# On Benefits of Network Coding in Bidirected Networks and Hyper-networks

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***Abstract*—Network coding is a technique that allows information flows to be encoded while routed across a data network. It was shown that network coding helps increase the throughput and reduce the cost of data transmission, especially for one-to-many multicast applications. An important direction in network coding research is to understand and quantify the *coding advantage* and *cost advantage*, *i.e.*, the potential benefits of network coding, as compared to routing, in terms of increasing throughput and reducing transmission cost, respectively. Two classic network models were considered in previous studies of coding advantage: directed networks and undirected networks. The study of coding advantage in this work further focuses on two types of parameterized networks, including bidirected networks and hyper-networks, which generalizes the directed and the undirected network models, respectively. With proper parameter setting, more realistic modeling of networks in practice can be achieved. We prove upper-bounds and lower-bounds on the coding advantage for multicast in these models. Some of our bounds are new and unknown before, some improve upon previously proven bounds, and some answer open questions in the literature.**

## I. Introduction

Network coding, originally proposed by Ahlswede *et al.* in the field of information theory [1], is a technique that allows information flows to be encoded when they meet within a data network, besides merely being forwarded and replicated. Such a departure from the classic store-and-forward principle has proven effective in increasing the network capacity. Higher end-to-end throughput, particularly for one-to-many multicast data transmission, is witnessed in a number of network scenarios [1], [2], [3]. Multicast represents an increasingly more important class of applications on the Internet, encompassing traditional and emerging one-to-many data dissemination applications, such as software patch distribution, live media streaming and video conferencing.

A fundamental direction of network coding research focuses on quantifying the benefits of network coding over routing, known as the *coding advantage*, measured as the ratio of the achievable throughput with network coding over that with routing. Without network coding, a multicast routing solution is based on a multicast tree, or a packing of multicast trees [3], [4].

In the directed network setting, where each link has a pre-selected direction, there exists a combination network pattern in which the coding advantage is unbounded as the network size grows [2]. However, in the undirected network setting, where capacity at each link can be shared flexibly between the two directions, a contrasting result is proved: the coding advantage is upper-bounded by a small constant of 2 [4], [5], [6]. That is, network coding can at most double multicast throughput.

Directed and undirected graphs are classic subjects of study in theoretical computer science. While simple and easy to apply, they do not faithfully depict the wireline or wireless network topologies in practice. For example, large coding advantages in the directed setting are observed in contrived, extremely asymmetric topologies that favors network coding over tree packing, with links existing in one direction only between neighboring nodes. This is apparently different from the picture of the Internet, where pair-wise router interconnections are mostly bidirectional, *i.e.*, if a router $A$ can transmit to a neighbor router $B$, so can $B$ transmit to $A$ [7].

In this work, we propose to study the coding advantage in two types of parameterized networks, with richer modeling power. The first is the bidirected network model, parameterized with $\alpha$, the highest ratio of opposite link capacities between neighboring nodes. When $\alpha = 1$, we have a (completely) balanced network, fairly close to core of the Internet [7], [8]. When $\alpha = \infty$, we arrive at the directed network model. Results obtained for the bidirected model directly carry over to special cases, including the two extreme ones above. Another motivation for studying bidirected networks is that, interestingly, it provides a tool for analyzing the coding advantage in hyper-networks, our second model of study in this work.

In the *hyper-network* model, a *hyper-link* connects two or more nodes, and a transmission from one of them through the hyper-link reaches all others. A hyper-network $H$ is parameterized with $\beta$, the max number of nodes a hyper-link may connect. An undirected network can be viewed as a special case with $\beta = 2$. Hence results obtained for hyper-networks directly apply in undirected networks. Furthermore, hyper-links are natural candidates for modeling wireless transmissions, since they capture the broadcast advantage of omnidirectional antennas [9]. An Ethernet bus or a multicast switch can also be viewed as a hyper-link, in that one transmission reaches all nodes simultaneously.

In this work, our main problem of study is: *how large can the coding advantage be, in bidirected networks and*

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*hyper-networks?* We explore how the coding advantage is related to the link symmetry of a network, and how an (approximately) balanced network behaves differently from a directed or undirected network. For hyper-networks, we aim to prove the first upper-bound on the coding advantage. Our work advances the state-of-the-art of network coding research through proving new bounds on the coding advantage, improving existing bounds, as well as resolving open questions in the field. We now provide an overview of contributions made in this work.

Combining the technique of link splitting in Eulerian graphs with Edmonds' Theorem on spanning tree packing, we prove that, for a node-balanced multicast network where each node has symmetrical transmit and receive capacities, the coding advantage is always 1. This implies that the coding advantage is 1 for link-balanced networks, improving the existing upper-bound of 4 [10]. We show that in either link-balanced or node-balanced networks, the max multicast throughput is always feasible without any information processing (replication or encoding) at relay nodes. If one assumes Internet routers have symmetrical inbound and outbound capacities available, then we can conclude that in-network processing (IP-multicast or network coding) is unnecessary, and end system multicast suffices. The tight upper-bound of 1 further implies that multicast tree packing, NP-hard in general, becomes polynomial-time computable in balanced networks. We discuss such an efficient algorithm, analyze its complexity, and show that it outperforms previously adopted algorithms [11].

An $\alpha$-balanced bidirected network relaxes link symmetry from 1 to $\alpha \geq 1$, and captures a larger class of networks in practice. We prove that the coding advantage in an $\alpha$-balanced network is at most $\alpha$, improving the best known upper-bound of $2(\alpha + 1)$ in existing literature [10]. We further prove a first lower-bound of $\Omega(\sqrt{\alpha})$. These two bounds for $\alpha$-balanced networks unify previous bounds for balanced networks and directed networks that can be arbitrarily imbalanced, revealing a connection between the asymmetry of a network and its coding advantage.

For hyper-networks, Li *et al.* [10] proposed the following open problem: *does there exist a constant upper-bound for multicast coding advantage in hyper-networks?* We provide a negative answer, by proving a lower-bound of $\Omega(\log(\beta))$ on the coding advantage, through generalizing the 3-layer combination network pattern [12] into the hyper-netwok paradigm. We further prove an upper-bound of $\beta$, through a network transformation technique that relates hyper-networks to balanced networks. The proof reveals an interesting connection between the two seemingly distinct network models. The result immediately implies the well-known upper-bound of 2 for multicast coding advantage in undirected networks [4], [5], [6].

