{\mathrm{op}}$ has partial products given by the coproduct in SLat (which is the same as the product).

The fundamental functor

Every restriction category has a "fundamental" restriction functor to $\mathsf{sSLat}^{\mathrm{op}}$:

$$R : \mathbf{X} \longrightarrow \mathsf{sSLat}^{\mathrm{op}}; \quad \begin{array}{c} X \\ {\scriptstyle f}\big\downarrow \\ Y \end{array} \quad \mapsto \quad \begin{array}{c} R(X) \\ \big\uparrow{\scriptstyle R(f)} \\ R(Y) \end{array}$$

where $R(X) = \{e : X \longrightarrow X \mid e = \overline{e}\}$ and

$$R(f) : R(Y) \longrightarrow R(X); e \mapsto \overline{ef}$$

Note that $\overline{f} = R(f)(1_Y)$.

Open maps in sSLat

While sSLat is, topologically speaking, a poor cousin it does have a good notion of an open map.

Definition 3. *in* sSLat, $f : X \longrightarrow Y$ *is an* open map *in case there is a restricted left adjoint* $f_! : Y \longrightarrow X$*, that is an order preserving map* $f_! : Y \longrightarrow X$ *such that:*

- $f_!(x) \wedge y \leq f_!(x \wedge f(y))$ *(Frobenius reciprocity)*
- $f_!(f(x)) \leq x$ *(counit)*
- $x \wedge f(\top) \leq f(f_!(x))$ *(restricted unit)*

Note that $f_!(x) = f_!(x \wedge f(\top))$ is an immediate consequence of these axioms.
In fact $f_!$ is uniquely determined by f as $f_!$ is the real left adjoint to f on $\{x \mid x \leq f(\top)\}$ (the "codomain" of f).
Open maps compose, all restricted isomorphisms are open maps, in particular, restriction idempotents are open maps.

Range restriction categories

What does it mean for a restriction category $\mathbf{X}$ to have its fundamental functor taking values in open maps?
A restriction category is a *range restriction category* if it is equipped with a range operator:

$$\frac{X \xrightarrow{f} Y}{Y \xrightarrow[\widehat{f}]{} Y}$$

satisfying:

[RR.1] $\overline{\widehat{f}} = \widehat{f}$

[RR.2] $\widehat{f} f = f$

[RR.3] $\widehat{\overline{g} f} = \overline{g}\,\widehat{f}$

[RR.4] $\widehat{g \widehat{f}} = \widehat{g f}$

Idea is that each map not only has a domain of definition given by the restriction but also a range of definition. Cearly there is a factorization involved ....

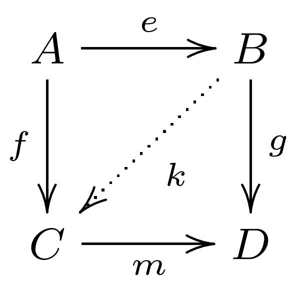
Examples of range restriction categories

1. The category of sets and partial maps.
2. The category of partial recursive functions on the natural numbers.
3. The category of topological spaces with partial maps defined on an open subset and an open map on that subset.
4. The category $\mathsf{sSLat}^{\mathrm{op}}$ with open maps is a range restriction category.
5. Any restriction category has a maximal subcategory which is a range restriction category consisting of those maps f such that $R(f)$ is an open map in sSLat.

Factorization systems

An $(\mathcal{E}, \mathcal{M})$-*factorization system* on a category $\mathbf{X}$ consists of two classes of map $\mathcal{E}$ and $\mathcal{M}$ such that

- Both $\mathcal{E}$ and $\mathcal{M}$ are closed to composition and contain all isomorphisms
- Every map f admits a factorization $f = me$ where $e \in \mathcal{E}$ and $m \in \mathcal{M}$
- $\mathcal{E}$ is orthogonal to $\mathcal{M}$ in the sense that in the following square whenever $e \in \mathcal{E}$ and $m \in \mathcal{M}$ then there is a unique cross-map k which makes both triangles commute:

