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Min-Cost Multicast of Selfish Information Flows

Zongpeng Li

Abstract— **We study multicast in a non-cooperative environment where information flows selfishly route themselves through the cheapest paths available. The main challenge is to enforce such selfish multicast flows to stabilize at a socially optimal operating point incurring minimum total edge cost, through appropriate cost allocation and other economic measures, with replicable and encodable properties of information flows considered. We show that known cost allocation schemes are not sufficient. We provide a shadow-price based cost allocation for networks without capacity limits, and show it enforces minimum-cost multicast. This improves previous result where a 2-approximate multicast flow is enforced. For capacitied networks, computing cost allocation by ignoring edge capacities will not yield correct results. We show that an edge tax scheme can be combined with a cost allocation to strictly enforce optimal multicast flows in this more realistic case. If taxes are not desirable, they can be returned to flows while maintaining weak enforcement of the optimal flow. We relate the taxes to VCG payment schemes and discuss an efficient primal-dual algorithm that simultaneously computes the taxes, the cost allocation, and the optimal multicast flow.**

Index Terms: Game Theory, Graph Theory, Information Theory, Mathematical Programming/Optimization.

I. INTRODUCTION

The classic min-cost flow problem [1], [2] studies the min-cost transmission of commodity flows across a flow network, where each unit edge capacity utilization incurs a cost, and the goal is to minimize the total edge cost while sustaining a target end-to-end flow rate. We consider in this paper the min-cost transmission of information flows in a data network, and focus on multicast transmissions where common data is disseminated from a source to multiple destinations. Multicast models real-world applications such as media streaming or the dissemination of popular files. The cost of a communication link is an abstraction of real-world cost including delay latency, power consumption, and monetary charges.

While both commodity flows and information flows need to confine to the network topology and respect link capacity limits, information flows are unique in that they are *replicable* and *encodable*. Replication and encoding are in general necessary in achieving the full capacity of a data network, and such coding operations applied at potentially any node in the network are referred to as *network coding* [3], [4]. Traditional models of multicast are usually based on Steiner trees, in which either maximizing multicast rate or minimizing multicast cost is computationally intractable [5], [6]. Recent research in network coding reveals a dramatically different structure of information multicast: in a directed network, a multicast rate d is feasible if and only if it is feasible as an independent unicast from the sender to each receiver [3], [4].

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Based on the above result, efficient optimization algorithms for both multicast rate and multicast cost have been successfully designed in the cooperative environment, by exploiting the underlying network flow structure [7], [8]. We consider in this paper a non-cooperative environment where flows selfishly minimize their own cost by routing themselves through the cheapest available paths. Such selfish behaviour is a well-studied phenomenon in game theory, and the stable state of such selfish routing games is characterized as a *Nash Equilibrium*, where no flow has incentive to deviate from its current route alone, assuming the rest of the flows stick to their routes. It is known that such Nash solutions can lead to very bad performances [9], [10] in general. Therefore it is necessary to impose calculated economic regulations in the network, such that individual decisions of selfish flows jointly lead to a socially desirable operation state of the network.

Previous research has considered the enforcement of optimal multicommodity flows [11], [12]. While they also employ economic measures to regulate the routing of flows among potential paths, it is unique and vital in the multicast problem to encourage flow cost sharing. It is well-known that cost sharing can be achieved through combining individual unicast paths into multicast trees. However, two problems are evident along the traditional multicast tree direction. First, the cost may not be really minimum, since a different multicast flow with lower cost may be found by also exploiting the encodable property of information flows [7], [6] (an example is shown in Fig. 1). Second, since optimal routing based on multicast trees are NP-hard to compute, it is unlikely that it would be *exactly* enforced by any efficiently computable regulation scheme, in general network topologies [13]. The best result prior to this work is a bicriteria analysis by Bhadra *et al.* [14], which shows that there always exist cost sharing schemes to enforce a 2-approximate multicast flow.

In this paper, we study the enforcement of optimal multicast flows in general network topologies, through proper edge cost sharing and other economic measures. We first consider the simple case where edges do not have capacity limits, and show that the well-known Shapley value [15] method can not enforce optimal multicast flows. Instead, we formulate the min-cost multicast problem into a pair of primal and dual linear programs, based on the aforementioned multicast feasibility result with network coding due to Ahlswede *et al.* [3], and propose to allocate edge costs based on the shadow prices of flow merging constraints. We show that Nash Equilibrium exists, and that any optimal multicast flow has a corresponding cost allocation which makes it a Nash flow. The flow-cost pair at Nash Equilibrium also achieves a balanced budget, *i.e.*, total charges to flows exactly equal the total cost incurred at edges across the network.

For the more realistic case where edges have finite capacity limits, we show that ignoring edge capacities and taking the solution above will not enforce the optimal multicast flow. We instead propose to further establish a nonnegative tax at each edge. We show that each optimal multicast flow can be *strictly* enforced by a pair of edge tax and cost allocation vectors, such that the solution remains stable even if the capacity limits on a subset of edges are relaxed. In cases where charging edge taxes is infeasible or undesirable, we prove that there always exists a tax return procedure, after which flows pay true edge costs only and the optimal multicast flow is still Nash. However, the optimal solution is now *weakly* enforced, and is sensitive to edge capacity fluctuations. We point out connections of the proposed edge taxes to the 'added value' concept in the Vickrey-Clarke-Groves (VCG) edge payment scheme and to strategyproof multicast [16].

The existential proofs mentioned above rely on path-based linear programming models of the min-cost multicast problem. These LPs, while convenient for analysis purposes, are not suitable for computing the solutions in practice, since they in general contain exponentially many variables/constraints. We finally reformulate the min-cost multicast LPs to reduce their sizes, and design an efficient primal-dual algorithm that simultaneously computes the edge taxes, the cost shares, and the optimal multicast flow they enforce. This is achieved by using Lagrange relaxation to remove coupling constraints among flows to distinct receivers, and decomposing the entire optimization into a series of shortest-path computations through the subgradient method [2]. Our algorithm outperforms general linear programming solution methods in running time and allows distributed implementations. We also show that it represents a dramatic improvement in the complexity of computing VCG payments to selfish edges.

The rest of the paper is organized as follows. We review previous research in Sec. II and discuss the network model together with preliminaries in linear programming in Sec. III. We study uncapacitied networks in Sec. IV, capacitied networks in Sec.V, and design solution algorithms in Sec. VI. The paper is concluded in Sec. VII.