Besides benefits in improving throughput, we also study the benefit of network coding in terms of reducing multicast cost, under the classic linear link cost criterion. Throughput advantage and cost advantage of network coding follow a loose primal-dual relation [5], [12]. The latter is more general, and depends on not only link capacities but also link costs. For a bidirected network with both balanced link capacities and balanced link costs, we prove that the cost advantage is at most 2. When the ratio of cost between a pair of nodes is between 1 and $\alpha'$, we prove a generalized upper-bound of $2\alpha'$, unifying the unbounded cost advantage in directed networks and the upper-bound of 2 in undirected networks. For hyper-networks, we prove instead that the cost advantage is at most $\beta$, which directly implies the bound of 2 for cost advantage in undirected networks [5].

The rest of the paper is organized as follows. We review previous research in Sec. II, and define models and notations in Sec. III. Coding advantage is analyzed for bidirected networks in Sec. IV, and for hyper-networks in Sec. V. Cost advantage is studied in Sec. VI. Sec. VII concludes the paper.

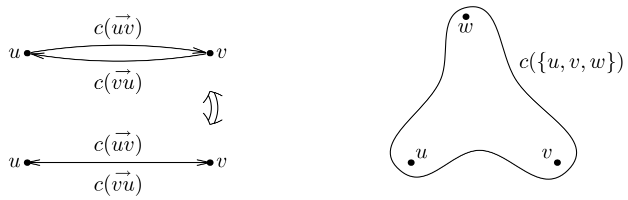
## II. Previous Research

Li *et al.* studied the coding advantage in undirected networks [4], [10]. Applying graph theory techniques, they first prove that the coding advantage is upper-bounded by a constant of 2. The same bound is later proved using linear programming techniques [5] and different graph theory techniques [6]. This work was partly inspired the work of Li *et al.*. In particular, applying graph splitting for handling relay nodes still plays a role in some of our analysis, although in directed not undirected fashion. Li *et al.* further proposed as an open question, whether a constant upper-bound exists for multicast in hyper-networks [10]. We resolve this open problem, and generalize the upper-bound of 2 for undirected networks to an upper-bound of $\beta$ for all hyper-networks.

In the core of the Internet, most links are roughly balanced in the two directions. For example, the six ISP topologies in the Rocketfuel project are all completely balanced [11]. Internet structure studies often apply a single value for specifying link capacities in both directions [7] . Fraleigh *et al.* [13] report that the closer to the Internet backbone, the more symmetric the communication traffic is. For the links they examined (OC-48), the opposite traffic ratio is always between $1 : 1$ and $5 : 1$. John and Tafvelin [8] tap a backbone link (OC-192), and report that no significant difference between opposite link flow volumes can be observed. Such empirical observations have motivated our studies of the balanced and $\alpha$-balanced network models.

In Peer-to-Peer and overlay networks, the coding advantage has been studied by Chiu *et al.* [14], and by Shao and Li [15]. In the latter work, the authors model an overlay network as a node-capacitated network, where the sum rate of all inbound and outbound traffic at a node is capped. They derive sufficient conditions for determining when network coding does not improve the achievable throughput.

The benefits of network coding in wireless networks are first studied by Liu *et al.* [16], and then extended by Keshavarz-Haddadt and Riedi [17]. They consider wireless interference and geometric properties in modeling a wireless network, and prove that both coding and cost advantages are upper bounded by constant factors.



(a) Bidirected Link (b) Hyper-link

Fig. 1. Communication links in (a) bidirected networks and (b) hyper-networks. In bidirected networks, we use a double-headed arrow to represent a pair of opposite links for simplicity.

For multicast throughput, we adopt the standard definition of symmetric throughput in the literature, *i.e.*, all receivers are required to receive and recover information flows transmitted from the source at an equal rate. It is also possible to relax the symmetry requirement, by allowing different receivers to receive at different rates. Under such a model, coding advantage is studied by Checkuri *et al.* [18], for average throughput across all receivers.

## III. Network Models and Notations

**Undirected Networks.** We represent an undirected network with a capacitated graph $G(V, E, c)$, with node set $V$ and edge set $E$. The vector $c \in \mathcal{Z}_+^E$ stores the capacities of links in $E$. Here $\mathcal{Z}_+$ is the set of positive integers. An undirected link $uv$ with total capacity $c(uv)$ can support a flow $f(\vec{uv})$ from $u$ to $v$ and a flow $f(\vec{vu})$ from $v$ to $u$ simultaneously, as long as the total flow rate form both directions does not exceed the available link capacity, *i.e.*, $f(\vec{uv}) + f(\vec{vu}) \le c(uv)$.

**Directed Networks.** We represent a directed network with a directed graph $D(V, A, c)$, with node set $V$ and link set $A$. The vector $c \in \mathcal{Z}_+^A$ stores the capacities of links in $A$. A directed link $\vec{uv}$ can be employed to transmit a flow from $u$ to $v$ only, and the flow rate can not exceed the link capacity, *i.e.*, $f(\vec{uv}) \le c(\vec{uv})$.

**Bidirected and $\alpha$-Balanced Networks.** In a bidirected network, node connection is always bidirectional (Fig. 1 (a)), in that for any pair of nodes $u$ and $v$ in $V$, $\vec{uv} \in A$ implies $\vec{vu} \in A$. Given a bidirected network $B(V, A, c)$, we define the *link capacity balance parameter* as

$$\alpha \triangleq \max_{\vec{uv} \in A} \frac{c(\vec{uv})}{c(\vec{vu})}.$$

We refer to $B(V, A, c)$ as an *$\alpha$-balanced network*. A special case is $\alpha = 1$, *i.e.*, link connections are entirely symmetrical, $c(\vec{uv}) = c(\vec{vu}), \forall\ \vec{uv} \in A$. We refer to such a network as a *(completely) balanced network*.

As a classic technique in graph theory, a link with integral capacity can be replaced with multiple unit-capacity links between the same pair of nodes. Such a transformation results in a directed multi-graph, and does not change the network connectivity between any pair of nodes.

**Hyper-Networks.** A generalization of the undirected network model is the *hyper-network* model, where each *hyper-link* connects two or more nodes (Fig. 1 (b)). Specifically, we model a hyper-network with a hypergraph $H(V, \mathcal{E})$, where $\mathcal{E}$ is the set of hyper-edges. Each hyper-edge $e \in \mathcal{E}$ 'covers' a set of two or more nodes. The *size* of a hyper-edge is the number of nodes it connects. We denote the maximum hyper-edge size as $\beta$. An undirected network can be viewed as a special type of hyper-network with $\beta = 2$.

When one node transmits an information flow through a hyper-link, the flow reaches all the other nodes adjacent to the hyper-link simultaneously. The total flow on a hyper-link at a given time is upper-bound by the hyper-link capacity. For example, if $e$ covers three nodes $u, v, w$, we have $f(u \to vw) + f(v \to uw) + f(w \to uv) \le c(e)$, where $f(u \to vw)$ denotes the amount of information transmitted from $u$.

