CPSC 453 – 2D Transformations

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Translation

- To move a point P(x, y)

  T_x units in x

  T_y units is y

  $x' = x + T_x \qquad y' = y + T_y$

- gives new location P' (x', y')
- in column vectors

  $P = \begin{vmatrix} x \\ y \end{vmatrix} \qquad P' = \begin{vmatrix} x' \\ y' \end{vmatrix} \qquad T = \begin{vmatrix} T_x \\ T_y \end{vmatrix}$

- can be stated P' = P + T

Scaling

- Points can be scaled along x and y axis by multiplication

$$x' = S_x \cdot x \qquad y' = S_y \cdot y$$

$$P = \begin{vmatrix} x' \\ y' \end{vmatrix} = \begin{vmatrix} S_x & 0 \\ 0 & S_y \end{vmatrix} \cdot \begin{vmatrix} x \\ y \end{vmatrix}$$

$$P' = S \cdot P$$

  - differential scaling

- scaling scaling w.r.t. origin - needs a fixed point
- S > 1 larger, 0<S<1 smaller, S<0 reflection

Rotation

- Rotation about the origin
- write x and y in polar coords

$$x = \rho\cos\alpha \qquad y = \rho\sin\alpha$$

$$x' = \rho\cos\alpha\cos\theta - \rho\sin\alpha\sin\theta = x\cos\theta - y\sin\theta$$

$$y' = \rho\cos\alpha\sin\theta - \rho\sin\alpha\cos\theta = x\sin\theta + y\cos\theta$$

$$\begin{vmatrix} x' \\ y' \end{vmatrix} = \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} \cdot \begin{vmatrix} x \\ y \end{vmatrix}$$

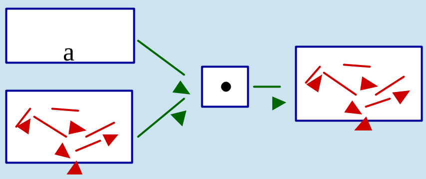
$$P' = R \cdot P$$

Types transformations

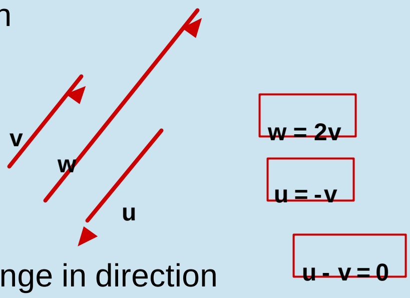
- Rigid body transformations
  - Shape of object stays the same
  - preserves – lengths, angles, parallelism of lines
- affine transformations
  - preserve parallelism of lines
  - not shape, length or angles

Properties

- scalar-vector multiplication



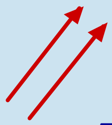
- scalar-vector multiplication



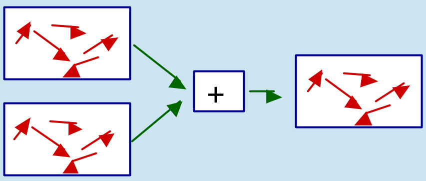
- result is a vector
- change in length
- if scalar is positive no change in direction
- if scalar is negative → inverse direction

Operations, Properties

- Vectors
  - direction,
  - magnitude
  - No location

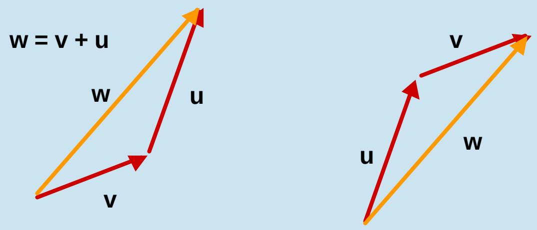


- vector-vector addition
  - results in a vector



Can make new vectors out of vector scalar expressions

r = a**w** + b**u** - **v**



Algebraic Interpretation

- A vector is an n-tuple
  - $v = (v_1, v_2, \ldots, v_n)$
- Scalar multiplication

  $\alpha v = (\alpha v_1, \alpha v_2, \ldots, \alpha v_n)$
- Vector – vector addition
  - $v + w = (v_1, v_2, \ldots, v_n) + (w_1, w_2, \ldots, w_n)$