II. Previous Research

The seminal work of Ahlswede *et al.* [3] initiated the research on network coding, and showed its necessity in achieving maximum network capacity. An important result they proved is that *in a directed network, a multicast rate* d *is feasible iff it is feasible to each receiver independently*. Koetter and Médard also derived this result in an algebraic framework [4]. This new characterization of multicast rate feasibility (with network coding) dramatically changed the underlying structure of the multicast problem, namely from multicast trees to a union of conceptual network flows [17]. Consequently, breakthroughs were made in efficient multicast algorithm design in directed networks [7], undirected networks [17], [8], and wireless ad hoc networks [18], [19], assuming a cooperative environment. In this paper, we instead study how to achieve min-cost multicast when information flows are selfish.

Edge pricing schemes that enforce minimum-delay multicommodity flows are studied in [12] and [11]. Both compute taxes/tolls on edges to guide the selfish routing process. [12] computes the taxes by solving a nonlinear complementary problem, while [11] takes the linear programming approach. Our work in this paper was partly inspired by them, and is similar in that edge taxes are also considered as part of the flow regulation measures. The edge taxes we introduce allow more intuitive interpretations and can be eventually returned to the multicast flows without jeopardizing their stability. Another important difference is that we need to also design appropriate cost sharing schemes that induce desired path sharing among multicast flows. Edge pricing is also recently discussed in [20] for combating routing anomalies that arise in unregulated interactions between multiple selfish overlays.

Feigenbaum *et al.* studied the sharing of multicast cost in the context of designing strategyproof multicast for selfish receivers with private utility information [13]. Along the multicast tree direction, they show that optimal welfare is NP-hard to approximate within any constant ratio in general networks. They consider acyclic network topologies instead, where the multicast cost is submodular and therefore both Shapley value [15] and marginal cost sharing lead to strategyproofness. Lower bounds on communication overhead are derived for these two sharing schemes. In this paper, we perform multicast cost sharing with network coding explicitly considered. As a result, optimal sharing schemes can be computed efficiently in general network topologies.

Bhadra *et al.* also studied the multicast of selfish information flows in [14]. They use monomial cost functions to approximate edge capacity limits and differentiable relaxations to approximate the $\max$ function in edge flow computation. After these approximations, the min-cost multicast problem is modelled as a non-linear optimization problem with differentiable objective function and constraints, and cost shares for enforcing the optimal multicast flow are derived based on Karush-Kuhn-Tucker (KKT) optimality conditions for nonlinear programs [21]. This result applies to power-law edge cost functions only. For the linear cost model and other convex cost functions, they employ a bicriteria approach introduced by Roughgarden and Tardos [10] and design cost shares that enforce a *sub-optimal* multicast flow, which has a cost lower than any optimal multicast flow achieving twice the throughput. In this paper, we use exact models for edge capacity limits and edge flow computation, and devise economic mechanisms that enforce optimal multicast flows.

Wang et al. [16] studied min-cost multicast in networks with selfish edges, where edge cost is private information and each edge reports a cost value at its own choice to maximize its utility. They show that the celebrated VCG payment scheme fails to induce strategyproofness, if optimal multicast routing, NP-hard without network coding, is approximated with schemes such as pruning min-spanning tree or link weighted Steiner tree. We establish underlying connections between edge taxes introduced in this paper and the 'added value' concept in VCG payments. We argue that paying each edge its declared cost plus such edge taxes according to VCG successfully induces strategyproofness at selfish edges. Therefore the failure of VCG identified in [16] is due to the fact that approximate multicast algorithms are employed rather than being inherent in the multicast problem itself.

III. Network Model and Preliminaries

The data network is modelled as a directed graph $G = (V, E)$. Let $S \in V$ be the multicast sender and $\mathcal{T} = \{T_1 \dots T_k\} \subseteq V$ be the set of receivers. Here k is the number of multicast receivers.

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We use vectors $c, w \in Q_+^E$ to store the capacity limit and cost of edges, respectively. Here Q_+ is the set of non-negative rational numbers. Note that $w(e)$ denotes the cost of a unit flow on e, and the total cost of flow $f(e)$ on e is $w(e)f(e)$. Let $\mathcal{P}_i$ be the set of distinct paths available from S to T_i. For $p \in \mathcal{P}_i$, $f(p)$ represents the amount of $S \rightarrow T_i$ flow carried by p. The total flow amount on an edge e is denoted as $f(e)$. A scalar d stores the target multicast rate. In this non-cooperative multicast routing game, each agent corresponds to an infinitesimally small amount of some $S \rightarrow T_i$ flow, the change of whose routing decision alone does not lead to observable variations in any edge state. This models the situation where each data packet makes its own routing decision.

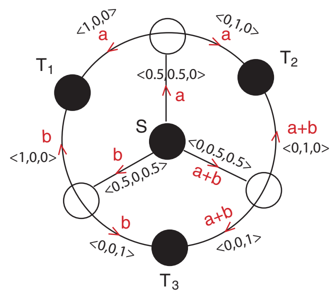

Fig. 1. An example of enforcing min-cost multicast flows. Each link has cost 1 and is labelled with the information flow it carries and its cost share vector $< y_1, y_2, y_3 >$. Each edge flow has rate 0.5. Total cost is $0.5 \times 9 = 4.5$ and is optimal. A minimum multicast tree has cost 5.

Fig. 1 illustrates some of the concepts we discussed using a multicast network with one sender and three receivers. The target multicast rate is 1 and each edge has cost 1 per unit flow. We ship on each edge an information flow of rate 0.5, of either a, b, or $a + b$ as labelled. Here $+$ is performed over a finite field and is equivalent to bit-wise exclusive-or. Each receiver may successfully recover the two original flows a and b from what it receives. A cost allocation is given in the figure, under which the multicast flow shown is Nash. Here $y_i(e)$ is the cost of a unit flow to T_i on e. For example, the $S \rightarrow T_1$ flow has total rate 1, with two paths each carrying 0.5. Its cost on each path is $0.5 \times 0.5 + 1 \times 0.5 = 0.75$, and the total $S \rightarrow T_1$ flow cost is 1.5. From S to each T_i, there are two different paths with identical cost 1.5. Flows do not have incentive to switch between paths, and the routing and cost allocation together constitute a Nash Equilibrium. Constructing a detailed coding scheme upon a given multicast flow is known as *code construction* [22], [23] and is orthogonal to our current topic. In the rest of the paper, we assume that code construction can be efficiently done and omit its further details.