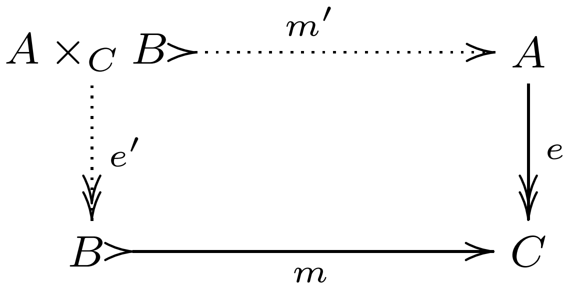


Example: the epic/monic factorization in of sets. More generally the regular epic/monic factorization in any regular category. The factorization of a commutative ring homomorphism into a localization followed by a morphism which inverts no elements.

Completeness of range restriction categories

An $(\mathcal{E}, \mathcal{M})$-factorization system is $\mathcal{M}$-*stable* in case:

- Every $m \in \mathcal{M}$ is monic
- Pullbacks along $\mathcal{M}$-maps always exist
- An $\mathcal{E}$-maps pulled back along an $\mathcal{M}$-map is an $\mathcal{E}$-map.



Theorem 4. *(Cockett-Guo) Every range restriction category has a full structure preserving embedding into an $\mathcal{M}$-partial map category of a category with an $\mathcal{M}$-stable factorization system.*

Locales and stable maps

Another example of a restriction category, sLoc. This is the dual of the category of locales with stable frame maps.

The category sLoc is defined by:

Objects: Locales $(X, \wedge, \top, \bigvee)$

Maps: $f : X \longrightarrow Y$ is a *stable frame map* that is a binary meet, arbitrary join preserving map $f : Y \longrightarrow X$

Restriction: Is given by the corestriction of the fram map $f : Y \longrightarrow X$

$$\overline{f} : X \longrightarrow X; x \mapsto f(\top) \wedge x$$

The category of locales is closely connected to the category of topological spaces (e.g. spatial locales are equivalent to Sober spaces). Locales are often referred to as "pointless topological spaces".

Join restriction categories

When does a restriction category have its fundamental functor landing in locales?

Morally: when it is a join restriction category!

$f \smile g$ iff $g\overline{f} = f\overline{g}$ (f is *compatible* with g: they agree where both are defined).

A set X of parallel maps is *compatible* if every pair of maps in the set is compatible.

A *join restriction category* has joins of compatible sets, X, such that:

$$\overline{\bigvee_{x \in X} x} = \bigvee_{x \in X} \overline{x} \qquad f(\bigvee_{x \in X} x)g = \bigvee_{x \in X} (fxg)$$

In any join restriction category:

(i) $R(X)$ for an object X is a locale.

(ii) $R(f) : R(X) \longrightarrow R(Y)$ is a stable locale map.

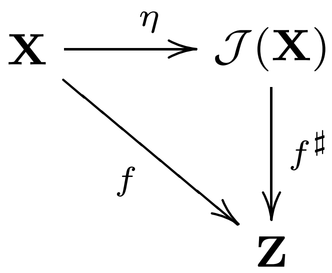
(iii) The subcategory of open maps is a join restriction category.

Examples of join restriction categories

1. The category of sets and partial maps.
2. The partial maps of any Grothendieck topos.
3. The category of topological spaces with partial maps defined on an open subsets.
4. The subcategory of open maps of topological spaces with open maps defined on open subsets.
5. The category $\mathsf{sLoc}^{\mathrm{op}}$.
6. The subcategory of open maps of the above.

The join completion

Theorem 5. *For every restriction category* $\mathbf{X}$ *there is a structure preserving embedding into its join completion* $\mathcal{J}(\mathbf{X})$ *which has the following universal property:*



Here f and η are restriction functors, $\mathbf{Z}$ and $\mathcal{J}(\mathbf{X})$ are join restriction categories and $f^\sharp$ is a unique restriction functor which preserves joins.
If $\mathbf{X}$ has partial products so does $\mathcal{J}(\mathbf{X})$.