    $= ((v_1 + w_1), (v_2 + w_2), , \ldots, (v_n + w_n))$

- Linear combination
  - $v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n$

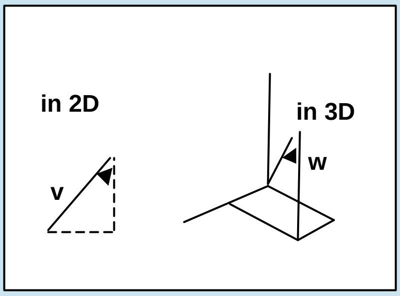
- Linearly independent
  - If the only set of scalars that results in
  - $0 = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n$
  - is $0 = \alpha_1 = \alpha_2 = \ldots = \alpha_n$

Algebraic Interpretation

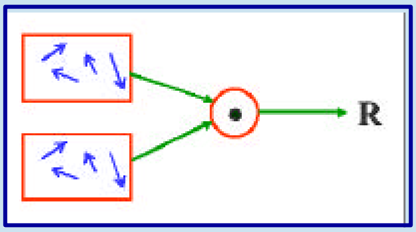
- Dimension
  - The greatest number of linearly independent vectors is the dimension of the vector space
  - If the dimension is **n**, then **n** linearly independent vectors form a **basis**
- From a Basis set
  - A vector can be specified
  - $v = \beta_1 v_1 + \beta_2 v_2 + \ldots + \beta_n v_n$
  - The scalars β_i → give the representation of v with respect to that basis
- Representation
  - Different basis – different representation

Magnitude

- Length or magnitude of a vector
  - In 2D
  $|v| = \sqrt{v_1^2 + v_2^2}$
  - in 3D
  $|v| = \sqrt{v_1^2 + v_2^2 + v_3^2}$
- Unit vector
  - Has a magnitude of length 1
- Normalized vector
  - Is a vector whose magnitude has been adjusted to be length 1
  - Multiply by the reciprocal of its length
  $(1/|v|)\ v = \ ((1/|v|)\ v_1^2 + (1/|v|)\ v_2^2 + \ldots.+ (1/|v|)\ v_3^2$
- Direction



Dot Product



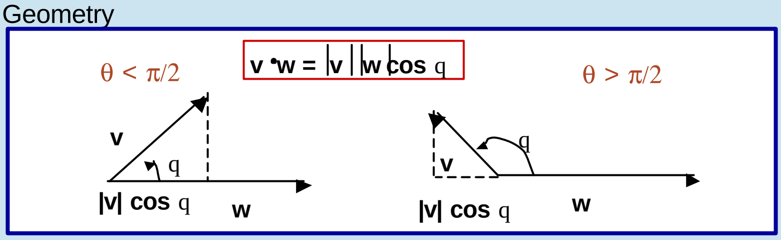
- Dot product or inner product
- Result is a scalar

  $v \bullet w = \alpha$

  $v \bullet w = (v_1, v_2, v_3) \bullet (w_1, w_2, w_3)$

  $= w_1 v_1 + w_2 v_2 + w_3 v_3$

- Geometry



- Properties

  cos q = (v •w) / |v| |w|

  θ = π/2 cos q = 0 = (v •w)

Angle Between Two Vectors

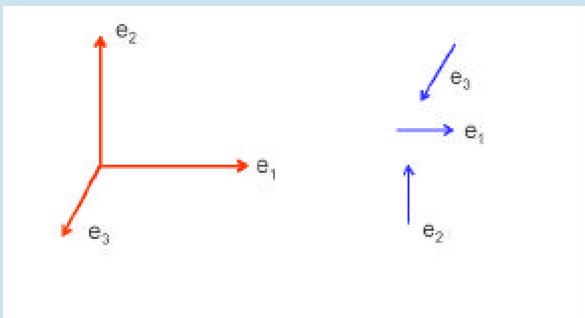
- Angle between two vectors

  cos q = (v •w) / |v| |w|

- Orthogonality condition

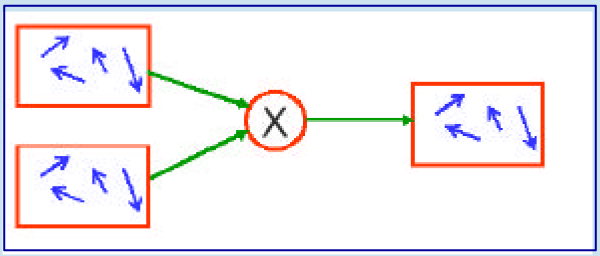
  θ = π/2 cos q = 0 = (v w)

