

SYNTHETIC ∞ -CATEGORY THEORY

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AIM: GIVE A PRECISE AND MODEL
INDEPENDENT ACCOUNT OF
 ∞ -CATEGORY THEORY.

RESOURCES:

" ∞ -CATEGORY THEORY FROM SCRATCH" (2016)

<https://arxiv.org/abs/1608.05314>

"ELEMENTS OF ∞ -CATEGORY THEORY" (2019)

<http://www.math.jhu.edu/~eriehl/elements.pdf>

$(\infty, 1)$ - CATEGORIES

SCHEMATIC

"DEFINITION"

AN $(\infty, 1)$ -CATEGORY IS
A CATEGORY WEAKLY
ENRICHED IN SPACES

∞ -GROUPOIDS

MODELS

QUASI-CATEGORIES
COMPLETE SEGAL SPACES
SEGAL CATEGORIES
MARKED QUASI-CATEGORIES

ALSO FIBRED
VARIANTS OF
THESE NOTIONS

WHAT IS A QUILLEN MODEL CATEGORY?

OR AT LEAST
HOW DO THEY RELATE
TO ∞ -CATEGORIES?

∞ - CATEGORIES

AXILIARY AIM: BUILD A FRAMEWORK THAT CAN BE EXTENDED TO GIVE FOUNDATION'S FOR THE CATEGORY THEORY OF (∞, n) - CATEGORIES

MODELS:

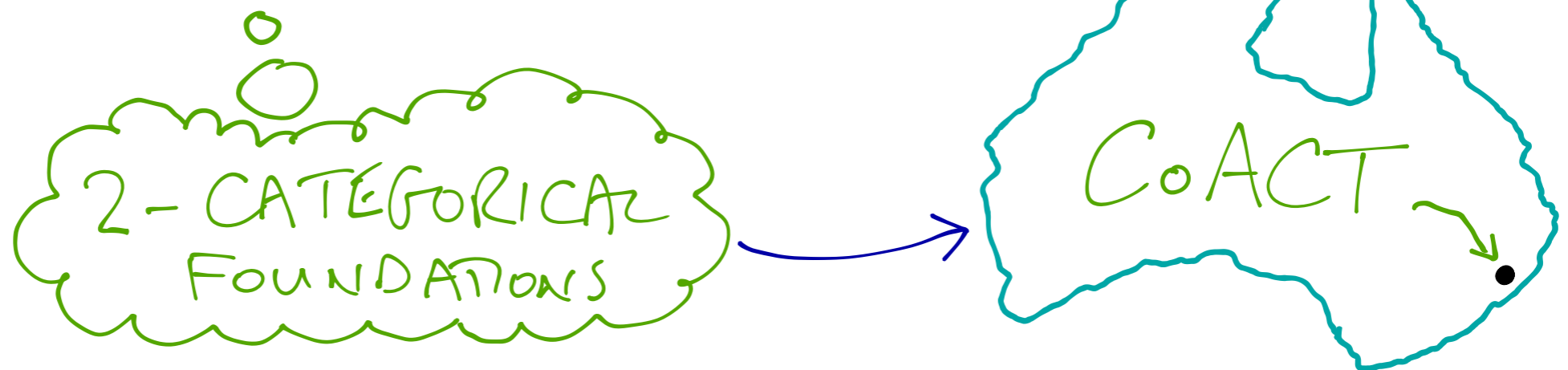
COMPICIAL SETS
ITERATED SEGAL
 \mathbb{H}_n -SETS AND SPACES

CATEGORIES WEAKLY ENRICHED IN $(\infty, n-1)$ -CATEGORIES

WE WILL USE THE TERM ∞ -CATEGORY TO MEAN ANY STRUCTURE THAT INHABITS AN INSTANCE OF THE FRAMEWORK WE BUILD HERE.

BENEFITS

-) GIVE **SYNTHETIC** PROOFS OF ∞ -CATEGORICAL RESULTS, WHICH THEN APPLY IN ALL MODELS.
-) TRANSFER CATEGORICAL RESULTS DERIVED **ANALYTICALLY** IN ONE MODEL TO RELATED MODELS.
-) ADAPT AND APPLY INTUITIONS FROM "META-CATEGORY" THEORY



PLAN

LECTURE 1: AN AXIOMATIC FRAMEWORK FOR ∞ -CATEGORY THEORY.

LECTURE 2: LIMITS AND ADJUNCTIONS.

LECTURE 3: HOMOTOPY COHERENT MONADS AND BECK MONADICITY

LECTURE 4: ?

DIVING IN: 2-CATEGORIES

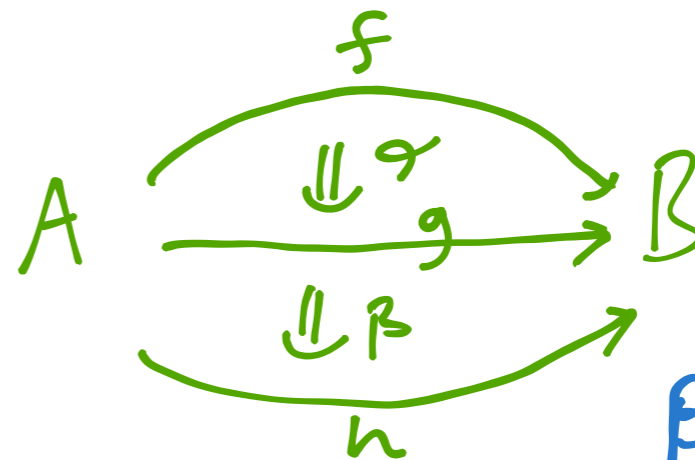
MODELED ON THE
TOTALITY OF CATEGORIES,
FUNCTORS + NATURAL
TRANSFORMATIONS

OF THE STRICT
VARIETY

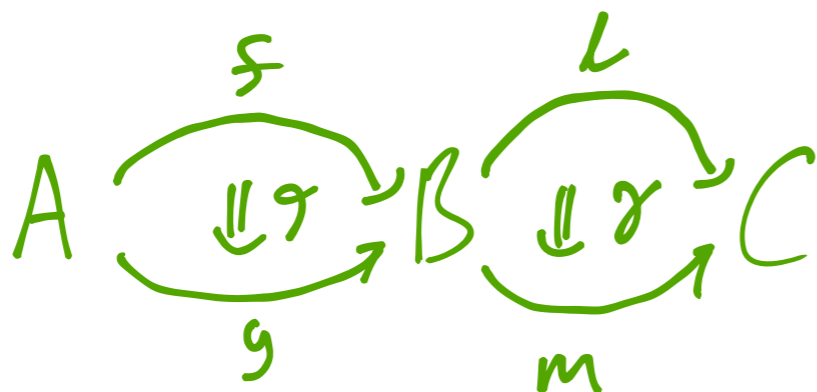
AKA
CATEGORIES
ENRICHED
IN $(\text{Cat}, \times, \mathbb{1})$