TABLE I
Summary of Notation

| | |
|---|---|
| $G(V,E)$ | Undirected graph $G$ with node set $V$ and edge set $E$ |
| $D(V,A)$ | Directed graph $D$ with link set $A$ |
| $B(V,A)$ | Bidirected graph $B$ |
| $H(V,\mathcal{E})$ | Undirected hypergraph $H$ with hyper-edge set $\mathcal{E}$ |
| $c(e)$ | Capacity of link $e$ |
| $w(e)$ | Cost per unit flow at link $e$ |
| $\alpha$ | Capacity balance parameter of a bidirected network |
| $\alpha'$ | Cost balance parameter of a bidirected network |
| $\beta$ | Max hyper-edge size of a hyper-network |
| $s$ | Source node |
| $T$ | Set of receiver nodes $\{t_1, t_2, \cdots, t_{\lvert T \rvert}\}$ |
| $\tau$ | A multicast tree that connect $s$ to the receivers $T$ |
| $\lambda(s,t)$ | Max flow from $s$ to $t$ |
| $\mathcal{R}_{tree}$ | Max multicast throughput with tree packing |
| $\mathcal{R}_{nc}$ | Max multicast throughput with network coding |
| $\mathcal{C}_{tree}$ | Min multicast cost with tree packing |
| $\mathcal{C}_{nc}$ | Min multicast cost with network coding |

## IV. Coding Advantage in Bidirected Networks

We now study the coding advantage in bidirected networks. We first provide a precise definition of the optimal multicast problem, both with and without network coding. We then prove bounds on the coding advantage in completely balanced networks, and generalize to $\alpha$-balanced networks.

### A. Optimal Multicast in Bidirected Networks

Multicast models the dissemination of information from a common source to a set of receivers within the same network. Given a (bi)directed network $D(V, A, c)$, let $s \in V$ be the multicast source, and $T \subset V \backslash \{s\}$ be the set of multicast receivers. Note that unicast (one-to-one) and broadcast (one-to-all) can be viewed as special cases of multicast, with $|T| = 1$ and $|T| = |V| - 1$, respectively. A multicast rate (throughput) $r$ is achieved if each receiver can receive and recover information flows from the source at rate $r$.

Let $\lambda_D(u, v)$ be the maximum flow rate from node $u$ to node $v$. By the max-flow min-cut theorem in network flows, $\lambda_D(u, v)$ equals the capacity of the min cut between $u$ and $v$. The celebrated result on multicast rate feasibility in directed networks extends the max-flow min-cut theorem from one-to-one unicast to one-to-many multicast: *with network coding, a multicast rate $r$ is feasible in a directed network if and*

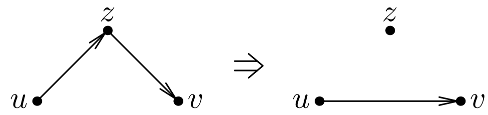



Fig. 2. Directed splitting operation at $z$.

*only if it is feasible as a unicast rate to each receiver* $t \in T$ *independently*, *i.e.*, the min-cut between $s$ and every $t \in T$ has capacity at least $r$. Therefore, the maximum multicast rate with network coding is $\mathcal{R}_{nc}(D, s, T) = \min_{t \in T} \lambda_D(s, t)$. Parameters $(D, s, T)$ are dropped when clear from context.

Without network coding, *i.e.*, with routing and information replication only, a multicast solution can be decomposed into a set of multicast trees, since each information flow propagates along a tree rooted at the multicast source, covering all multicast receivers. In this context, achieving the maximum multicast throughput is equivalent to the combinatorial problem of Steiner tree packing [4]. Specifically, let $\mathcal{T}$ denote the set of all possible multicast trees. For each multicast tree $\tau \in \mathcal{T}$, let $r_\tau$ be the rate of the information flow we wish to transmit along $\tau$. A tree packing solution is feasible if for each link $\overrightarrow{uv} \in A$, the total flow rate of trees containing $\overrightarrow{uv}$ is bounded by the link capacity $c(\overrightarrow{uv})$. The maximum throughput achieved by routing, denoted by $\mathcal{R}_{tree}(D, s, T)$, is the maximum aggregated flow rates of a feasible multicast tree packing, which can be formulated into a linear program:

$$\begin{array}{lll} \text{maximize} & \sum_{\tau \in \mathcal{T}} r_\tau & \\ \text{subject to :} & \sum_{\tau : \overrightarrow{uv} \in \tau} r_\tau \leq c(\overrightarrow{uv}), & \forall\, \overrightarrow{uv} \in A \\ & r_\tau \geq 0, & \forall \tau \in \mathcal{T} \end{array}$$

By definition, $\mathcal{R}_{tree} \leq \mathcal{R}_{nc}$ for any $(D, s, T)$, since a multicast tree packing solution is a special case of a network coding solution, with no coding applied. We define the *coding advantage* as $\mathcal{R}_{nc}/\mathcal{R}_{tree}$, *i.e.*, the ratio of maximum throughput with network coding over that without coding, which is always no less than 1.

### *B. Upper-bound for Completely Balanced Networks*

With the coding advantage well defined, we can now derive our first result on the coding advantage in completely balanced networks. It will latter help our studies in $\alpha$-balanced networks and hyper-networks.

**Theorem 1.** *For multicasting in a completely balanced network* $B$*,* $\mathcal{R}_{tree} = \mathcal{R}_{nc}$*. Furthermore, the optimal multicast throughput can be achieved with integral link flow rates, and without coding or replication at relay nodes.*

*Proof:* We consider the equivalent directed multigraph, where a link with capacity larger than 1 is replaced with parallel links each with unit capacity. We prove that there exist $\mathcal{R}_{nc}$ link-disjoint multicast trees, each of which may contribute a multicast throughput of 1.

We simplify the multicast network by applying splitting operations to remove relay nodes. As illustrated in Fig. 2, a directed splitting at a node $z$ refers to the replacement of two links $\overrightarrow{uz}, \overrightarrow{zv}$ with a new link $\overrightarrow{uv}$. The first graph theory tool we need is the directed splitting lemma:

*Frank and Jackson [19]: Let* $D = (V + z, A)$ *be a directed multigraph with symmetrical in-degree and out-degree at each node. For each link* $\overrightarrow{uz} \in A$*, there exists a link* $\overrightarrow{zv} \in A$*, such that after splitting off* $\overrightarrow{uz}, \overrightarrow{zv}$*, the link connectivity between every pair of nodes in* $V$ *is unchanged.*

A completely balanced network satisfies the node-symmetric condition in the splitting lemma above. After each splitting operation, the node symmetric property remains true, although link symmetry may no longer hold. Consequently, we can completely isolate a node $z$ by repeatedly applying splitting operations at $z$, while preserving the maximum flow rate between any two other nodes.