The analysis and algorithm design presented in this paper depend heavily on linear programming duality results, including primal and dual LPs and their correspondence, complementary slackness conditions, and Lagrange duality. These results are discussed in many linear programming textbooks, *e.g.*, [24]. We provide here only a sketched description due to space limitations. Every (primal) LP has an equivalent dual LP. Each variable in the primal maps to a constraint in the dual, and vice versa. Equality constraints map to free variables and inequality constraints map to non-negative variables. A variable-constraint pair is *complementary* if either the variable is zero or the constraint is tight, *i.e.*, satisfied at equality. If both primal and dual LPs are feasible then they have pair-wise complementary optimal solutions. Conversely, if a pair of primal and dual solutions are feasible and complementary then they are both optimal. A *Lagrange relaxation* of an LP removes a group of its constraints (let $\overrightarrow{x}$ be the variable vector they map to) while adding a corresponding penalty term into the objective function. Viewing the resulting (smaller) LP as a function of $\overrightarrow{x}$ and optimizing it over the feasible domain of $\overrightarrow{x}$ yields the same optimal solutions (if existent) as in the original LP.

In the rest of the paper, we use interchangeably the terms *cost allocation* and *cost sharing*, *edge* and *link*. A multicast flow at Nash Equilibrium is called a *Nash flow*. A network is referred to as an *uncapacitied network* if it has unlimited edge capacities or a *capacitied network* otherwise.

IV. Uncapacitied Networks

We begin our study with the simpler case where edges in the network G are assumed to always have sufficient bandwidth supply for the purpose of multicast routing. This allows us to focus on mechanisms that foster appropriate path sharing among flows. We handle realistic network scenarios with finite edge capacity limits in the subsequent section.

A. Cost Sharing Based on Shapley Value

The Shapley value is a well-known 'fair' value allocation method proposed by Lloyd Shapley [15]. It was introduced originally in the context of value sharing in coalitional games [25], where each player receives a share depending on how 'important' she is in the coalitions. It is the only scheme that simultaneously achieves *budget balance*, *anonymity* (player labelling is irrelevant) and *additivity* (a player's shares in two coalitional games sum up to its share in the combination of these two games). In multicast tree cost sharing, it is known that according to Shapley value, each edge is paid for equally by all its downstream receivers [13].

For cost sharing of multicast with network coding, an accurate definition of the Shapley value can be formulated as:

$$y_i(e) = \frac{w(e)}{f_i(e)} \sum_{T_i \notin \mathcal{H} \subseteq \mathcal{T}} \frac{h!(k-h-1)!}{k!} \left(\max_{j \in \mathcal{H} \cup i} f_j(e) - \max_{j \in \mathcal{H}} f_j(e) \right)$$

where $y_i(e)$ is the cost share of a unit $S \rightarrow T_i$ flow on e, $f_j(e) = \sum_{e \in p \in \mathcal{P}_j} f(p)$ is the total amount of $S \rightarrow T_j$ flow on e, and h is the cardinality of $\mathcal{H}$. Intuitively, the Shapley value method enumerates all possible orders that flows to different receivers may occur on an edge e, calculates the added cost of each flow upon its appearance, and takes the average as the final share. It satisfies $\sum_i y_i(e) f_i(e) = w(e) f(e)$, *i.e.*, the total flow cost on e is shared exactly, achieving a balanced budget.

Unfortunately, the Shapley value based cost sharing does not successfully enforce min-cost multicast flows. As an example, the multicast session shown in Fig. 2 has a min-cost of 6, and is not Nash under Shapley cost sharing because for $S \rightarrow M \rightarrow T_1$ flows, there is a cheaper direct path $S \rightarrow T_1$ to switch onto. A Nash flow after such switches has cost 7 and is not optimal.

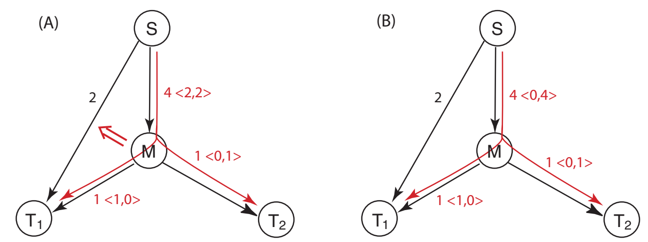

Fig. 2. An example multicast network with its optimal multicast flow. (A) The optimal flow with Shapley value cost sharing is not Nash. (B) The cost allocation under which the optimal flow is Nash. Each non-zero edge flow has rate d. Each edge is labelled with its cost and sharing $w(e) < y_1(e), y_2(e) >$. The double arrow indicates the direction of potential path switches.

The reason that the Shapley value method is no longer appropriate lies in the fact that it is inherently a local cost sharing method. While it may work fine for a single edge or for a simple tree topology, global attention needs to be paid for multicast flows in general network topologies, to handle the competition among potential flow paths. We now proceed to derive cost shares within the linear programming framework, and show that it is always possible to use shadow-priced based cost shares to enforce optimal multicast flows in any network topology.

B. LP Formulations

We first formulate the min-cost multicast problem into a pair of path-based LPs as follows:

Path-based min-cost multicast LP (primal)

Minimize $\sum_e w(e)f(e)$

Subject to:

$$\begin{cases} \sum_{p\in\mathcal{P}_i} f(p) = d & \forall i & \leftrightarrow x_i \\ f(e) \geq \sum_{p\in\mathcal{P}_i: e\in p} f(p) & \forall i, \forall e & \leftrightarrow y_i(e) \end{cases}$$

$$f(e), f(p) \geq 0 \qquad \forall e, \forall p$$

Path-based min-cost multicast LP (dual)

Maximize $\sum_i x_i d$

Subject to:

$$\begin{cases} \sum_i y_i(e) \leq w(e) & \forall e & \leftrightarrow f(e) \\ x_i \leq \sum_{e\in p} y_i(e) & \forall i, \forall p \in \mathcal{P}_i & \leftrightarrow f(p) \end{cases}$$

$$x_i <> 0; y_i(e) \geq 0 \qquad \forall i, \forall e$$

The formulation of the above LPs are based on the fact that a multicast rate d can be achieved by setting up a network flow of rate d from sender S to each receiver T_i; these network flows are *conceptual* in that they share instead of compete for available bandwidth on an edge [17]. In other words, if the $S\rightarrow T_i$ and $S\rightarrow T_j$ network flows have rates $f_i(e)$ and $f_j(e)$ on an edge e respectively, then their combined effective flow rate is $\max(f_i(e), f_j(e))$ instead of $f_i(e) + f_j(e)$. This is possible due to the unique encodable and replicable properties of information flows.