- OBJECTS A, B, C
- 1-CELLS $A \xrightarrow{f} B$
- 2-CELLS $A \begin{matrix} \xrightarrow{f} \\ \Downarrow \eta \\ \xrightarrow{g} \end{matrix} B$

2-CELLS COMPOSE
VERTICALLY



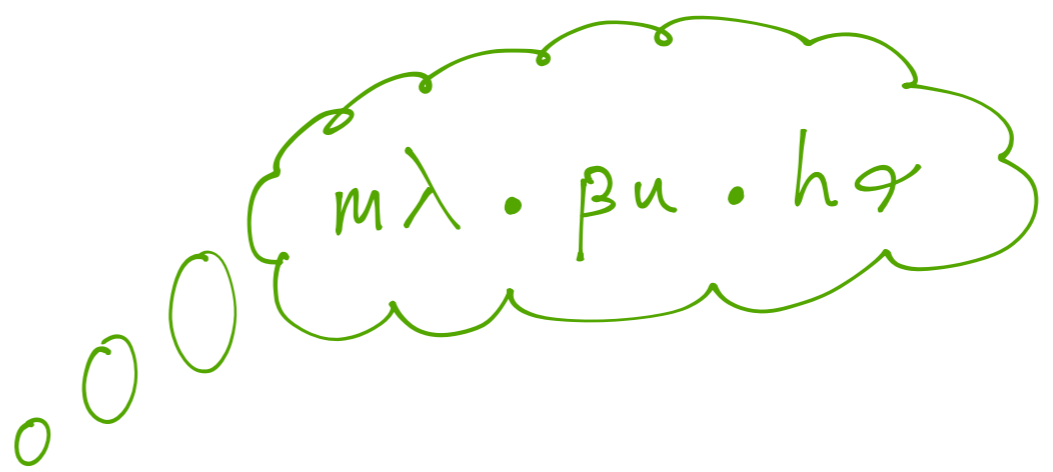
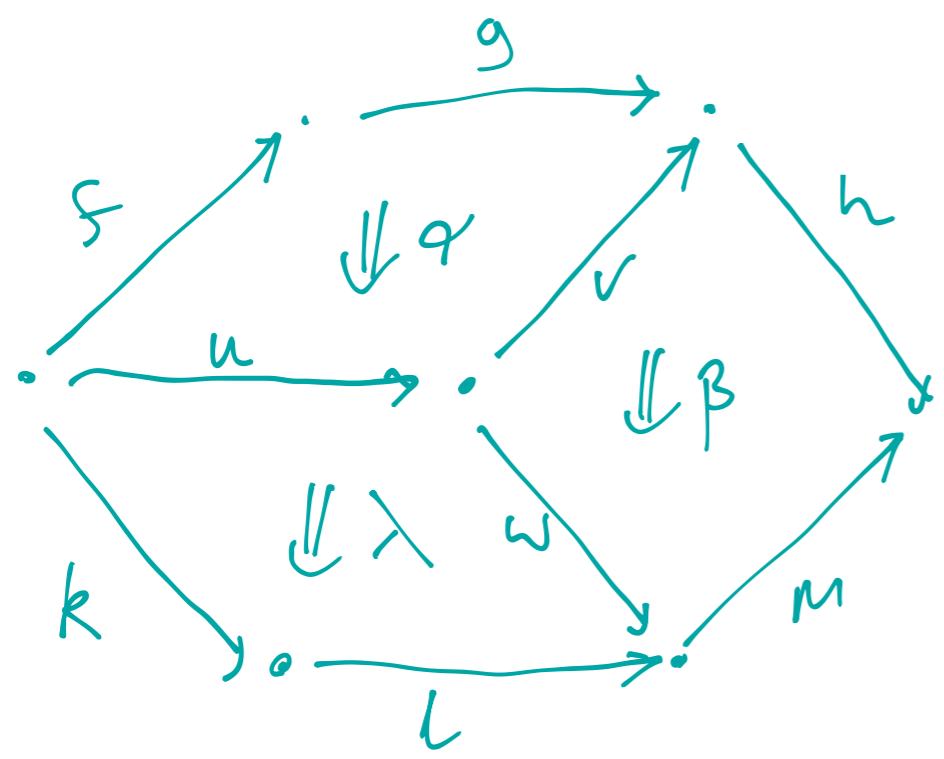
HORIZONTAL



$\gamma \circ \eta$

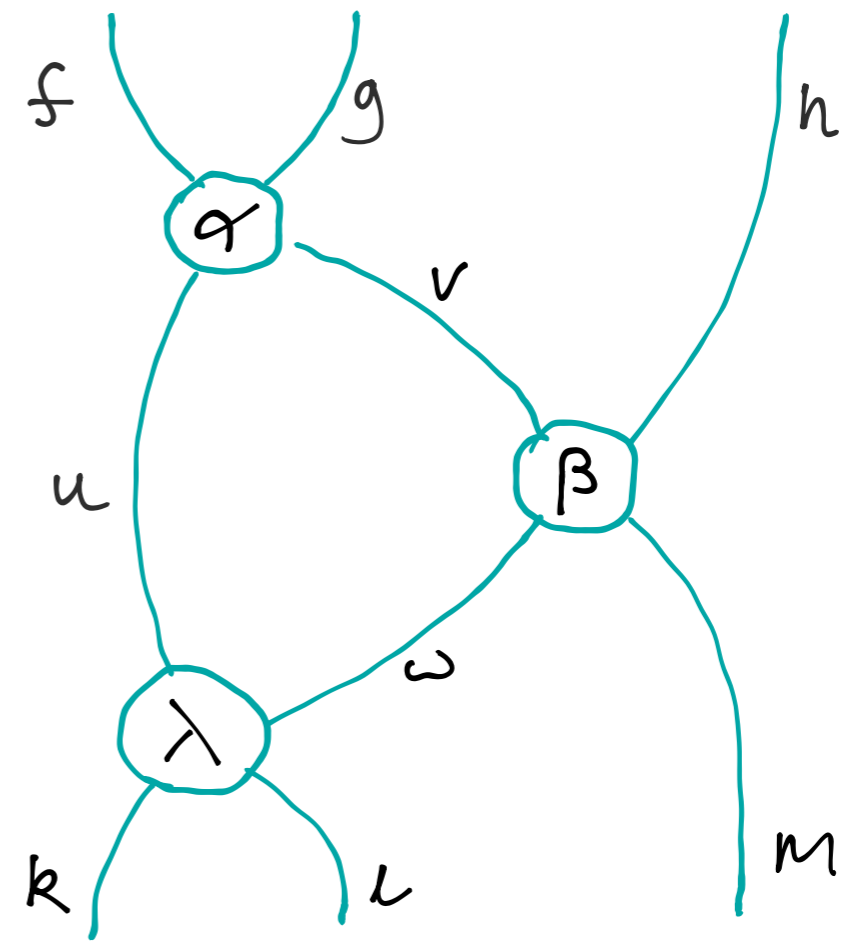
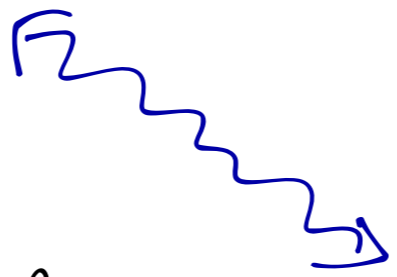
BOTH STRICTLY
ASSOCIATIVE