- Orthonormal



Cross Product

- Just for 3D



- cross product, outer product, vector product
- Given 2 non-parallel vectors, v and w, cross product can determine a third vector n that is orthogonal to both v and w.
- n = v X w

Cross Product

- formula

  $u \times v = (u_1, u_2, u_3) \times (v_1, v_2, v_3)$

  $= ((u_2 v_3 - u_3 v_2), (u_3 v_1 - u_1 v_3), (u_1 v_2 - u_2 v_1))$

- Determinate notation

  $u \times v = (u_1, u_2, u_3) \times (v_1, v_2, v_3)$

$$\begin{matrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{matrix}$$

$$= \left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, -\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right)$$

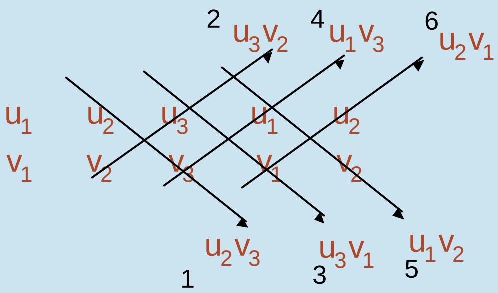
$= ((u_2 v_3 - u_3 v_2), (u_3 v_1 - u_1 v_3), (u_1 v_2 - u_2 v_1))$

Cross Product

- memory aid?

$u \times v = (u_1,\ u_2,\ u_3) \times (v_1,\ v_2,\ v_3)$

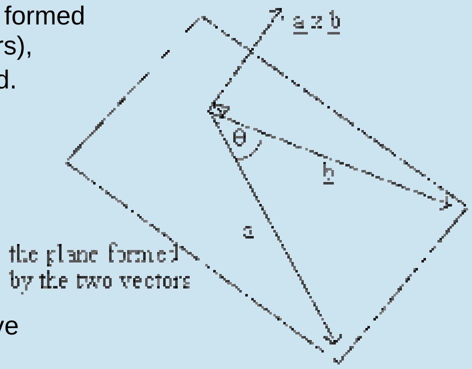
$= (\ (u_2 v_3 - u_3 v_2),\ (u_3 v_1 - u_1 v_3),\ (u_1 v_2 - u_2 v_1)\ )$



$= (\ (u_2 v_3 - u_3 v_2),\ (u_3 v_1 - u_1 v_3),\ (u_1 v_2 - u_2 v_1)\ )$

Cross Product

- Geometric interpretation
- two vectors that this could be - one on either side of the plane formed by the two vectors),
- right handed triad.



- anti - commutative

  a x b = - b x a
- cross product of parallel vectors is null vector,

  a x a = 0

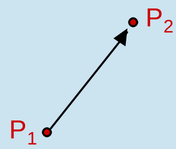
Points

- Geometric objects are based on points
- Points have a location in plane or space
- Point-vector addition
  - Given a point P_1 and a vector v

    $P_1 + v = P_2$

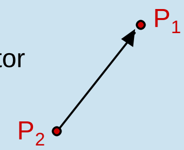
    addition of a point and a vector results in a point

    vector v displaces P_1 to a new location P_2



- Point-point subtraction
  - 2 points define a line segment
  - Subtraction of point P_1 from point yields a vector

    $V = P_1 - P_2$



- Point-point addition?!

Comparison

vectors	points
Add and scale	Do not have
Direction and length	Do not have
Do not have	A fixed position
Unaffected by translation	Moved by translation
u,v,w,…	P,Q,R,…

Could we add points?

- Arbitrary linear combinations of points doesn't have a clear meaning
- Line as linear combinations of two points
- consider

  $P(\alpha) = P_0 + \alpha v$

- P_0 is an a given point, v is any vector (any direction) and α is a scalar ranging over some values
- For any value of α, $P(\alpha)$ yields a point
- Results in a line – parametric equation of a line
- A ray starting from P_0 and going in the direction of v
  - Line – infinite, ray – infinite in 1 direction, segment – finite