In the primal LP, the first constraint requires the total network flow rate to each receiver be the desired multicast throughput d, the second constraint requires total edge flow $f(e)$ to be no less than each conceptual flow rate $f_i(e)$. Note that the smallest $f(e)$ satisfying this constraint is $f(e) = \max_i f_i(e)$. The objective is to minimize total multicast flow cost $\sum_e w(e)f(e)$. Following each constraint we also list its corresponding dual variable. The dual LP can be interpreted as follows. The first constraint allocates the total cost of an edge $w(e)$ to the k conceptual flows. The second constraint implies that the cost x_i from S to T_i should be no more than the total cost of any path p from S to T_i. Note that the largest value x_i satisfying this constraint is $x_i = \min_{p\in\mathcal{P}_i} \sum_{e\in p} y_i(e)$. The dual objective is to maximize $\sum_i x_i d$.

C. Cost Sharing Based on Shadow Prices

We say that a cost allocation y *enforces* a multicast flow f if f and y together satisfy: (1) *stability*, f is a Nash flow under y, (2) *budget balance*, $\forall e, w(e)f(e) = \sum_i y_i(e)f_i(e)$, and (3) *fairness*, no flow has a cost share $y_i(e)$ of more than total edge cost $w(e)$ at any edge e. Requirements in (2) and (3) preclude trivial solutions such as assigning a cost $y_i(e) = a$ wherever $f_i(e) > 0$ and $y_i(e) = b$ otherwise, for some constants $a < b$.

We now prove that the optimal solutions in the dual LP, also known as *shadow prices*, may serve as cost allocations that enforce optimal multicast flows.

Theorem 1. If f^* and (x^*, y^*) are a pair of optimal solutions to the primal and dual min-cost multicast LPs respectively, then y^* enforces f^*.

Proof: We first show stability. By linear programming duality, f^* and (x^*, y^*) together satisfy the complementary slackness conditions. By dual complementary slackness, we have:

$$p \in \mathcal{P}_i, f^*(p) > 0 \longrightarrow x_i^* = \sum_{e\in p} y_i^*(e) \qquad \forall i$$

Since x^* maximizes $\sum_i x_i^* d$, we can conclude that (1) for any receiver T_i, $x_i^* = \min_{p\in\mathcal{P}_i} \sum_{e\in p} y_i^*(e)$; and (2) if $p_1, p_2 \in \mathcal{P}_i$ and $f^*(p_1) > 0$, then $\sum_{e\in p_1} y_i^*(e) \leq \sum_{e\in p_2} y_i^*(e)$. In other words, if we use y^* as the cost allocation, then every path p with a non-zero flow is a shortest path, and flows on that path have no incentive of switching to a different path, which makes f^* a Nash flow.

We next show budget balance. By primal complementary slackness:

$$y_i^*(e) > 0 \longrightarrow f_i^*(e) = f^*(e) \qquad \forall i \forall e$$

i.e., a $S\rightarrow T_i$ flow has a positive cost on e only if $f_i^*(e) = \max_j f_j^*(e)$. Further, dual complementary slackness implies:

$$f^*(e) > 0 \longrightarrow \sum_i y_i^*(e) = w(e) \qquad \forall e$$

Let $\mathcal{I}_e = \{i | f_i^*(e) > 0\}$, then we have $\sum_i y_i^*(e) f_i^*(e) = \sum_{\imath\in\mathcal{I}_e} y_i^*(e) f_i^*(e) = \sum_{\imath\in\mathcal{I}_e} y_i^*(e) f^*(e) = w(e)f(e)$.

We finally show fairness. Dual feasibility requires $\forall e, \sum_i y_i^*(e) \leq w(e)$, since entries in y^* are non-negative, we have $\forall i \forall e, y_i^*(e) \leq w(e)$. □

Corollary 1. Every optimal multicast flow is enforceable.

Proof: If a primal LP is feasible and has an optimal solution then so does its dual [24]. Therefore every optimal primal solution f^* of the min-cost multicast LP has a corresponding optimal dual solution (x^*, y^*). This combined with Theorem 1 implies the corollary. □

D. Discussions

We proved that every optimal multicast flow f^* can be enforced by a shadow price vector y^*. However, we point out that there may exist under y^* a different Nash flow f' that is not optimal. An example can be derived in Fig. 1 by rearranging flow rates f_1 on the two different $S \to T_1$ paths from $(0.5, 0.5)$ to, say, $(0.2, 0.8)$. Therefore, the shadow price method guarantees the *price of stability* (ratio of *best* Nash flow cost to optimal flow cost) [26] to be 1, but the *price of anarchy* (ratio of *worst* Nash flow cost to optimal flow cost) [27] may by higher than 1.

There are heuristic improvements to the shadow price method. For example, if we adopt a two-tier charging scheme and set $y'_i(e) = y^*_i(e)$ for the portion of $f_i(e) \le f^*(e)$, and $y'_i(e) = w(e)$ for the portion of $f_i(e) > f^*(e)$, then the optimal multicast flow in Fig. 1 will become the unique enforced flow, achieving a price of anarchy of 1 with a balanced budget. We leave it as future research to decide whether there exists a cost allocation scheme that guarantees a price of anarchy of 1 in general.

V. Capacitied Networks

In practice, communication links have bounded bandwidth and this constitutes a new dimension of constraint in multicast routing. We now study how to regulate the selfish multicast routing game in this more realistic scenario. Before presenting the solution, we note that simply ignoring edge capacity limits and applying the shadow price based cost allocation described in the previous section will not yield correct results.