Construction of the join completion

The construction of $\mathcal{J}(\mathbf{X})$ is as follows:

Objects: $X \in \mathbf{X}$

Maps: Down closed sets of compatible parallel arrows $S = \downarrow S \subseteq \mathbf{X}(X, Y)$

Identities: $R(X) \subseteq \mathbf{X}(X, X)$

Composition: $TS = \downarrow \{ts \mid t \in T, s \in S\}$

Restriction: $\overline{S} = \{\overline{s} \mid s \in S\}$

Join: $\bigvee_i S_i = \bigcup_i S_i$

$\mathcal{J}(\mathbf{X})$ is a join restriction category and $\eta : \mathbf{X} \longrightarrow \mathcal{J}(\mathbf{X})$ is a faithful restriction functor.

Not everything has joins ...

Two examples of a restriction categories which are *not* a join restriction categories:

1. $\mathsf{Par}(\mathsf{CRing}^{\mathrm{op}}, \mathsf{Loc})$: the partial map category with respect to localizations Loc of the opposite of the the category of commutative rings, CRing.

   Think algebraic geometry!

   $\mathcal{J}(\mathsf{Par}(\mathsf{CRing}^{\mathrm{op}}, \mathsf{Loc}))$ is a join restriction category and it is a category of schemes ...

2. $\mathsf{Par}(\mathsf{ffSet}, \mathsf{monic})$: the category of finitely fibered partial maps between sets. f is *finitely fibred* in case $f^{-1}(x)$ is finite for each x.

   $\mathcal{J}(\mathsf{Par}(\mathsf{ffSet}, \mathsf{monic}))$ is in this case $\mathsf{Par}(\mathsf{Set}, \mathsf{monic}))$.

Closed and proper maps

Closed maps in locales are determined by;.

Definition 6. *A stable frame map between locales is* closed *in case* $f : Y \longrightarrow X$*, as a map of frames, has a restricted right adjoint* $f_* : X \longrightarrow Y$ *such that*

- $f_*(x) \leq f_*(f(\top))$ *(restricted)*
- $f(f^*(x)) \leq x \wedge f(\top)$ *(unit)*
- $y \leq f_*(f(y))$ *(unit)*
- $f_*(x \vee f(y)) \leq f_*(x) \vee y$ *(dual of Frobenius reciprocity)*

It is proper *in case* f^* *preserves directed joins* $f^*(\curlyvee_i x_i) = \curlyvee_i f^*(x_i)$*.*

For frames these partial adjoints alway exist thus we are really looking for the Frobenius reciprocity.

Closed maps compose but the pullback of a closed map is not necessarily closed. Closed maps whose pullbacks are always closed are proper maps.

Topological properties

Definition 7. *In a join restriction category with partial products*

(i) *A* discrete object *is an object with* $\Delta : X \longrightarrow X \otimes X$ *an open map.*

(ii) *A* Haussdorf object *is an object with* $\Delta : X \longrightarrow X \otimes X$ *a closed map.*

(iii) *A* Compact object *is an object* X *with* $1 = \bigvee_{i \in I} u_i$ *(directed join) then* $1 = u_j$ *for some* $j \in I$.

(iv) *A map* $f : X \longrightarrow Y$ *is a* local homeomorphism *in case there are a set of restriction idempotents* $U \subseteq R(Y)$ *which cover* Y*, that is* $\bigvee_{u \in U} u = 1_Y$ *so that* fu *is a restricted isomorphism for each* $u \in U$.

Question: is this definition of compactness the same as demanding that $X \xrightarrow{!} 1$ is proper? Problem is that 1 can have significant structure ..

Other developments

1. (Marco Grandis) The *manifold construction*: given a join restriction category $\mathbf{X}$ constructs a new join restriction category $\mathsf{Man}(\mathbf{X})$ which has all manifolds made from atlases in $\mathbf{X}$.
   For each object $X \in \mathsf{Man}(\mathbf{X})$ the category of local homeomorpisms to X is a topos of sheaves.
2. (Robin Cockett and Dorette Pronk ... in progress) The *obifold construction*: given a join restriction category $\mathbf{X}$ constructs a new join restriction category $\mathsf{Orb}(\mathbf{X})$ which has all orbifolds made from atlases in $\mathbf{X}$.
3. (Day dreams ...) How do you do differentiable manifolds?
4. (Dreams ...) Simplectic manifolds?

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