PASTING



STRING DIAGRAMS

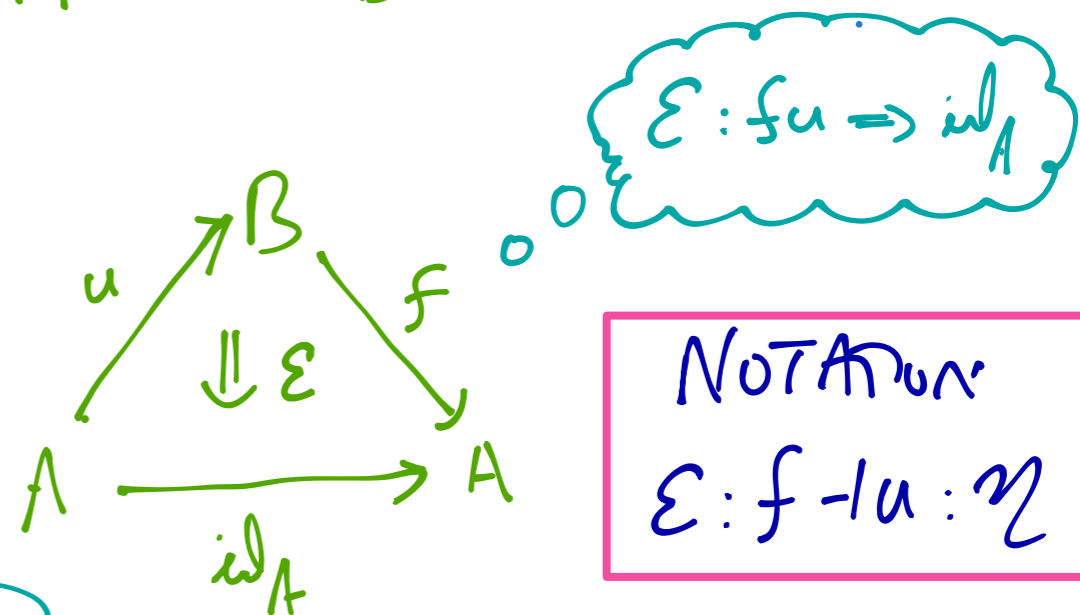
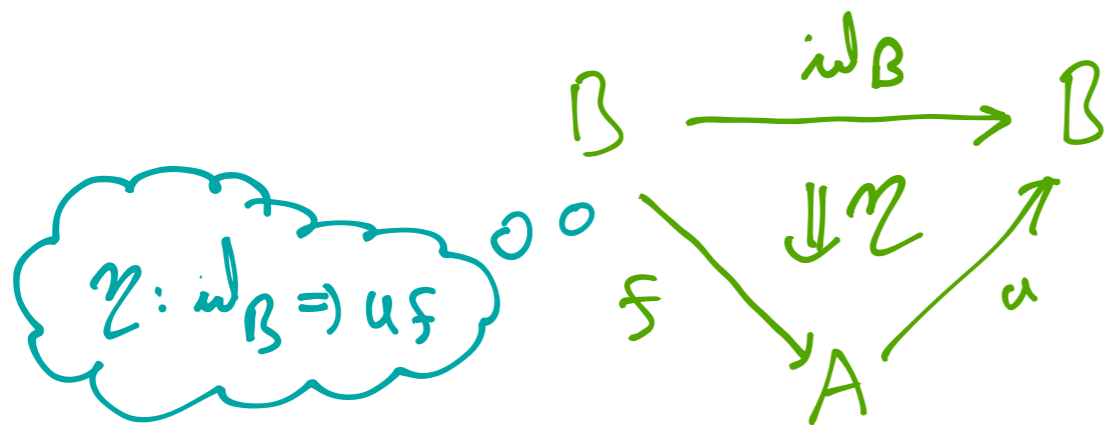
PLANAR
DUAL