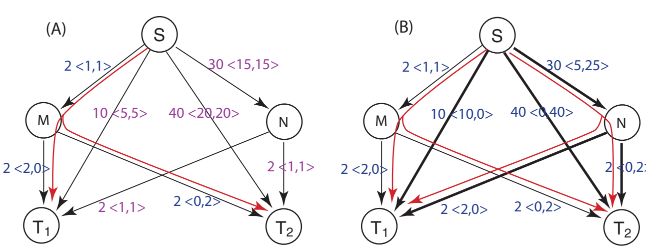

Fig. 3. The same cost allocation y that enforces an optimal multicast flow in the uncapacitied network may fail when edges have limited capacities. (A) y and its enforced optimal multicast flow f, when the network is uncapacitied. (B) The capacitied case. Multicast rate $d = 2$. Thin edges have capacity 1 and bold edges have 3. Each non-zero edge flow is of rate 1. New cost allocation y' that enforces the optimal flow f' is shown. f' is not enforced by y.

Fig. 3 shows a counter example: (A) depicts an uncapacitied network and (B) depicts a capacitied one, over the same topology. An optimal multicast flow f and its enforcing cost allocation vector y are shown in (A). The only optimal multicast flow f' with the specified edge capacity configuration is given in (B). We can see that f' is not a Nash flow under y because flows on $S \to N \to T_1$ have a better path $S \to T_1$ to switch onto. The problem is that in the uncapacitied case, the cost allocation y may focus only on the atomic multicast topology with the minimum cost, and provide little guarantee on the rest of the edges. We now proceed to remodel the min-cost multicast problem with finite edge capacities introduced into the picture.

A. LP formulations

The min-cost multicast problem where each edge has a finite constant bandwidth limit can be formulated into a pair of primal and dual LPs as follows.

Capacitied min-cost multicast LP (primal)

Minimize $\quad \sum_e w(e) f(e)$

Subject to:

$$\begin{cases} \sum_{p \in \mathcal{P}_i} f(p) = d & \forall i & \leftrightarrow x_i \\ f(e) \ge \sum_{p \in \mathcal{P}_i : e \in p} f(p) & \forall i, \forall e & \leftrightarrow y_i(e) \\ f(e) \le c(e) & \forall e & \leftrightarrow t(e) \end{cases}$$

$$f(e), f(p) \ge 0 \qquad \forall e, \forall p$$

Capacitied min-cost multicast LP (dual)

Maximize $\quad \sum_i x_i d - \sum_e c(e) t(e)$

Subject to:

$$\begin{cases} \sum_i y_i(e) \le w(e) + t(e) & \forall e & \leftrightarrow f(e) \\ x_i \le \sum_{e \in p} y_i(e) & \forall i, \forall p \in \mathcal{P}_i & \leftrightarrow f(p) \end{cases}$$

$$x_i <> 0; y_i(e), t(e) \ge 0 \qquad \forall i, \forall e$$

The primal LP has an extra constraint $f(e) < c(e)$ than in the uncapacitied case to model finite edge capacity limits. Recall that c is the edge capacity vector. The dual LP has a corresponding extra variable vector t. From an economic point of view, $t(e)$ reflects how critical bandwidth supply is at edge e. This is in line with our algorithm for solving y and t in Sec. VI-C: $t(e)$ will be adjusted higher whenever capacity on e is over-used and lower whenever it is under-used.

B. Cost Allocation with Edge Taxes

We say that an edge tax and cost sharing pair (t, y) *enforces* a multicast flow f, if y enforces f with $w' = w + t$ as the new edge cost vector, and that (t, y) *strictly enforces* f if (t, y) enforces f with the edge capacity vector c replaced by any new vector $c' \ge c$. Intuitively, a strictly enforced multicast flow remains stable even if some edge capacity limits are relaxed. We now prove that charging each edge a tax according to t may lead to strict enforcement of optimal multicast flows in capacitied networks.

Theorem 2. If f^* and (x^*, y^*, t^*) are optimal solutions to the primal and dual min-cost multicast LPs respectively, then (y^*, t^*) strictly enforces f^*.

Proof: Since f^* and (x^*, y^*, t^*) are corresponding primal and dual optimal solutions, dual complementary slackness requires that:

$$p \in \mathcal{P}_i, f^*(p) > 0 \longrightarrow x^*_i = \sum_{e \in p} y^*_i(e) \qquad \forall i$$

The dual LP maximizes $\sum_i x_i d - \sum_e c(e)t(e)$. Given a fixed vector t, it strives to maximize each x_i in order to maximize $\sum_i x_i d$. Therefore we know $x_i^* = \min_{p\in\mathcal{P}_i} \sum_{e\in p} y_i^*(e)$, and:

$$p_1, p_2 \in \mathcal{P}_i, f^*(p_1) > 0 \longrightarrow \sum_{e\in p_1} y_i^*(e) \le \sum_{e\in p_2} y_i^*(e) \quad \forall i$$

Therefore, towards each receiver T_i, every path with non-zero flow has the minimum after-tax cost, and no flow has incentive to deviate from its current path. In other words, the tax t^* and the cost allocation y^* together makes f^* a Nash flow. The proof of budget balance (with $w + t$ as edge cost) and fairness are essentially the same as that in the uncapacitied case and is omitted.

If capacities of a subset of edges are increased, all flows in f^* are still feasible, and all paths carrying non-zero flows are still shortest after taxes, therefore f remains stable. We conclude that y^* and t^* strictly enforces f^*. □

Corollary 2. In a capacitied network, every optimal multicast flow can be strictly enforced by an edge tax scheme combined with a cost allocation scheme.

Proof: This follows from the result in Theorem 2 and the fact that every primal optimal solution has a corresponding optimal dual solution. □

C. Tax Return

Charging extra taxes from flows may not always be feasible or desirable. We now show that it is possible to return taxes to the multicast flows, such that the optimal multicast flow is still enforced by the resulting cost allocation. The challenge is to return taxes to flows that paid them only, and to ensure that the multicast flow is still Nash and the budget is still balanced after the return. Our overall solution for tax-free cost sharing in capacitied networks is a two-stage one: compute with-tax cost shares first, then return the taxes appropriately.

Theorem 3. In a capacitied network, every optimal multicast flow f^* can be enforced by a cost allocation scheme y'.

Proof: The main idea is to return edge taxes to flows, while maintaining the condition that no path flow has a lower-cost path to switch onto. This is possible partly due to the fact that edges have finite capacities. In particular, if a conceptual network flow f_i is already using the full capacity on an edge e, then even if the cost $y_i(e)$ is reduced due to tax return, it is still infeasible for the rest of f_i flows to switch onto e since there is no residual edge capacity available.