We apply complete splitting to relay nodes in $V \backslash (\{s\} \cup T)$ sequentially, and denote the resulting digraph as $D$. By the splitting lemma, $\mathcal{R}_{nc}(B) = \min_{t \in T} \lambda_B(s, t) = \min_{t \in T} \lambda_D(s, t) = \mathcal{R}_{nc}(D)$, *i.e.*, the optimal throughput achieved by network coding is not affected. We do not explicitly distinguish these two notations in the rest of the proof.

We next introduce a result by Edmonds on spanning tree packing, for characterizing the coding advantage of a broadcast session in directed networks:

*Edmonds[20]: In a directed multigraph* $D = (V, A)$*, the maximum number of arc-disjoint spanning trees rooted at* $s \in V$ *is* $\min_{t \in V - s} \lambda_D(s, t)$*.*

This result implies that network coding can not improve the throughput of a broadcast session in a directed network. Since there is no relay node in $D$, we claim that there exist $\mathcal{R}_{nc}$ broadcast trees in $D$.

The splitting operations at the relay nodes indicate how to *forward* the received information. No replication or encoding is required at the nodes being split off. We can achieve the optimal throughput $\mathcal{R}_{nc}$ by applying reverse splitting operations on broadcast trees in $D$, and using the resulting multicast trees for transmission. As a result, we only need to replicate information at the source and receiver nodes. ■

**Polynomial Time Algorithm for Steiner Tree Packing.** Steiner Tree Packing in general graphs is a well known NP-hard problem. Interestingly, for completely balanced networks, a polynomial time tree packing algorithm can be extracted from the proof of Theorem 1:

- Step 1: Split off all the relay nodes to obtain a directed graph $D$;
- Step 2: Apply the polynomial time algorithm of Wu *et al.* [21] to compute a spanning tree packing in $D$;
- Step 3: Translate the spanning trees in $D$ to Steiner trees in $B$, by reversing the split operations.

For step 1, brute-force search for splittable link pairs takes O($|T||E||V|^3$) time. Step 2 can be done in time O($\mathcal{R}_{nc}^3|T||E|$) [21]. Step 3 can be done in time O($|E|$) with the split operations in step 1 stored in memory. Thus the overall complexity is O($|T||E||V|^3 + \mathcal{R}_{nc}^3|T||E|$).

Wu *et al.* [11] designed heuristic algorithms for Steiner tree packing for the six ISP topologies they study, which are all link-balanced. Our algorithm above outperforms their heuristic algorithms in that the former guarantees optimal solutions, while the latter finds sub-optimal solutions.

**Discussions.** The "link-balanced" condition of the theorem can be relaxed to "node-balanced", which guarantees the existence of the desired directed splitting operation. We emphasize that the optimal throughput can be achieved by packing multicast trees with integer flow rates, while in general, fractional packing outperforms integral packing [5]. For example, in the classic butterfly network [1], [22], fractional packing can achieve a multicast rate 1.5, while integral packing can only achieve a rate 1.

While access networks at the edge of the Internet may not be balanced, the core of the Internet is rather close to a balanced network [13]. Then by Theorem 1, in such type of balanced networks, optimal multicast is feasible without IP-multicast or network coding requirements at routers in the middle of the network. Furthermore, the fact that fractional flows are unnecessary for achieving the maximum multicast throughput helps reduce the overhead in traffic splitting and management for achieving optimal multicast.

### *C. Upper-bound for $\alpha$-Balanced Networks*

Networks in practice often exhibit approximate rather than absolute symmetry in opposite link capacities. Such networks can be characterized using our $\alpha$-balanced network model. The coding advantage in $\alpha$-balanced networks was previously analyzed by Li *et al.* [10], where it was proved to be upper bounded by $2(\alpha+1)$. In this subsection, we apply theorem 1 to improve this upper-bound to $\alpha$.

**Theorem 2.** *The coding advantage for multicast in an $\alpha$-balanced network is upper bounded by $\alpha$.*

*Proof:* Let $B = (V, A, c)$ be an $\alpha$-balanced network, let $B_1 = (V, A, \lfloor c \rfloor)$ denote the completely balanced graph induced from $B$ by truncating the larger link capacity to the smaller one, between each pair of neighboring nodes. We have $\mathcal{R}_{tree}(B) \geq \mathcal{R}_{tree}(B_1)$.

Let $\alpha B_1 = (V, A, \alpha\lfloor c \rfloor)$ be the digraph induced from $B_1$, by scaling each link's capacity by $\alpha$. Note that each link's capacity in $\alpha B_1$ is no less than in $B$, hence $\mathcal{R}_{nc}(\alpha B_1) \geq \mathcal{R}_{nc}(B)$. Therefore,

$$\begin{aligned}\mathcal{R}_{tree}(B) &\geq \mathcal{R}_{tree}(B_1) = \mathcal{R}_{nc}(B_1) \\ &= \frac{1}{\alpha}\mathcal{R}_{nc}(\alpha B_1) \geq \frac{1}{\alpha}\mathcal{R}_{nc}(B)\end{aligned}$$

The first equality $\mathcal{R}_{tree}(B_1) = \mathcal{R}_{nc}(B_1)$ is due to Theorem 1, since $B_1$ is a completely balanced network. The second equality $\mathcal{R}_{nc}(B_1) = \frac{1}{\alpha}\mathcal{R}_{nc}(\alpha B_1)$ holds because the maximum throughput scales linearly with the edge capacities. ■

### *D. Lower-bound for $\alpha$-Balanced Networks*

Given the improved upper-bound of $\alpha$ in Theorem 2, a natural question is: can this new upper-bound be further tightened? Apparently, the bound is tight for the case of $\alpha = 1$. For larger values of $\alpha$, we show that there exist $\alpha$-balanced networks with coding advantage of $\Omega(\sqrt{\alpha})$. The upper- and lower- bounds we prove are therefore approximately tight against each other, within a factor of $O(\sqrt{\alpha})$.

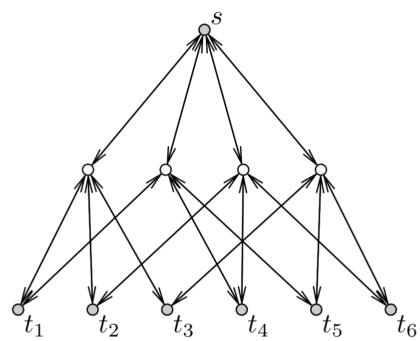



Fig. 3. The $\alpha$-balanced network $C_\alpha(n=4, k=2)$. Each downward link has capacity $\alpha$, each upward link has capacity 1.

Our approach to show the lower-bound of the coding advantage is to construct bidirected multicast networks with large coding advantages. A natural way is to transform existing examples in directed networks into $\alpha$-balanced networks. We begin with combinational networks $C(n, k)$ [2], which were used to show that the benefits of network coding can be arbitrarily large in directed networks.