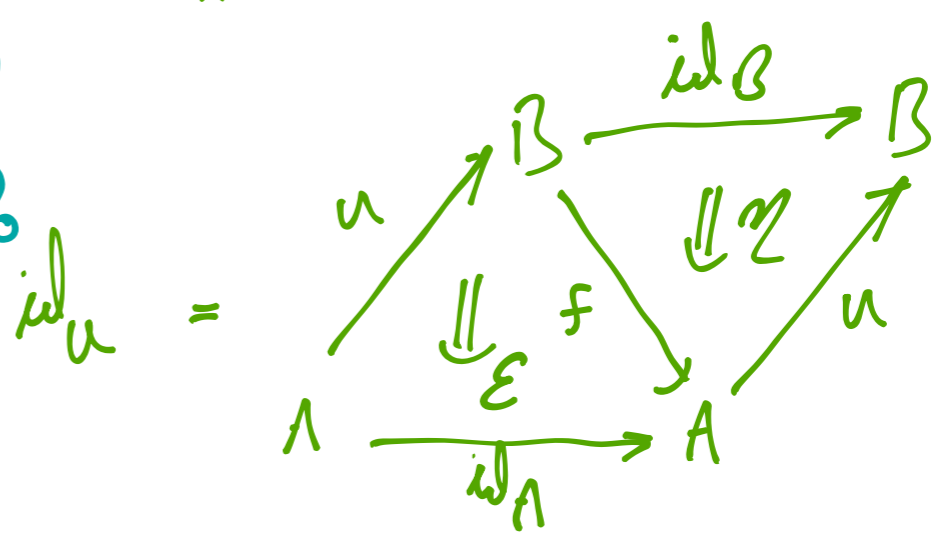
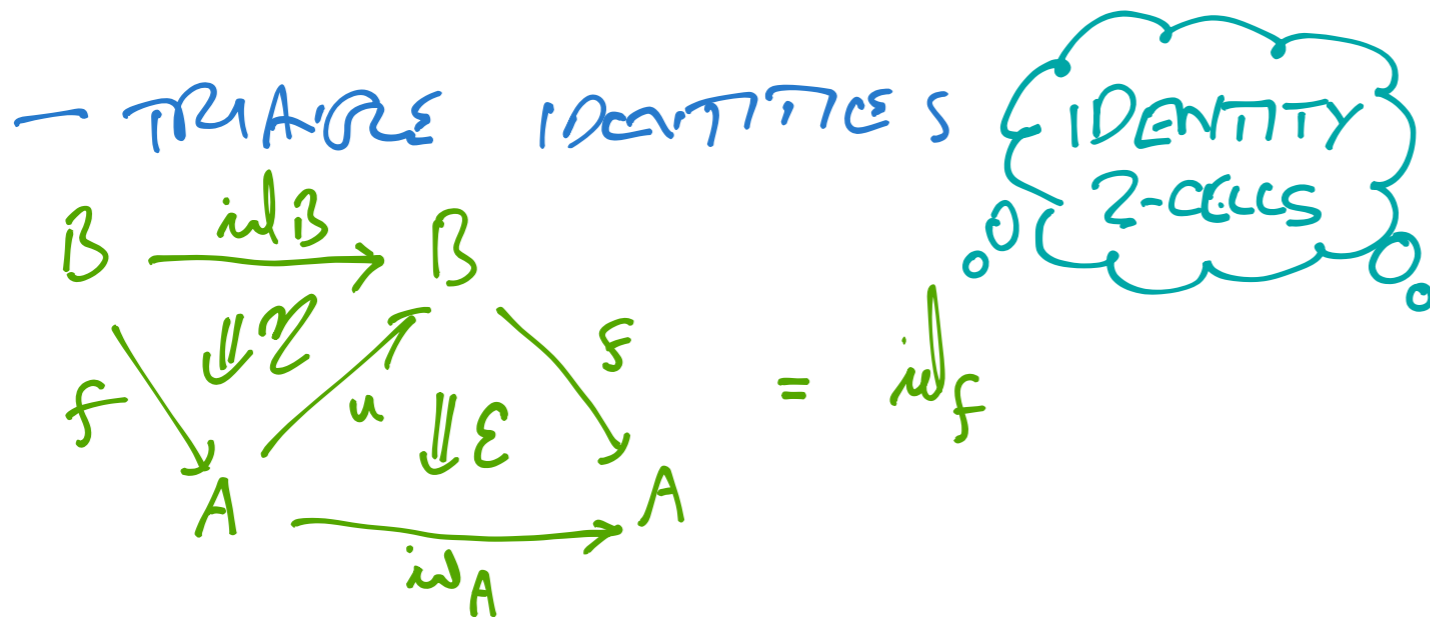
ADJUNCTIONS IN 2-CATEGORIES

DEFN AN ADJUNCTION IN A 2-CATEGORY COMPRISES

- A PAIR OF OBJECTS A AND B
- A PAIR OF 1-CELLS $A \xrightarrow{u} B$ AND $B \xrightarrow{f} A$
- A PAIR OF 2-CELLS:

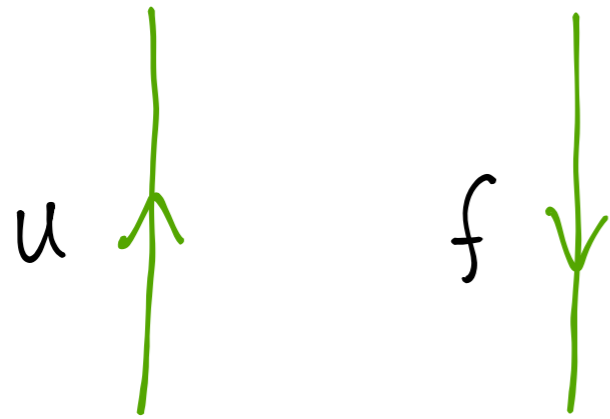


NOTATION
 $\epsilon: f \dashv u: \eta$

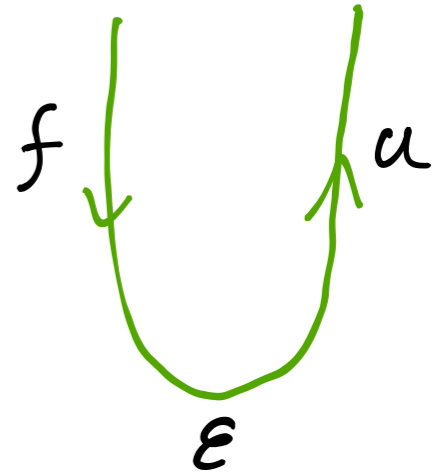
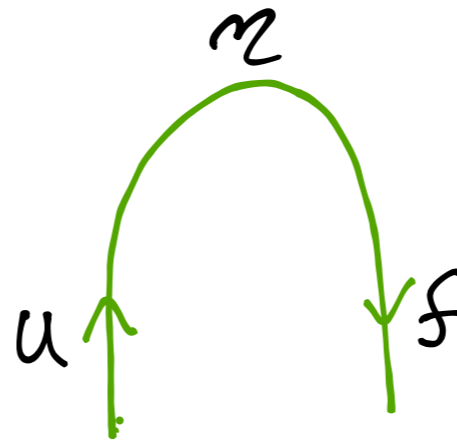