By Theorem 2, we know that every optimal multicast flow f^* can be strictly enforced by edge taxes t^* and cost allocation y^*, which are optimal dual solutions. Now, consider returning edge taxes as follows:

$$y_i'(e) = \frac{w(e)}{w(e) + t(e)} y_i^*(e) \qquad \forall e, \forall i$$

By primal complementary slackness, we have:

$$t^*(e) > 0 \longrightarrow f^*(e) = c(e) \qquad \forall e$$

$$y_i^*(e) > 0 \longrightarrow \sum_{p\in\mathcal{P}_i : e\in p} f^*(p) = f^*(e) \quad \forall i, \forall e$$

In other words, every edge e assigned a non-zero tax $t^*(e)$ has zero residual capacity under flow f^*, and every destination i that has a non-zero cost share $y_i^*(e)$ on edge e has the largest flow rate on e. After the tax return, if a non-zero path flow $f^*(p), p \in \mathcal{P}_i$ wishes to switch to another path $p' \in \mathcal{P}_i$, it must be the case that:

$$\exists e \in p', y_i'(e) < y_i^*(e),$$

which implies that $y_i^*(e) > 0$ and $t^*(e) > 0$. Then by the primal complementary slackness conditions presented above, we have:

$$\sum_{p\in\mathcal{P}_i : e\in p} f^*(p) = f^*(e) = c(e)$$

i.e., the edge e would have no residual capacity for a $S \rightarrow T_i$ path flow, and a $p \rightarrow p'$ switch is impossible. Therefore f^* is a Nash flow under y'. Budget balance and fairness follow from the fact that y^* enforces f^* with edge cost vector $w + t$, and that $y' = \frac{w}{w+t} y^*$. We conclude that the cost allocation y' enforces the optimal multicast flow f^*. □

D. Discussions

The tax return scheme can indeed be quite flexible. The requirements of stability, budget balance and fairness in the resulting cost allocation still leave ample space for manipulation. The proportional return described above is just one possible approach. It is an interesting direction to investigate whether more sophisticated tax return schemes can simultaneously achieve other desirable properties as well, such as improving the price of anarchy.

The complementary slackness conditions imply that taxes in t^* make all utilized $S \rightarrow T_i$ paths equally expensive. We plan to show in a forthcoming work that for a large class of network scenarios, the tax $t^*(e)$ is equivalent to the added value of an edge e, *i.e.*, the increment of total multicast cost that would occur if e is removed from the network; furthermore, if edge cost $w(e)$ is private information known to e only, and edges are selfish, then paying each edge $w'(e) + t^*(e)$ ($w'(e)$ is the declared cost of e) according to the VCG scheme makes e have no incentive to lie. It is a dominant strategy for each edge to report its true cost $w(e)$, leading to a *strategyproof* multicast system. The algorithm we present in the next section improves the time complexity of payment computation by a factor of $|E|$, compared to a straightforward algorithm using the definition of VCG.

VI. Algorithm Design

We have shown that shadow price based cost shares and taxes may enforce optimal multicast flows. We now proceed to discuss how these optimal solutions can be efficiently computed. First, note that the path-based LPs we have used for analysis purposes are not feasible to be solved directly. The number of distinct paths between two nodes in a general graph may be exponential to the graph size. Consequently, the path-based LP has exponentially many variables (primal) or constraints (dual) and is impractical for computing purposes.

In this section, we first present reformulated link-based min-cost multicast LPs with reduced polynomial sizes. We argue that they are equivalent to the path-based LPs, and in particular, a cost allocation y^* or cost-tax vector pair (y^*, t^*) is an optimal solution to the link-based dual LP if and only if it is an optimal solution to the path-based dual LP. We then discuss how the classic approach of Lagrange relaxation followed by subgradient optimization can be applied to derive effective solution algorithms, in both uncapacitied and capacitied networks. These algorithms are combinatorial in nature, consisting of mostly shortest-path computations, and are therefore amenable for distributed implementations.

A. Equivalence between Path-based and Link-based LPs

Again we start with the uncapacitied case. Presented below are the reformulated min-cost multicast linear program based on edge-flow variables, along with its dual. Here $\vec{uv}$ denotes an edge from node u to node v, $N_\downarrow(u)$ and $N_\uparrow(u)$ denote downstream and upstream neighbor set of u in G, respectively. We use $\vec{uv}$ instead of e to represent an edge here because it is helpful to have explicit connections between nodes and their adjacent edges. We assume there is a conceptual link from each T_i back to S with unlimited capacity, for succinct expression of flow conservation constraints.

Link-based min-cost multicast LP (primal)

Minimize $\quad \sum_{\vec{uv}} w(\vec{uv}) f(\vec{uv})$

Subject to:

$$\begin{cases} \sum_{v\in N_\downarrow(u)} f_i(\vec{uv}) = \sum_{v\in N_\uparrow(u)} f_i(\vec{vu}) & \forall i, \forall u \quad \leftrightarrow p_i(u) \\ f_i(\vec{T_iS}) = d & \forall i \quad \leftrightarrow x_i \\ f_i(\vec{uv}) \le f(\vec{uv}) & \forall i, \forall\, \vec{uv} \quad \leftrightarrow y_i(\vec{uv}) \end{cases}$$

$$f_i(\vec{uv}), f(\vec{uv}) \ge 0 \qquad \forall i, \forall\, \vec{uv}$$

Link-based min-cost multicast LP (dual)

Maximize $\quad \sum_i x_i d$

Subject to:

$$\begin{cases} p_i(u) - p_i(v) \le y_i(\vec{uv}) & \forall i \forall\, \vec{uv} \quad \leftrightarrow f_i(\vec{uv}) \\ p_i(T_i) - p_i(S) + x_i \le y_i(\vec{T_iS}) & \forall i \quad \leftrightarrow f_i(\vec{T_iS}) \\ \sum_i y_i(\vec{uv}) \le w(\vec{uv}) & \forall\, \vec{uv} \quad \leftrightarrow f(\vec{uv}) \end{cases}$$

$$p_i(u), x_i <> 0; y_i(\vec{uv}) \ge 0 \qquad \forall i, \forall u, \forall\, \vec{uv}$$

The link-based LPs and the path-based LPs model the same min-cost multicast problem, and are equivalent to each other. In particular, a pair of primal and dual solutions (f, y) is feasible/optimal in the link-based formulation if and only if it is feasible/optimal in the path-based formulation. In the primal problem, both LPs establish end-to-end network flows of rate d from S to each T_i. Flow conservation of each conceptual flow is implicit in the path-based formulation but explicit in the link-based formulation. In the dual program, both allocate edge cost w to y_i's, and use x_i to compute the shortest $S \rightarrow T_i$ path under distance vector y_i. The fact that x_i is the shortest path length in any optimal dual solution is less explicit in the link-based LP, but can be deduced based on the following facts. Variables in p can be interpreted as the *altitude* of nodes [1]. The first dual constraint bounds the altitude difference of two neighboring nodes in the network with y_i, and the second dual constraint bounds x_i with the altitude difference between S and T_i. The equivalence of the two different LP formulations in the capacitied case is similar.