More specifically, we construct an $\alpha$-balanced network $C_\alpha(n, k)$ from $C(n, k)$ by first scaling each link's capacity to $\alpha$, and then adding opposite links of unit capacity to make the network bidirected. Fig. 3 illustrates the network $C_\alpha(n = 4, k = 2)$.

**Claim 1.** *The coding advantage of $C_\alpha(n, k)$ is $\Omega(\log \alpha)$.*

*Proof:* For a multicast tree to cover all receivers, it needs to include at least $n - k + 1$ intermediate nodes. Otherwise, there is a receiver who is connected to $k$ uncovered intermediate nodes and is not covered by the tree. Therefore, each tree has at least $n - k + 1$ arcs entering the intermediate nodes. The total number of such arcs is upper bounded by $\alpha n + n\binom{n-1}{k-1}$. Thus, the maximum number of trees we can fractionally pack is

$$\mathcal{R}_{tree} \leq \frac{\alpha n + n\binom{n-1}{k-1}}{n-k+1}$$

On the other hand, note that $\mathcal{R}_{nc} = \alpha k$. Therefore,

$$\frac{\mathcal{R}_{nc}}{\mathcal{R}_{tree}} \geq \frac{\alpha k}{\alpha + \binom{n-1}{k-1}} \cdot \frac{n-k+1}{n}$$

Let $n = 2k$ and $\alpha = 2^n$, we have

$$\frac{\mathcal{R}_{nc}}{\mathcal{R}_{tree}} > \frac{\alpha k}{\alpha + 2^n} \cdot 2 \geq k = \frac{1}{2}\log \alpha$$

■

From the above analysis of $C_\alpha(n, k)$, it can be seen that the utilization of upstream links at each receiver substantially increases routing throughput. As the number of receivers in

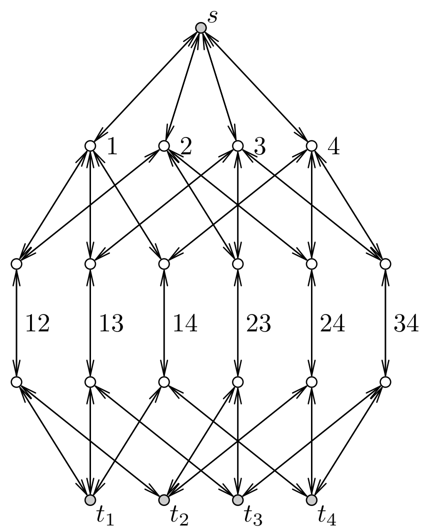



Fig. 4. The $\alpha$-balanced network $ZK_\alpha(p = 2, N = 4)$. Each downward link has capacity $\alpha$, each upward link has capacity 1.

the combinational networks is exponential to the parameter $n$ and $k$, the coding advantage becomes relatively small. With this observation, we turn to the class of directed networks $ZK(p, N)$, introduced by Chekuri *et al.* [18] in their investigation of the coding advantage for average multicast throughput. The $ZK(p, N)$ networks turn out to be more powerful than combination networks, in terms of providing higher coding advantage with less receivers.

Specifically, a $ZK(p, N)$ network consists of a source $s$, $N$ receivers $t_1, t_2, \cdots, t_N$, and three layers of relay nodes. The first layer, layer $A$, has $\binom{N}{p-1}$ nodes, which can be indexed as $A_U$, with $U \subset \{1, 2, \cdots, N\}$ and $|U| = p - 1$. The next two layers $B$ and $C$ each has $\binom{N}{p}$ nodes, which can be indexed as $B_W$ and $C_W$ respectively, with $W \subset \{1, 2, \cdots, N\}$ and $|W| = p$. These nodes are connected in the following way: for any nodes $A_U, B_W, C_W$, there is a link from $s$ to $A_U$, a link from $A_U$ to $B_W$ if $U \subset W$, a link from $B_W$ to $C_W$, and a link from $C_W$ to $t_i$ if $i \in W$. So far each link is unidirectional. To construct the $\alpha$-balanced network $ZK_\alpha(p, N)$ from $ZK(p, N)$, we add an opposite link of unit capacity for each link in $ZK(p, N)$, and set all the original links' capacity to $\alpha$. Fig. 4 illustrates the $ZK_\alpha(p = 2, N = 4)$ network.

**Theorem 3.** *The coding advantage in $ZK_\alpha(p, N)$ is of order $\Omega(\sqrt{\alpha})$.*

*Proof:* For each rooted Steiner tree $\tau$, let $a_\tau$ denote the number of layer $A$ nodes in $\tau$, $c_\tau$ denote the number of layer $C$ nodes with its predecessor in layer $B$, and $d_\tau$ denote the number of layer $C$ nodes with its predecessor being a receiver. For a fractional tree packing $\mathcal{T} = \{\tau_1, \tau_2, \cdots, \tau_{|\mathcal{T}|}\}$ with tree flow rates $r_1, r_2, \cdots, r_{|\mathcal{T}|}$, consider the aggregated link capacity constraints at nodes counted in $a_\tau$, $c_\tau$, and $d_\tau$ respectively,

$$\sum_{\tau\in\mathcal{T}} r_\tau a_\tau \leq \alpha\binom{N}{p-1} + p\binom{N}{p} \quad (1)$$

$$\sum_{\tau\in\mathcal{T}} r_\tau c_\tau \leq \alpha\binom{N}{p} \quad (2)$$

$$\sum_{\tau\in\mathcal{T}} r_\tau d_\tau \leq p\binom{N}{p} \quad (3)$$

The maximum number of receivers directly covered by nodes counted in $c_\tau$ is $a_\tau(p-1)+c_\tau$. For each node in $d_\tau$, it can cover at most $p - 1$ receivers. Therefore, to cover all the receivers,

$$a_\tau(p-1) + c_\tau + d_\tau(p-1) \geq N$$

Multiply both sides with $\sum_{\tau\in\mathcal{T}} r_\tau$, we have:

$$N\sum_{\tau\in\mathcal{T}} r_\tau \leq \sum_{\tau\in\mathcal{T}} r_\tau\big(a_\tau(p-1) + c_\tau + d_\tau(p-1)\big) \quad (4)$$

Combining the above inequalities (1–4), we have that the maximum rate for fractional routing

$$\mathcal{R}_{tree} \leq \frac{1}{N}\left[\alpha(p-1)\binom{N}{p-1} + \alpha\binom{N}{p} + 2p(p-1)\binom{N}{p}\right]$$

Furthermore, $\mathcal{R}_{nc} = \alpha\binom{N-1}{p-1}$. Therefore,

$$\frac{\mathcal{R}_{tree}}{\mathcal{R}_{nc}} \leq \frac{p-1}{N-p+1} + \frac{1}{p} + \frac{2(p-1)}{\alpha}$$

Let $p - 1 = \sqrt{\alpha}$ and $N = \alpha$, we have

$$\frac{\mathcal{R}_{tree}}{\mathcal{R}_{nc}} < \frac{1}{\sqrt{\alpha}-1} + \frac{3}{\sqrt{\alpha}}$$

which completes the proof. ■

## V. Coding Advantage in Hyper-Networks

Wireless networks represent a particularly promising paradigm for the application of network coding. Intuitively, due to the broadcast nature of a wireless transmission, an encoded packet can be delivered to multiple neighbors in a single transmission, helping more than one of them in a bandwidth efficient manner. A hyper-link is a natural candidate for modeling such an one-to-many transmission in an atomic step [9].