REPRISE WITH STRINGS

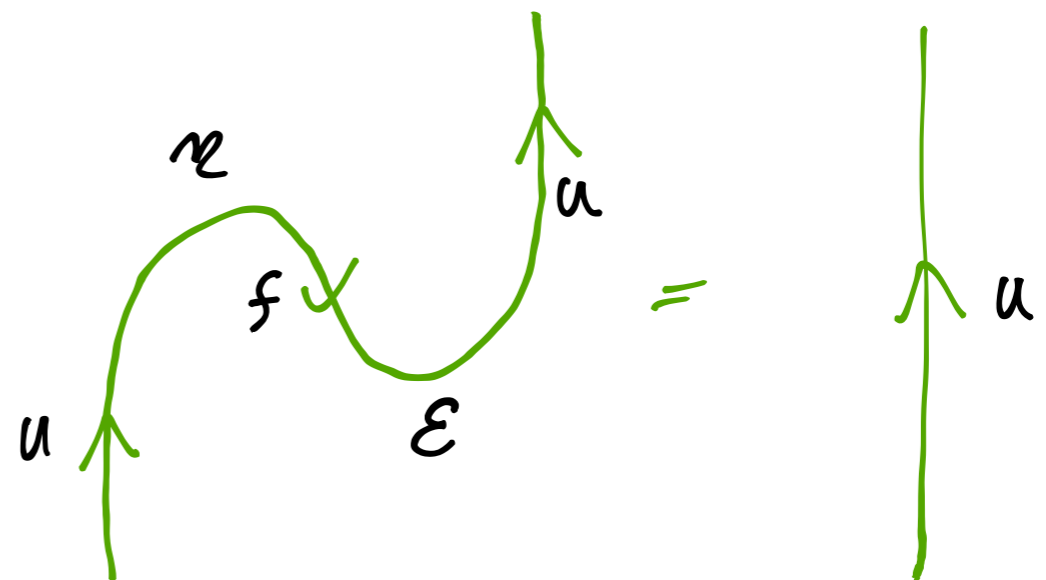
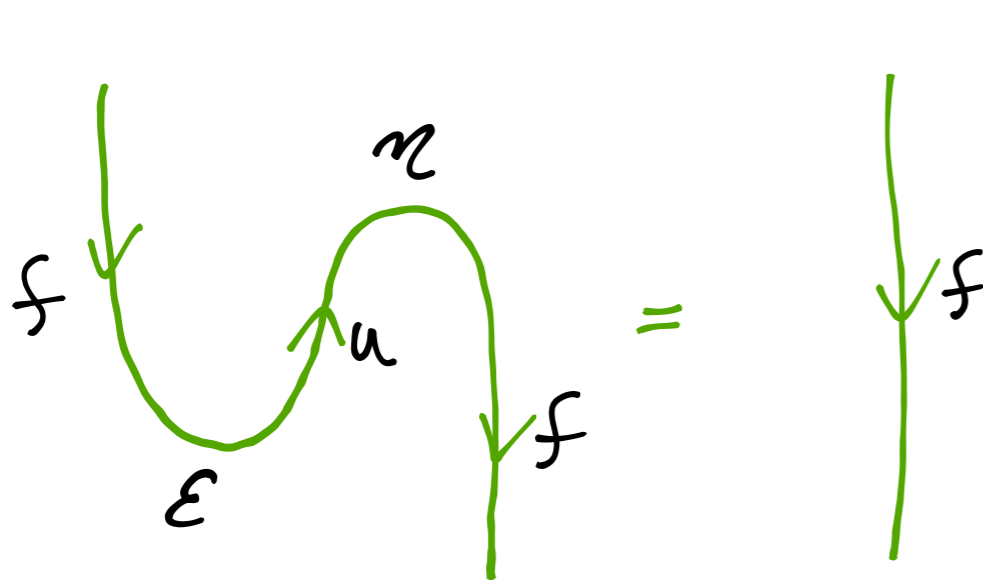
1-CELLS



2-CELLS



TRIANGLE IDENTITIES



SOME CONSEQUENCES

- COMPOSITION OF ADJUNCTIONS
- ADJOINTS ARE UNIQUE UPTO, AND STABLE UNDER, ISOMORPHISM.
- EQUIVALENCES MAY BE PROMOTED TO ADJOINT EQUIVALENCES.
- ADJUNCTIONS ARE STABLE UNDER EQUIVALENCES.

SLOGAN

CATEGORIES LIVE

INSIDE 2-CATEGORIES^{*}

- ⊛ IT IS MORE ACCURATE TO SAY THAT 2-CATEGORIES PROVIDES A NATURAL META-FRAMEWORK IN WHICH TO DEVELOP CATEGORY THEORY.

∞ -SLOGAN

∞ -CATEGORIES LIVE

INSIDE $(\infty, 2)$ -CATEGORIES^{*}

- * IT IS MORE ACCURATE TO SAY THAT $(\infty, 2)$ -CATEGORIES PROVIDES A NATURAL META-FRAMEWORK IN WHICH TO DEVELOP ∞ -CATEGORY THEORY.

$(\infty, 2)$ -CATEGORIES REALLY?!!

THE TRICK HERE IS TO:

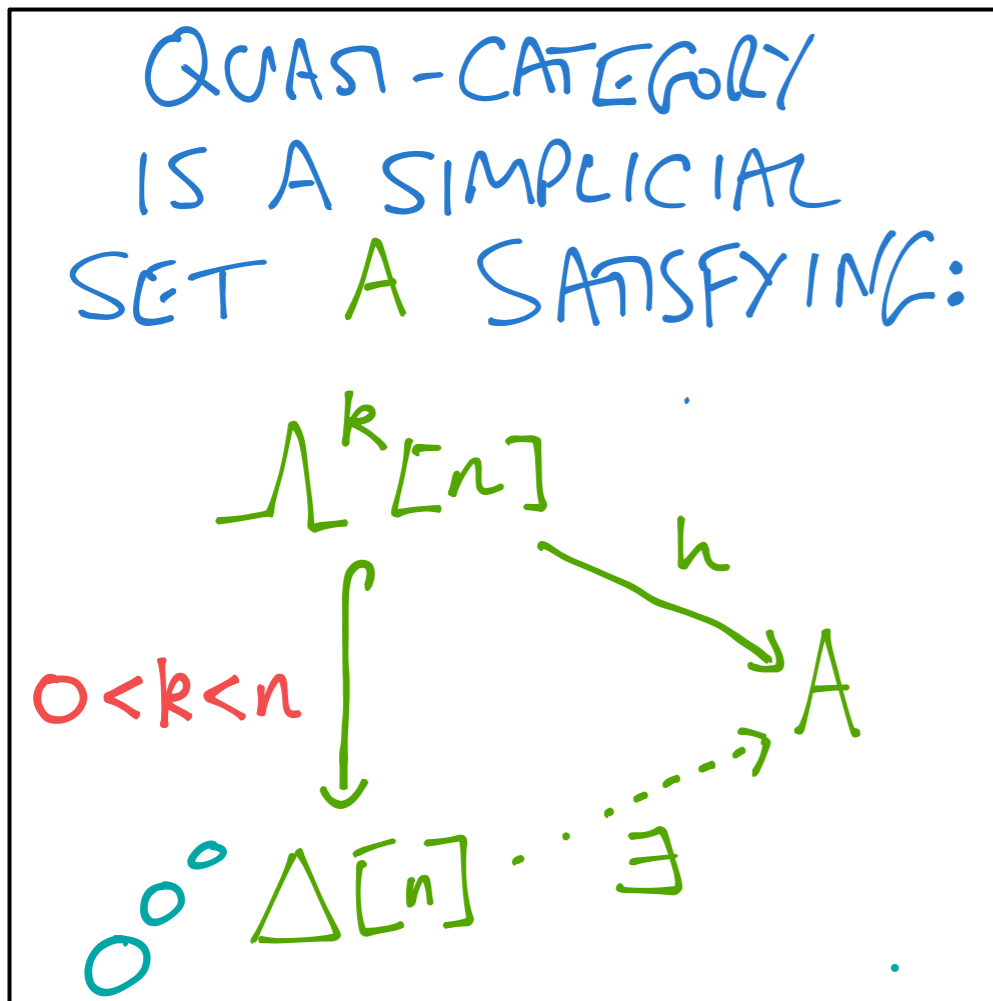
- PICK A MODEL OF $(\infty, 2)$ -CATEGORIES THAT IS FAMILIAR.
- SIMPLIFY MANY ARGUMENTS BY WORKING, WHERE POSSIBLE, IN THE HOMOTOPY 2-CATEGORY ASSOCIATED WITH OUR AMBIENT $(\infty, 2)$ -CATEGORY.

