NURBS

❖ Who’s afraid of NURBS, Anyway?
Overhead of a bus at SIGGRAPH ’98

The most important modeling technique in CAD
Another professional software:
Rhinoceros NURBS modeling for windows
Design sketch
Wireframe Model

Rendered
Motivation

- It is hard to produce exact circle with B-splines
- It is important for rotational objects
- We need more control over the curve
Motivation

- Different weights of $P_i$, to have different attraction factors

- Weighted B-spline

  - Candidate:
    \[ \sum_{i=0}^{m} P_i w_i N_{i,k}(u) \]

  - Is not a valid affine operation
  - Need to be normalized in homogenous coordinate
  - Normalization factor: \[ \sum_{i=0}^{m} w_i N_{i,k}(u) \]
Perspective Projection

- The situation is exactly as the perspective projection

Derivation of a Perspective Projection

- P: the given 3D point
- View plane: positioned at d distance
- O: (0,0,0) is the center of perspective
- P’: the designed projected point
Closed form of Perspective Transformation

- \( x_p = x \cdot \frac{d}{z} \)
- \( y_p = y \cdot \frac{d}{z} \)
- \( z_p = d \)

- Nonlinear formula?!
- Matrix form?!

Modifying Homogenous Coordinates

- \((x,y,x,w)\)
- \(w=0\): vector
- \(w=1\): point
- Other values?
- \((wx,wy,wz,w) = (x,y,z,1)\)
- Perspective matrix

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & \frac{1}{d} & 0
\end{bmatrix}
\]
NURBS

- Non-Uniform Rational B-spline

\[ R(u) = \frac{\sum_{i=0}^{m} P_i w_i N_{i,k}(u)}{\sum_{i=0}^{m} w_i N_{i,k}(u)} , \ w_i : \ weight \ of \ P_i \]

- Denominator is a normalization factor

Lifting and projection concept

3D homogenous coordinate \rightarrow 3D B-spline curve

2D control points with weights \rightarrow 2D NURBS curve
**Lifting and projection concept**

**3D NURBS**

- 4D homogenous coordinate
- 4D B-spline curve
- 3D control points with weights
- 3D NURBS curve

---

**NURBS**

\[ R(u) = \frac{\sum_{i=0}^{m} P_i w_i N_{i,k}(u)}{\sum_{i=0}^{m} w_i N_{i,k}(u)} , \quad w_i : \text{weight of } P_i \]

- Denominator is a normalization factor
- If the weights are set to 1 the fraction become B-spline
- Higher weights \(\Rightarrow\) more attraction
- It has local effect
- Very useful for representing the conic section exactly
Rational Basis Functions

\[ R_{i,k}(u) = \frac{N_{i,k}(u)w_i}{\sum_{j=0}^{m} N_{j,k}(u)w_j} \]

\[ R(u) = \sum_{i=0}^{m} R_{i,k}(u)P_i \]

\{R_{i,k}(u)\} are Rational basis functions

Properties of Rational Basis Functions

- With the standard knot sequence:
  - Non negativity: \( R_{i,k}(u) \geq 0 \), all \( i, k, u \in [0, 1] \)
  - Unit summation: \( \sum_{i=0}^{m} R_{i,k}(u) = 1 \), for all \( u \in [0, 1] \)
  - \( R_{0,k}(0) = R_{m,k}(1) = 1 \)
  - Local support: \( R_{i,k}(u) = 0 \), \( u \notin (u_i, u_{i+k}) \)
  - \( w_i = 1 \) for all \( i \), then \( R_{i,k}(u) = N_{i,k}(u) \)
  - How about \( w_i = \alpha \)
The Characteristic of NURBS Curves

- $R(0) = P_0$, $R(1) = P_n$
- Affine invariance
- NURBS curves are also invariant under perspective projections
- Strong convex hull property
  
  \[
  \text{if } u \in [u_i, u_{i+1}) \text{ then } R(u) \text{ lies within the convex hull of the control points } P_{i-k+1}, \ldots, P_i
  \]
- Local control: if the control point $P_i$ is moved, or the weight $w_i$ is changed, it effects only that portion of the curve on the interval $u \in [u_i, u_{i+k})$

NURBS Flexibility

- Control points (same as non-rational)
- Knot movement (same as non-rational)
- Multiple knot (same as non-rational)
- Weight control (a new option)
- Conic sections are produced exactly by NURBS (a good reason for CAD industry!)

\[
U = \{0, 0, 0, 1/4, 1/4, 1/2, 3/4, 3/4, 1, 1, 1\}
\]

\[
W = \{1, \frac{\sqrt{2}}{2}, 1, \frac{\sqrt{2}}{2}, 1, \ldots\}
\]
NURBS and Linear Combination

\[ R(u) = \frac{\sum P_i w_i N_{i,k}(u)}{\sum w_i N_{i,k}(u)}, \quad P_i = (x_i, y_i, z_i) \]

- Expanded version

\[ R(u) = \left( \frac{\sum x_i w_i N_{i,k}(u)}{\sum w_i N_{i,k}(u)}, \frac{\sum y_i w_i N_{i,k}(u)}{\sum w_i N_{i,k}(u)}, \frac{\sum z_i w_i N_{i,k}(u)}{\sum w_i N_{i,k}(u)} \right) \]

- We need 4 times of:

\[ \sum E_i N_{i,k}(u) \]

Computing sums of B-splines

\[ S(u) = \sum_i E_i B_{i,k}(u) \]

- It appears in curve, surface, NURBS and their derivatives

- \( u \) is a given and fixed parameter value

- :the given real numbers (such as \( E_i \))