In this section, we prove the first upper-bound for the coding advantage in hyper-networks, and then analyze the tightness of the upper-bound by proving a corresponding lower-bound for the same.

### A. Optimal Multicast in Hyper-networks

Let $H = (V, \mathcal{E})$ be an (undirected) hypergraph. In this section, we use $e$ to denote a hyper-edge in $\mathcal{E}$. A path connecting $s$ and $t$ is a sequence of hyper-edges $(e_1, e_2, \cdots, e_n)$, such that $s \in e_1$, $t \in e_n$, $e_i \cap e_{i+1} \neq \emptyset, i = 1, 2, \cdots, n-1$, and $e_i \neq e_j, \forall i \neq j$. The edge connectivity $\lambda_H(s, t)$ is defined as the maximum number of hyper-edge disjoint paths between $s$ and $t$. Assume each hyper-edge has unit capacity, then

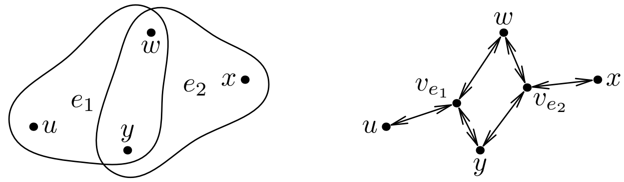



Fig. 5. A hyper-network (left) and its corresponding completely balanced bipartite network (right).

the maximum throughput from $s$ to $t$ is upper bounded by $\lambda_H(s,t)$, *i.e.*, $\mathcal{R}_{nc}(H,s,T) \le \min_{t\in T}\lambda_H(s,t)$.

An orientation of a hypergraph is obtained by assigning a direction to each hyper-edge $e$, via identifying a tail for $e$, indicating which node is sending messages through this hyper-link.

For $s \in V, T \subset V\backslash\{s\}$, define an $s$-$T$ hyper-tree as a set of hyper-edges $\tau$, for which there exists an orientation so that there is at least one directed path from $s$ to $t$, $\forall t \in T$. For routing in the hyper-network model, the trace of a message from $s$ to all receivers forms an $s$-$T$ hyper-tree. Therefore, the max throughput achieved by routing is the max packing of $s$-$T$ hyper-trees.

Let $\mathcal{T}$ denote the set of all possible $s$-$T$ hyper-trees. Similar to the bidirected network case, $\mathcal{R}_{tree}(H,s,T)$ is the optimal value of the following linear program:

$$\begin{array}{lll} \text{maximize} & \sum_{\tau\in\mathcal{T}} r_\tau & \\ \text{subject to :} & \sum_{\tau:e\in\tau} r_\tau \le c(e), & \forall e\in\mathcal{E} \\ & r_\tau \ge 0, & \forall \tau\in\mathcal{T} \end{array}$$

### *B. Upper-bound for Hyper-networks*

Now with the coding advantage $\mathcal{R}_{nc}/\mathcal{R}_{tree}$ well defined in hyper-networks, we are ready to prove an upper-bound on it.

**Theorem 4.** *In an undirected hypergraph $H = (V,\mathcal{E})$, the coding advantage $\mathcal{R}_{nc}/\mathcal{R}_{tree}$ is upper bounded by the maximum hyper-edge size $\beta$.*

*Proof:* As the maximum throughput $\mathcal{R}_{nc}$ is upper bounded by $\lambda_H(s,T) = \min_{t\in T}\lambda_H(s,t)$, it is sufficient to prove that $\mathcal{R}_{tree} \ge \frac{1}{\beta}\lambda_H(s,T)$.

As illustrated in Fig. 5, given the hypergraph $H = (V,\mathcal{E})$, we construct a completely balanced bipartite digraph $B(V \cup V_{\mathcal{E}}, A)$, where $V$ is the original node set in $H$, and $V_{\mathcal{E}} = \{v_e \mid \forall e \in \mathcal{E}\}$. Connect $v \in V$ and $v_e \in V_{\mathcal{E}}$ if $v$ is an end node of $e$. In other words, $A = \{\overrightarrow{vv_e}, \overrightarrow{v_ev} \mid \forall v \in e\}$. For any pair of nodes $u, v \in V$, $\lambda_B(u,v) \ge \lambda_H(u,v)$, because for any $n$ hyper-edge disjoint $s$-$t$ paths $\mathcal{P}_1, \mathcal{P}_2, \cdots, \mathcal{P}_n$ in $H$, we can find $n$ edge disjoint s-t paths $P_1, P_2, \cdots, P_n$ in $B$ by the following method: let $\mathcal{P}_i = (e_1, e_2, \cdots, e_n)$, we choose a sequence of intermediate nodes $v_i \in e_i \cap e_{i+1}$, so the path can be rewritten as $s = v_0, e_1, v_1, e_2, \cdots, v_{n-1}, e_n, v_n = t$. Then we construct $P_i$ as $(v_0, v_{e_1}, v_1, \cdots, v_{e_n}, v_n)$. Because $e_i$ appears only once in $\mathcal{P}_1, \mathcal{P}_2, \cdots, \mathcal{P}_n$, $P_1, P_2, \cdots, P_n$ are edge disjoint.