COMMITTING TO AN $(\infty, 2)$ -CATEGORICAL MODEL

AN ∞ -COSMOS IS, IN ESSENCE,
A CATEGORY OF FIBRANT
OBJECTS ENRICHED IN THE
JOYAL MODEL STRUCTURE

QUASI-CATEGORIES

RECALL:

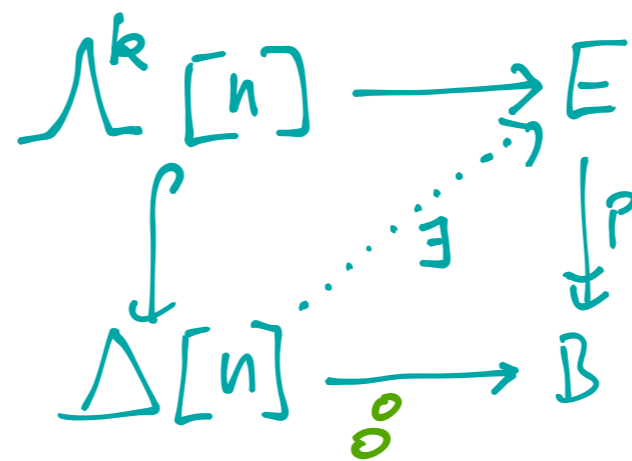


INNER HORN

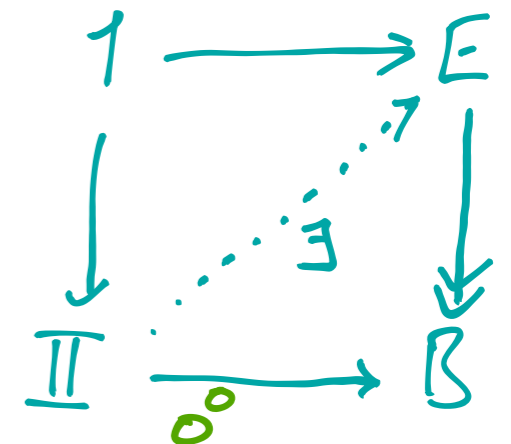
JOYAL MODEL STRUCTURE

ON SSet HAS

-) FIBRANT OBJECTS THE QUASI-CATEGORIES
-) FIBRATIONS, CALLED ISOFIBRATIONS



LIFTING OF COMPOSITES



LIFTING OF ISOMORPHISMS
 $II := \{ \bullet \cong \bullet \}$

-) COFIBRATIONS \equiv MONOS

∞-COSMOS IN ONE PAGE...

A SIMPLICIAL
CATEGORY K

- EQUIPPED WITH
CLASSES OF
- ISOFIBRATIONS
 - WEAK EQUIVALENCES

PRODUCTS AND
PULLBACKS OF
ISOFIBRATIONS

COFIBRANT
REPLACEMENT
 $A_c \xrightarrow{\sim} A$

ALL OBJECTS
FIBRANT $!A \rightarrow 1$

FIBRATION NOTIONS
STABLE UNDER
PULLBACK

ALL LIMITS
SIMPLICIALLY
ENRICHED

COTENSORS SATISFYING "SM7"
WRT JOYAL MODEL STRUCTURE

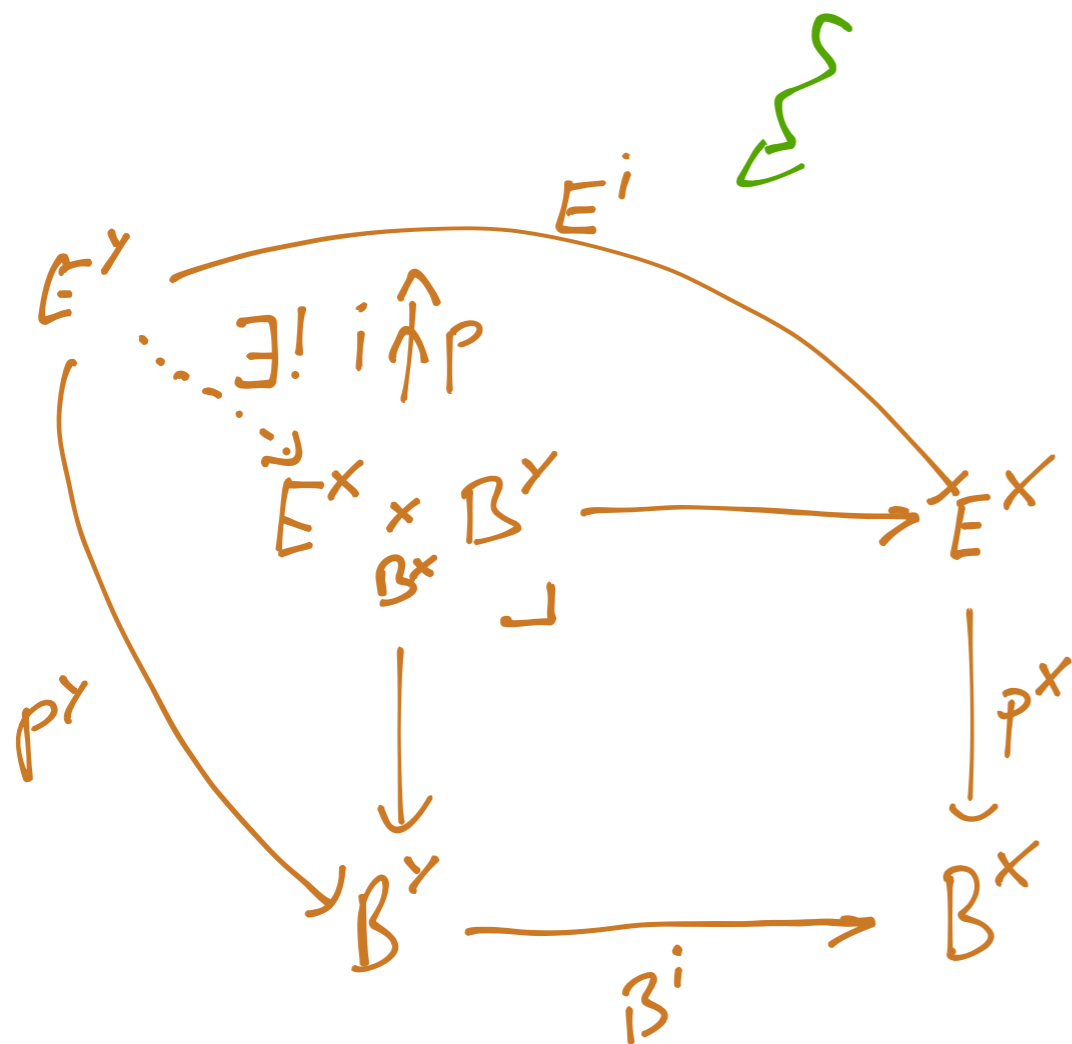
FUNCTOR SPACE

AXIOM "JM7"

$$\text{Fun}_{\mathcal{K}}(A, B)^X \cong \text{Fun}_{\mathcal{K}}(A, B^X)$$

COTENSOR

INCLUSION $i: X \hookrightarrow Y \in \underline{\text{Set}}$
 ISOFIBRATION $p: E \rightarrow B \in \mathcal{K}$



"JM7"

$i\hat{\wedge} p$ IS AN ISOFIBRATION AND IT IS A TRIVIAL FIBRATION IF i IS A WEAK EQUIVALENCE IN THE JOYAL MODEL STRUCTURE OR p IS A TRIVIAL FIBRATION IN \mathcal{K} .

