Introduction to Fractals

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Ways of creating graphic models

- Constructed solid geometry
- Digital scan
- Implicit
- Particles
- Physics
- Procedural – grammars
  - fractals
Procedural models

- Objects defined by procedures and algorithms rather than being defined by the primitives
- Objects are generated at the rendering stage when it is possible to determine what primitives are necessary
- Can be used in conjunction with a random number generator
  - Can vary details of a given object
  - Good for real world objects such as plants and terrains
  - Provides variation within structure

Defining dimension

- What is the dimension of a
  - Line
  - Curve

- What is the dimension of a
  - Plane
  - Surface
An Aluminum sheet

• Bend and fold the sheet gradually
• Continuous deformation of shape
• Discontinuous variation of “dimension”
• Extension of “dimension” to fraction numbers or any real numbers

Effective factors

• Measuring the wiggleness and wrinkles
• More wiggleness means more dimension
• “Self-similarity” small portion of the object, when magnified, can reproduce a large portion
Self Similarity

- “self similarity” in nature
- zoom sequence of a coastline
- clouds, mountains, trees and plants

Fractals examples
Fractals and Self-similarity

Sierpinski Triangle, ST, composed of three congruent figures, each 1/2 the size of ST magnifying any of the 3 pieces by a factor of 2, gives an exact replica the original form.

ST consists of 3 self-similar copies of itself, each with magnification factor 2.

ST consists of 9 self-similar copies with magnification factor 3.

ST consists of 27 self-similar pieces, magnification factor 8.

ST consists of 3^{n} self-similar pieces, magnified by a factor of 2^{n}, recreates entire figure.

Fractals self-similarity at all scales

Fractal Dimension

• Lets go back to the familiar
  • line has dimension 1,
  • a plane dimension 2, and
  • a cube dimension 3.

• But why is this?
  • line has dimension 1 because there is only 1 way to move on a line
    • backward and forward? 2 directions in a line and infinitely many in the plane.
  • the plane has 2 dimensions because there are 2 linearly independent directions

• In more rigorous mathematical language?
  • A line segment
  • Can be divided into 4 self-similar intervals, each with the same length, and each of which when magnified by a factor of 4 yield original segment.
  • or a line segment into 7 self-similar pieces, each with magnification factor 7.
  • or N self-similar pieces, each with magnification factor N.
Fractal Dimension

A square
- Can be decompose a square into 4 self-similar sub-squares, and the magnification factor here is 2.
- or 9 self-similar pieces with magnification factor 3.
- or 25 self-similar pieces with magnification factor 5.
- the square may be broken into $N^2$ self-similar copies of itself, each of which must be magnified by a factor of $N$ to yield the original figure.

A cube
- Can be divided into $N^3$ self-similar pieces, each of which has magnification factor $N$.
- The dimension is the exponent of the number of self-similar pieces with magnification factor $N$ into which the figure may be broken.

Fractal Dimension

$$\text{Dimension} = \frac{\log (\text{number of self-similar pieces})}{\log (\text{magnification factor})}$$

Line
- Dimension = $\frac{\log N^1}{\log N} = 1$

Square
- Dimension = $\frac{\log N^2}{\log N} = 2$

Cube
- Dimension = $\frac{\log N^3}{\log N} = 3$

Fractal dimension = $\frac{\log (\text{number of self-similar pieces})}{\log (\text{magnification factor})}$
Fractal Dimension

Dimension of the Sierpinski Triangle

3 self-similar pieces, each with magnification factor 2.

Fractal Dimension = \( \frac{\log \text{(number of self-similar pieces)}}{\log \text{(magnification factor)}} \)

= \( \frac{\log 3}{\log 2} \)

= \( \approx 1.58 \)

(dimensions of S is somewhere between 1 and 2)

But the Sierpinski Triangle also consists of

9 self-similar pieces with magnification factor 4.

\[ D = \frac{\log \text{(number of self-similar pieces)}}{\log \text{(magnification factor)}} \]

= \( \frac{\log 9}{\log 4} \approx 1.58 \)

or

\[ = \frac{\log 3^2}{\log 2^2} \]

= \( \frac{2 \log 3}{2 \log 2} \)

= \( \frac{\log 3}{\log 2} \)

= \( \approx 1.58 \)

Fractal dimension is a measure of how "complicated" a self-similar figure is.

Von Koch Snowflake

1. Divide a given line into 3 equal line segments

2. Replace the middle line segment with two equal segments forming an equilateral triangle

3. Repeat steps 1 and 2 for all line segments
Fractal dimension of Koch Snowflake

- 4 self-similar pieces
- Magnification factor 3

\[ D_F = \log 4 / \log 3 = \approx 1.26 \]

Still really a line
Euclidean dimension is 1

More examples

- Quadratic Koch
- 8 self-similar pieces
- Magnification factor 4

\[ D_F = \log 8 / \log 4 = 3 \log 2 / 2 \log 2 = 1.5 \]

Peano’s Space Filling

- 9 self-similar pieces
- Magnification factor 3

\[ D_F = \log 9 / \log 3 = 2 \log 3 / \log 3 = 2 \]
Properties of fractals

- Complex shapes (strange lengths and volumes)
- Self similarity
- Good models for natural shapes
- Simple recursive algorithms

Recursive algorithm for fractals

```c
void divide_triangle(point a, point b, point c, int m)
{ /* triangle subdivision using vertex numbers */
    point v0, v1, v2;
    int j;
    if(m>0)
    {
        for(j=0; j<2; j++) v0[j]=(a[j]+b[j])/2;
        for(j=0; j<2; j++) v1[j]=(a[j]+c[j])/2;
        for(j=0; j<2; j++) v2[j]=(b[j]+c[j])/2;
        divide_triangle(a, v0, v1, m-1);
        divide_triangle(c, v1, v2, m-1);
        divide_triangle(b, v2, v0, m-1);
    }
    else(triangle(a,b,c)); /* draw triangle at end of recursion */
}
```
examples

- http://www.fractalism.com/fractals/geometry.htm

Credit: F. Griffin. 1997

Beehive pools fractal – part of the Mandelbrot set. Colours have been chosen to emphasize the structure and detail of the M-set rather than to make an artistic statement.

http://olpaimages.nsf.gov/viewimage.cfm?ImageNum=691&RecNum=2&PageType=all