According to Theorem 1, in the completely balanced network $B$, there is a rooted Steiner tree packing $\tau_1, \tau_2, \cdots, \tau_m$ with weights $r_1, r_2, \cdots, r_m$, such that

$$\sum_{j=1}^{m} r_j = \mathcal{R}_{tree}(B) = \mathcal{R}_{nc}(B) = \lambda_B(s,T) \ge \lambda_H(s,T).$$

We complete the proof by constructing $m$ hyper multicast trees $\tau'_1, \tau'_2, \cdots, \tau'_m$ in $H$ with weights $r_1/\beta, r_2/\beta, \cdots, r_m/\beta$. Let $\tau'_j = \{e \in \mathcal{E} \mid v_e \in \tau_j\}$. The orientation and $s$-$t$ directed path in $\tau'_j$ can be determined referring to the $s$-$t$ path in $\tau_j$, so $\tau'_j$ is an $s$-$T$ hyper-tree. For any hyper-edge $e \in \mathcal{E}$,

$$\begin{aligned} \sum_{j:e\in\tau'_j} r_j/\beta &= \sum_{j:v_e\in\tau_j} r_j/\beta && (5)\\ &= \sum_{j:\exists! v\in V, \overrightarrow{vv_e}\in\tau_j} r_j/\beta && (6)\\ &= \sum_{v\in N(v_e)} \sum_{j:\overrightarrow{vv_e}\in\tau_j} r_j/\beta && (7)\\ &\le \sum_{v\in N(v_e)} 1/\beta && (8)\\ &\le 1 && (9) \end{aligned}$$

where $N(v_e)$ denotes the neighbors of $v_e$ in $B$. In the above derivation, equation (5) is due to the construction of hyper-tree $\tau'_j$, and (6) holds because there is a unique parent $v$ for the intermediate node $v_e$, as $\tau_j$ is an $s$ rooted tree. In other words, $v_e \in \tau_j$ if and only if there is a unique link $\overrightarrow{vv_e} \in \tau_j$. Equation (7) rewrites the sum by enumerating the neighbors of $v_e$, and (8) holds because $\tau_1, \tau_2, \cdots, \tau_m$ with weights $r_1, r_2, \cdots, r_m$ form a feasible tree packing. (9) holds because each hyper-edge covers $\beta$ nodes at most, and the neighbors of $v_e$ in $B$ are the same nodes covered by $e$ in $H$.

Therefore, the hyper-trees $\tau'_1, \tau'_2, \cdots, \tau'_m$ with weights $r_1/\beta, r_2/\beta, \cdots, r_m/\beta$ constitute a feasible hyper-tree packing in $H$. Recall that $\sum_{j=1}^{m} r_j/\beta = \frac{1}{\beta}\lambda_B \ge \frac{1}{\beta}\lambda_H$, we have $\mathcal{R}_{tree}(H) \ge \frac{1}{\beta}\lambda_H(s,T) \ge \frac{1}{\beta}\mathcal{R}_{nc}(H)$. ■

### *C. Lower-bound for Hyper-networks*

It is postulated that the upper-bound of 2 for coding advantage in undirected networks is not tight [10]. The largest coding advantage observed in undirected networks is no more than 8/7. Closing the gap between 2 and 8/7 has remained as an important open problem in network coding research. For hyper-networks, we now prove that the max coding advantage is of order $\Omega(\log\beta)$. Our proof is a constructive one based on transforming combination networks [12] into hyper-networks.

**Theorem 5.** *The max coding advantage among hyper-networks of maximum hyper-edge size $\beta$ is of order $\Omega(\log\beta)$.*

*Proof:* Consider the hypergraph $HC(n,k)$ derived from the undirected combination network $C(n,k)$, by viewing the star topology centered at each relay node as a hyper-edge. More specifically, $HC(n,k)$ contains one source node, $\binom{n}{k}$ receiver nodes, and $n$ hyper-edges each of unit capacity. Label the receivers with the chosen set $M \subset \{1, 2, \cdots, n\}, |M| = k$, hyper-edge $e_i (i = 1, 2, \cdots, n)$ connects the source and all the receivers $M$ if $i \in M$. Each receiver is adjacent to $k$ hyper-edges, and the size of each hyper-edge is $\beta = \binom{n-1}{k-1} + 1$.

Let $\mathcal{T}$ denote the set of all possible $s$-$T$ hyper-trees, and $\{r_\tau | \tau \in \mathcal{T}\}$ be the optimal hyper-tree packing. For any hyper-tree $\tau \in \mathcal{T}$ which connects $s$ to all the receivers, $\tau$ has at least $n-k+1$ hyper-edges, because if there are $k$ hyper-edges not contained in $\tau$, the corresponding receiver only connecting to them is left uncovered. Sum all the inequalities of hyper-edge capacity constraints $\sum_{\tau : e_i \in \tau} r_\tau \leq 1, \quad i = 1, 2, \cdots, n$, we have $\sum_{\tau \in \mathcal{T}} r_\tau |\tau| \leq n$, where $|\tau|$ is the number of hyper-edges in $\tau$, which is no less than $n-k+1$. Therefore,

$$\mathcal{R}_{tree} = \sum_{\tau \in \mathcal{T}} r_\tau \leq \frac{n}{n-k+1}$$

On the other hand, $\mathcal{R}_{nc} = k$, since each receiver is directly connected to the source by $k$ hyper-edges. Let $n = 2k$, $\mathcal{R}_{nc}/\mathcal{R}_{tree} \geq k/2$. Recall that $\beta = \binom{n-1}{k-1} + 1 < 2^{2k}$, we have $\mathcal{R}_{nc}/\mathcal{R}_{tree} > \frac{1}{4} \log \beta$. ■

## VI. Cost Advantage

Besides throughput improvements, bandwidth and cost saving is another important benefit of network coding. With the help of network coding, less overall bandwidth consumption is required to achieve a target multicast rate. More generally, we assume a link cost vector $w \in \mathcal{Q}_+^A$, indicating the cost to transmit a unit flow through a link. Here $\mathcal{Q}_+$ is the set of positive rational numbers. For a multicast solution with an underlaying flow of rate $f(\overrightarrow{uv})$ through each link $\overrightarrow{uv}$, the total cost is $\sum_{\overrightarrow{uv} \in A} w(\overrightarrow{uv}) f(\overrightarrow{uv})$.

### A. Cost Advantage in Bidirected Networks

We start our study of the cost advantage within uncapacitated networks, networks that have sufficient bandwidth supply at each link, such that link capacities are not a limiting factor of optimal multicast cost. Given an uncapacitated bidirected graph $D(V, A)$ and a link cost vector $w$, let $\mathcal{C}_{tree}(D, w)$ and $\mathcal{C}_{nc}(D, w)$ be the minimum cost to achieve a unit multicast rate with routing and network coding respectively. The cost advantage is defined as $\mathcal{C}_{tree}(D, w)/\mathcal{C}_{nc}(D, w)$, and is no less than 1.

With link capacity limits removed, an undirected network and a bidirected network differ only in that link weights in opposite directions are always equal in the former but not the latter. Similar to the study of coding advantage in $\alpha$-balanced networks, let $\alpha'$ denote the max ratio of link weights of the two directions, *i.e.*,

$$\alpha' \triangleq \max_{\overrightarrow{uv} \in A} \frac{w(\overrightarrow{uv})}{w(\overrightarrow{vu})}$$

and such weight vector is called $\alpha'$-balanced.