Since the link-based LPs have polynomial sizes, they are more practical to solve using general linear programming solution methods such as the simplex algorithm. Solving the dual LP yields optimal cost allocation y^* and tax t^*. However, past experiences suggest that much better scalability can be achieved by exploiting the specific structure of the multicast problem, if possible [6], [7]. Our experiments in large-scale network topologies [6] show that a well-designed subgradient algorithm may outperform both the simplex algorithm and the primal-dual interior-point algorithm by more than an order of magnitude, in terms of running time. Furthermore, general linear programming solution methods are inherently centralized, while distributed subgradient algorithm design are often possible. We now proceed to describe tailored subgradient algorithms for computing optimal cost sharing and taxes from the link-based LPs.

B. Solving the Uncapacitied Case

Lagrange relaxation coupled with subgradient optimization has proven effective in designing tailored efficient algorithms for classic combinatorial problems (including the network flow problem) with side constraints [6], [8], [7]. The side constraints can be relaxed so that efficient combinatorial algorithms can be applied to the remaining smaller problem. The price associated with the relaxation is that *a series of*, instead of *one*, smaller problems need to be solved. Lun *et al.* [7] studied a similar problem of min-cost multicast in cooperative networks, and presented a distributed optimization algorithm based on Lagrange relaxation. In a previous work [6], we also applied similar techniques in computing the optimal orientation of an undirected multicast network. The goal in this paper is to compute optimal shadow prices y^* and t^* instead of optimal primal solutions. This leads to subtle but important differences in dualization strategies and subgradient algorithm design.

In the uncapacitied case, we can relax the constraint group $f_i(\vec{uv}) \le f(\vec{uv})$ to obtain the following Lagrange dual problem:

Maximize $\quad L(y)$

Subject to:

$$\begin{cases} \sum_i y_i(\vec{uv}) \le w(e) & \forall\, \vec{uv} \\ y_i(\vec{uv}) \ge 0 & \forall i, \forall\, \vec{uv} \end{cases}$$

where

$$L(y) = \text{Min}_P \sum_i \sum_{\vec{uv}} f_i(\vec{uv}) y_i(\vec{uv})$$

with P being the polytope:

$$P: \begin{cases} \sum_{v\in N_\downarrow(u)} f_i(\vec{uv}) = \sum_{v\in N_\uparrow(u)} f_i(\vec{vu}) & \forall i, \forall u \\ f_i(\vec{T_iS}) = d & \forall i \\ f_i(\vec{uv}) \ge 0 & \forall i, \forall\, \vec{uv} \end{cases}$$

The Lagrange duality theorem [24] assures that the above problem has the same optimal solutions as in the original linear program. Two reasons associated with this dualization strategy are: first, the inter-flow coupling constraints are removed, allowing simple combinatorial computations in subsequent subgradient iterations; second, the corresponding dual variable y is what the subgradient algorithm finds an optimal solution for, and is also what we are trying to solve for. We note that solving this Lagrange dual is indeed a special case of the solution provided in [7], which computes optimal y^* and f^* in capacitied multicast networks. The difference is that each subproblem during primal update degrades from a min-cost flow problem to a shortest path problem, due to the lack of edge capacity limits. The optimal multicast flow f^* is recovered by a linear combination of intermediate values in f during the subgradient iterations [28], [7].

C. Solving the Capacitied Case

In the capacitied case, we need to compute both optimal cost shares y^* and optimal edge taxes t^*. This is different than the case in [7], where the subgradient iterations converge at y^*, f^* is obtained through primal recovery, and t^* is unsolved. We take a different approach to let the subgradient algorithm converge at both y^* and t^* simultaneously. In order to do so, we relax two groups of constraints $f_i \le f \le c$ from the primal LP

Link-based min-cost multicast LP (capacitied)

Minimize $\sum_{\vec{uv}} w(\vec{uv})f(\vec{uv})$

Subject to:

$$\begin{cases} \sum_{v\in N_\downarrow(u)} f_i(\vec{uv}) = \sum_{v\in N_\uparrow(u)} f_i(\vec{vu}) & \forall i, \forall u \\ f_i(\vec{T_iS}) = d & \forall i \\ f_i(\vec{uv}) \le f(\vec{uv}) & \forall i, \forall\, \vec{uv} \quad \leftrightarrow y_i(\vec{uv}) \\ f(\vec{uv}) \le c(\vec{uv}) & \forall\, \vec{uv} \ne \vec{T_iS} \quad \leftrightarrow t(\vec{uv}) \end{cases}$$

$$f_i(\vec{uv}), f(\vec{uv}) \ge 0 \qquad \forall i, \forall\, \vec{uv}$$

and introduce both y and t into the new objective function:

$$\sum_{\vec{uv}} w(\vec{uv})f(\vec{uv}) + \sum_i \sum_{\vec{uv}} y_i(\vec{uv})(f_i(\vec{uv}) - f(\vec{uv})) + \sum_i t_i(\vec{uv})(f(\vec{uv}) - c(\vec{uv}))$$

$$= \sum_{\vec{uv}} f(\vec{uv})(w(\vec{uv}) + t(\vec{uv}) - \sum_i y_i(\vec{uv})) + \sum_i \sum_{\vec{uv}} y_i(\vec{uv}) f_i(\vec{uv}) - \sum_i c_i(\vec{uv})t_i(\vec{uv})$$