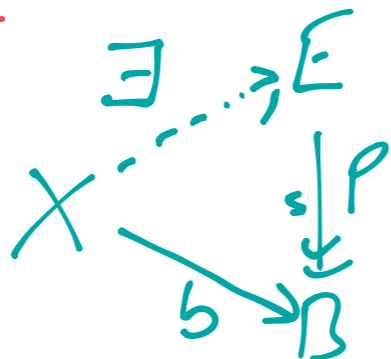
∞ -COSMOS EXAMPLES

$(\infty, 1)$ -MODELS

- QUASI-CATEGORIES
- SEGAL CATEGORIES
- COMPLETE SEGAL SPACES
- MARKED QUASI-CATS

MOST COMMON
EXAMPLES HAVE
ALL OBJECTS

COFIBRANT



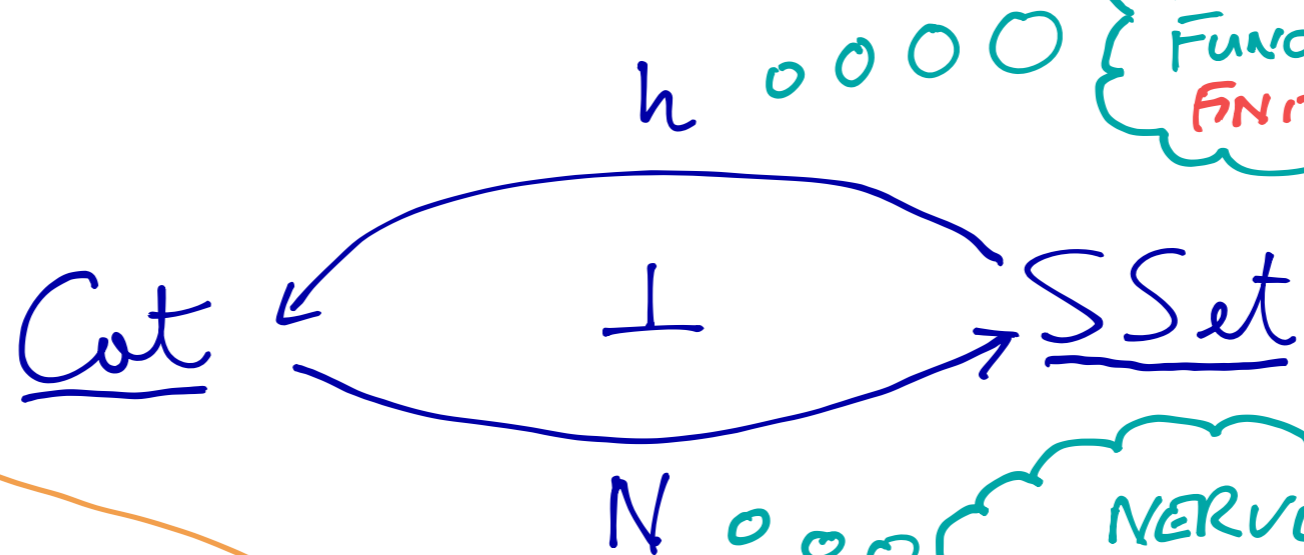
(∞, n) -MODELS

- SEGAL (∞, n) -SPACES
- \mathbb{H}_n -SETS + SPACES
- n -COMPLICIAL SETS

DERIVED EXAMPLES

- SLICES
- ∞ -CATEGORIES FIBRED OVER A BASE.
- (CO)COMPLETE ∞ -CATS

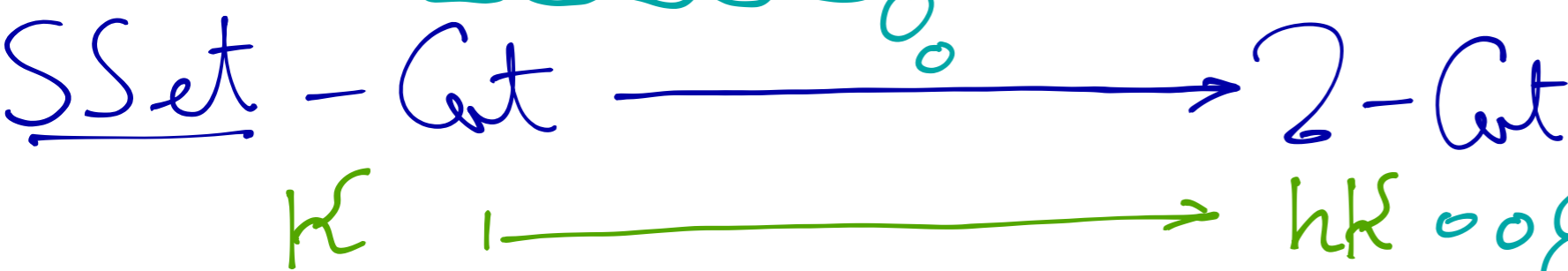
HOMOTOPY 2-CATEGORIES



HOMOTOPY CATEGORY
FUNCTOR - PRESERVES
FINITE PRODUCTS

NERVE
"ALL CATEGORIES
ARE SIMPLICIAL SETS"

APPLY h TO
HOM-SPACES



THE HOMOTOPY
2-CATEGORY OF K

HOMOTOPY 2-CATEGORIES OF

∞ -COSMOS

WHEN A IS COFIBRANT
IN K THEN ANY HOMSPACE
 $\text{Fun}_K(A, B)$ IS A QUASI-CAT



WE SHALL RESTRICT
THE HTY 2-CAT hK
OF AN ∞ -COSMOS
TO COFIBRANT OBJECTS

EXPLICIT DESCRIPTION
OF hK

- 0-CELLS THE COFIBRANT OBJECTS IN K
- 1-CELLS ARROWS BETWEEN COF OBJECTS IN K
- 2-CELLS HOMOTOPY CLASSES OF 1-SIMPLICES, THAT IS OF $(\infty-)$ NATURAL TRANSFORMATIONS, IN $\text{Fun}_K(A, B)$.