When $\alpha' = 1$, we are essentially considering the max cost advantage in an undirected network, which was proved to be equal to the max coding advantage of undirected networks [5], [12], and therefore, is upper bounded by 2:

**Theorem 6.** *In an uncapacitated bidirected network with symmetric link weight, the cost advantage is upper bounded by* 2.

For the general case $\alpha' \geq 1$, we have the following result that echoes the the coding advantage.

**Theorem 7.** *In an uncapacitated bidirected network* $(B, w)$*, the cost advantage is upper bounded by* $2\alpha'$.

*Proof:* Let $\lfloor w \rfloor$ denote weight vector derived from $w$ by truncating the larger weight to the smaller one, so that $\lfloor w \rfloor$ is symmetric. As $w$ is $\alpha'$-balanced, $\alpha' \lfloor w \rfloor \geq w$.

$$\begin{aligned}\mathcal{C}_{nc}(B, w) &\geq \mathcal{C}_{nc}(B, \lfloor w \rfloor) \geq \frac{1}{2} \mathcal{C}_{tree}(B, \lfloor w \rfloor) \\ &= \frac{1}{2\alpha'} \mathcal{C}_{tree}(B, \alpha' \lfloor w \rfloor) \geq \frac{1}{2\alpha'} \mathcal{C}_{tree}(B, w).\end{aligned}$$

In the derivations above, the first and the last inequalities are due to $\lfloor w \rfloor \leq w \leq \alpha' \lfloor w \rfloor$, and the minimum cost is monotonic to the weight vector. The second inequality is due to theorem 6, since $\lfloor w \rfloor$ is symmetric. The only equality holds because the minimum cost scales linearly with the weight vector. ■

**Remarks.** If the network is not bidirected, the cost advantage is unbounded, as can be demonstrated in directed combination networks [2]. This can be viewed as a special case of $\alpha' = \infty$ (link costs are arbitrarily unbalanced).

It is interesting to note that unlike the duality between cost advantage and coding advantage in undirected networks, we have coding advantage of 1 for completely balanced networks, but cost advantage lower bounded by $9/8$ [12] with completely balanced link weights.

### B. Cost Advantage in Hyper-networks

The cost advantage analysis can be extended to hyper-networks. For a hyper-network $H$, assign each hyper-link a weight $w(e)$ indicating the cost to transmit a unit flow there, then the cost of a multicast solution is defined in the same way as in the bidirected model. We now prove the first upper-bound for the cost advantage in hyper-network.

**Theorem 8.** *In an uncapacitated hyper-network* $H = (V, \mathcal{E})$*, the cost advantage is upper bounded by the maximum hyper-link size* $\beta$.

*Proof:* Given a hypergraph $H = (V, \mathcal{E})$ and weight vector $w$, consider the undirected bipartite graph $G = (V \cup V_\mathcal{E}, E)$ constructed in a similar way as the bidirected graph in the proof of theorem 4, except that we connect nodes with undirected links instead of bidirected links. For the edges in $G$, we assign the same weight $w(e)$ to all the edges adjacent to node $v_e, \forall e \in \mathcal{E}$. Denote this weight vector as $w'$. According to the upper-bound of cost advantage in undirected networks, we have $\mathcal{C}_{tree}(G, w')/\mathcal{C}_{nc}(G, w') \leq 2$. To bound the cost advantage in $H$, we need to analyze the relationship between the minimum cost in $(H, w)$ and that in $(G, w')$.

For fractional routing solutions, the minimum cost can always be achieved by a single min cost multicast tree. Denote the min cost multicast tree in $G$ as $\tau'$. We can derive a multicast hyper-tree $\tau$ in $H$ from $\tau'$ as $\tau = \{e \in \mathcal{E} | v_e \in \tau'\}$. The cost of $\tau$ is at most half of the cost of $\tau'$, since for each node $v_e$ in $\tau'$, at least two edges adjacent to $v_e$ are counted in $\tau'$, but the corresponding hyper-edge $e$ is counted only once in $\tau$. So $\mathcal{C}_{tree}(H, w) \leq \frac{1}{2}\mathcal{C}_{tree}(G, w')$.

With network coding, let $f$ be a multicast flow of minimum cost in the hyper-network $H$. In the capacitated subgraph $H'$ induced by $f$, where each hyper-edge $e \in \mathcal{E}$ has capacity $f(e)$, the max flow from the source to each receiver is at least 1. In the bipartite graph $G$, we can obtain a capacitated subgraph $G'$ where link $\{v, v_e\} \in E$ has capacity $f(e)$. We can easily derive a flow solution $f'$ in $G'$ from $f$, and its cost is

$$\sum_{\{v,v_e\} \in E} f'(\{v, v_e\}) w'(\{v, v_e\}) = \sum_{e \in \mathcal{E}} \sum_{v \in e} f'(\{v, v_e\}) w(e)$$
$$\leq \sum_{e \in \mathcal{E}} \sum_{v \in e} f(e) w(e) \leq \sum_{e \in \mathcal{E}} \beta f(e) w(e) = \beta \mathcal{C}_{nc}(H, w)$$

where the first inequality is due to the capacity of link $\{v, v_e\}$ is $f(e)$ in $G'$. As the minimum cost of network coding solution in $G$ is no more than the cost of $f'$, we can see that $\mathcal{C}_{nc}(G, w') \leq \beta \mathcal{C}_{nc}(H, w)$.

Combining all these inequalities, we have

$$\frac{\mathcal{C}_{tree}(H, w)}{\mathcal{C}_{nc}(H, w)} \leq \frac{\frac{1}{2}\mathcal{C}_{tree}(G, w')}{\frac{1}{\beta}\mathcal{C}_{nc}(G, w')} \leq \beta.$$

■

## VII. Conclusion

We have focused on the benefits of network coding in two types of parameterized networks throughout this work, including bidirected networks and hyper-networks. Compared to simple directed and undirected network models, these networks are more powerful and flexible for characterizing real-world networks. We proved a number of new upper-bounds and lower-bounds on the potential benefits of network coding, in terms of improving multicast throughput and saving multicast cost. We also improved some existing results in the literature, by proving tighter bounds. Our theoretical analysis along the way has led to interesting observations, including that in-network information processing (network coding and IP-multicast) is unnecessary for achieving optimal multicast in balanced network topologies.

Similar to the case of undirected networks, the new upper-bounds and lower-bounds proved in this work are not exactly tight. Closing such gaps is left open. As future research, it is also interesting to study possible efficient multicast algorithm design for $\alpha$-balanced networks.

## Acknowledgment

This work was supported in part by the National Science Foundation of China under Grant 61171074, the National S&T Major Project of China under Grant 2010ZX03003-003-03 and SRFDP under Grant 200802461134. Xin Wang is the corresponding author.

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