With $f(\vec{uv})$ freely chosen from $[0,\infty)$, $\sum_{\vec{uv}} f(\vec{uv})(w(\vec{uv}) + t(\vec{uv}) - \sum_i y_i(\vec{uv}))$ is unbounded from below if $\sum_i y_i(\vec{uv}) > w(\vec{uv}) + t(\vec{uv})$. Therefore, primal feasibility requires $\sum_i y_i(\vec{uv}) \le w(\vec{uv}) + t(\vec{uv})$, and we obtain the following Lagrange dual:

Maximize $L(y,t)$

Subject to:

$$\begin{cases} \sum_i y_i(\vec{uv}) \le w(e) + t(e) & \forall\, \vec{uv} \\ y_i(\vec{uv}), t(e) \ge 0 & \forall i, \forall\, \vec{uv} \end{cases}$$

where

$$L(y,t) = \text{Min}_P \sum_{\vec{uv}} (\sum_i f_i(\vec{uv}) y_i(\vec{uv}) - c(\vec{uv})t(\vec{uv}))$$

Due to the relaxation of the inter-flow coupling constraints, the optimization of $L(y,t)$ is *separable*, *i.e.*, given a pair of fixed dual vectors (y,t):

$$\text{Min}_P \sum_{\vec{uv}} (\sum_i f_i(\vec{uv}) y_i(\vec{uv}) - c(\vec{uv})t(\vec{uv}))$$

$$= \sum_i \text{Min}_P \sum_{\vec{uv}} f_i(\vec{uv}) y_i(\vec{uv}) - \sum_{\vec{uv}} c(\vec{uv})t(\vec{uv})$$

During primal update in each iteration of the subgradient algorithm, the second term $\sum_{\vec{uv}} c(\vec{uv})t(\vec{uv})$ is irrelevant since both $c(\vec{uv})$ and $t(\vec{uv})$ are constant there. The first term $\sum_i \text{Min}_P \sum_{\vec{uv}} f_i(\vec{uv}) y_i(\vec{uv})$ can be solved through k shortest-path computations: for each i, find a shortest path from S to T_i and assign a flow rate $f_i(\vec{uv}) = d$ to every link $\vec{uv}$ on that path.

During dual update, given a fixed multicast flow f, we compute new values for y and t with the aid of two pre-scribed step size sequences θ_y, θ_t, and project them to the positive orthant:

$$\begin{cases} y'_i = y_i[k] + \theta_y[k] f_i[k] & \forall i \\ t' = \max(0, t[k] - \theta_t[k]c) \end{cases}$$

Here $y_i[k]$ denotes the value of y_i at round k of the subgradient iterations, similar for $f_i[k]$, $t[k]$ and $\theta[k]$. Values in y' and t' may not be dual feasible in general. In cases dual constraints are violated, we need to project (y', t') into the feasibility polytope. We discuss three alternative approaches here for the choice of practical implementations. First, we may choose the geometrically closest feasible dual solution [7], $(y[k+1], t[k+1]) = \text{argmin}_{y,t\ge 0; \sum_i y_i \le w+t} ||(y,t),(y',t')||$. Second, we may proportionally scale the dual variables [6]: $y[k+1] = \alpha y'$, $t[k+1] = \alpha t'$, where $\alpha = w/(\sum_i y'_i - t')$. Third, observe that when $\sum_i y_i(\vec{uv}) > w(\vec{uv}) + t(\vec{uv})$, the subproblem $L(y,t)$ could assign an arbitrarily large value to $f(\vec{uv})$ for the minimization purpose. This violates primal constraint $f(\vec{uv}) \le c(\vec{uv})$, and should incur a high penalty $t(\vec{uv})$. Therefore another approach is to increase t to satisfy dual constraints: $y[k+1] = y'$, $t[k+1] = \sum_i y_i - w$.

The subgradient iterations may start at any feasible dual variables, *e.g.*, $y_i = w/k, \forall i$ and $t = 0$. Each subgradient iteration consists of primal update followed by dual update. In order to converge, the step sizes should satisfy: $\theta \ge 0$, $\lim_{k\to\infty} \theta[k] = 0$ and $\sum_{k=1}^{\infty} \theta[k] = \infty$. Example sequences include $\theta[k] = a/\sqrt{k}$ or $\theta[k] = b/(ck+d)$ for some positive constants a, b, c and d. Upon convergence, we obtain optimal cost allocation vector y^* and optimal edge tax vector t^*. Then we may apply the tax return procedure introduced in Sec. V-C to obtain a tax-free solution y', or apply primal recovery techniques [28] to obtain f^*.

D. Discussions

Recall that the optimal edge taxes in t^* are in line with the added value component of VCG payment for selfish nodes. According to the original definition of VCG, we need to solve a min-cost multicast problem in network G as well as in $G - e$, for each $e \in E$. Applying the subgradient method we just discussed, all entries in t^* are recovered simultaneously, with the total amount of computation equivalent to that of solving just a single min-cost multicast problem in G. Therefore our algorithm represents an improvement in time complexity on the order of $|E|$.

VII. Conclusion and Discussions

We studied in this paper how to regulate selfish multicast flows to achieve minimum total cost. We consider explicitly the encodable property of information flows, and adopt the conceptual flow structure of multicast routing accordingly. We show that encouraging cost share is critical in enforcing min-cost multicast, and that traditional cost sharing fails to achieve this goal. We then prove that shadow price based cost sharing may enforce optimal multicast flows. Prior to this work, cost shares exist only to enforce sub-optimal multicast flows. With further complications of finite link capacity bounds, we propose to enforce the optimal multicast flow with edge taxing and cost sharing combined. We also show that it is possible to return taxes to multicast flows to obtain a tax-free solution. Finally we present efficient algorithms to compute the necessary cost shares and taxes. Our shadow price based method guarantees a minimum price of stability given selfish multicast flows. We leave as an open problem whether the price of anarchy can also be minimized through appropriate economic measures, for the same selfish min-cost multicast problem.

We finally bring to the reader's attention two insightful comments from the referees. First, unlike the Shapely value method, our solution loses the *anonymity* property by differentiating flows in cost sharing. Like many other mechanisms without anonymity, our shadow-price based cost sharing has a continuous neighborhood centered around it that all satisfy the stability, budget balance and fairness requirements. We plan to study whether the shadow-price solution indeed has a unique property that differentiates itself from the neighborhood. Second, our solution is designed for infinitesimally small flows as agents. It is an interesting direction to investigate how the solution need to change if each receiver is instead an agent, capable of re-routing a non-negligible amount of its flows. We point the readers to [29] for a related study.

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