∞ -CATEGORIES

$(\infty-)$ FUNCTORS

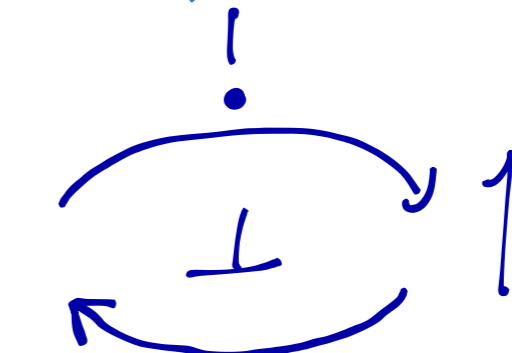
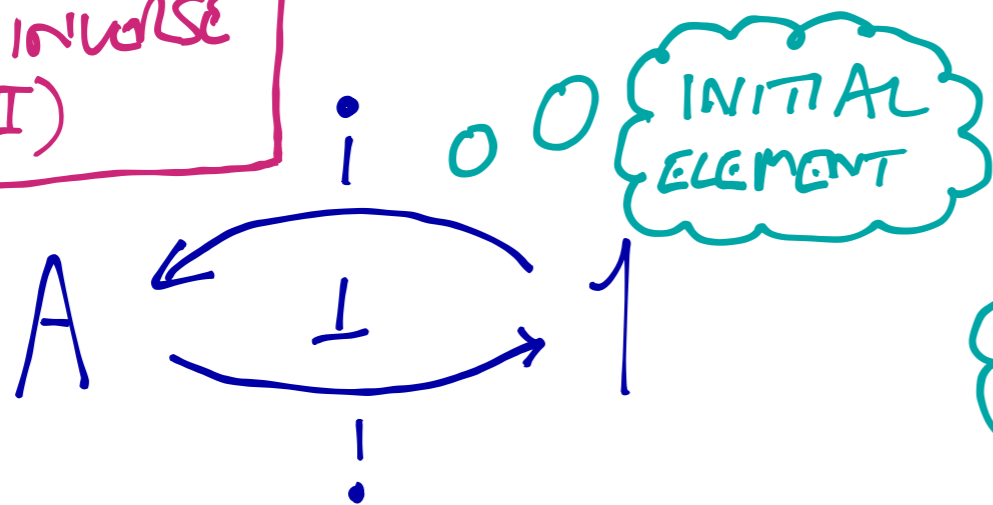
A WILD CONJECTURE!

2-CATEGORICAL ADJUNCTIONS
IN THE HOMOTOPY 2-CATEGORY
 hK OF AN ∞ -COSMOS K ARE
THE CORRECT NOTION WITH RESPECT
TO ∞ -CATEGORY THEORY.

FIRST STEPS

... TOWARDS ∞ -CATEGORY THEORY.

LEFT ADJOINT
RIGHT INVERSE
(LARI)



RIGHT ADJOINT
RIGHT INVERSE
(RARI)

MINIMAL DATA PRESENTING A TERMINAL ELEMENT

(i) AN ELEMENT $t : 1 \rightarrow A$

(ii) A 2-CELL $A \xrightarrow{id_A} A$

NATURAL FAMILY
OF MAPS FROM THE
ELEMENTS OF A TO t

(iii) $\epsilon t = id_t$

WHOSE COMPONENT
AT t IS THE IDENTITY

EXERCISE

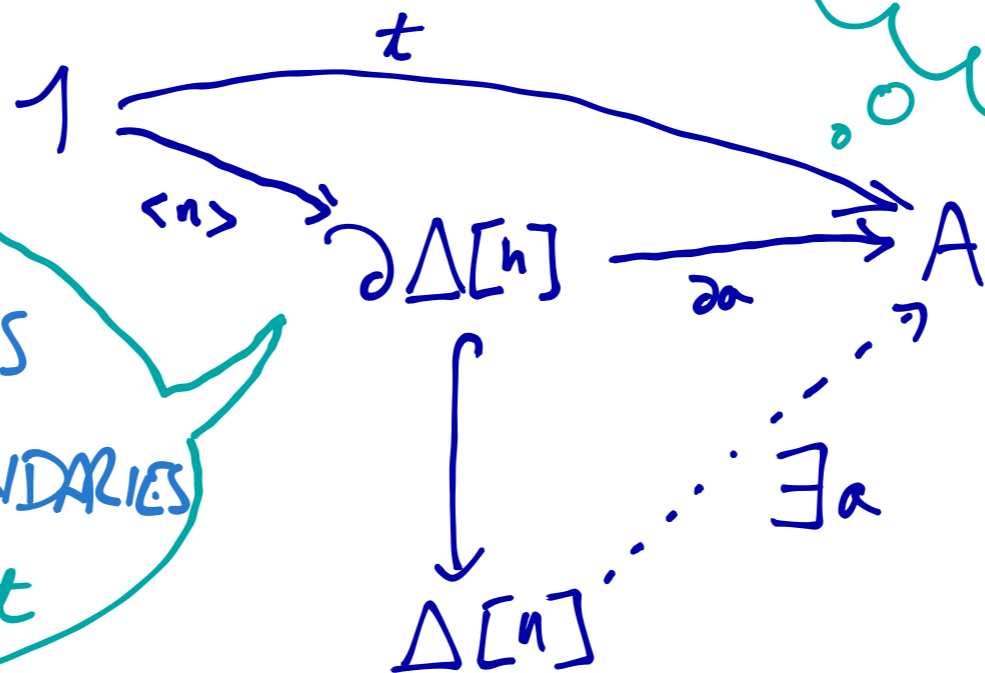
SUPPOSE THAT A IS A QUASI-CATEGORY;
AN OBJECT IN THE ∞ -COSMOS QC_{Cat} .
LET t BE A VERTEX OF A , THEN

$t : 1 \rightarrow A$ IS A TERMINAL ELEMENT



THIS IS JOYAL'S
TERMINAL OBJECT
NOTION

A ADMITS FILLERS
FOR SIMPLEX BOUNDARIES
WITH LAST VERTEX t



IMMEDIATE CONSEQUENCES

- TERMINAL ELEMENTS PRESERVED BY RIGHT ADJOINTS.

- $\tau: 1 \rightarrow A$ IS TERMINAL IFF FOR ALL ∞ -CATS B AND FUNCTORS $f: B \rightarrow A$ WE HAVE

$$\begin{array}{ccc}
 B & \xrightarrow{f} & A \\
 \searrow & \Downarrow \exists! & \nearrow \\
 ! & \rightarrow 1 & \xrightarrow{\tau}
 \end{array}$$

- $\tau: 1 \rightarrow A$ IS TERMINAL IFF FOR ALL ∞ -CATS B THE GENERALISED ELEMENT $B \xrightarrow{!} 1 \xrightarrow{\tau} A$ IS TERMINAL IN THE QUASI-CAT $\text{Fun}_K(X, A)$.

LIMITS AND COLIMITS

$$\begin{array}{ccc} \underline{\text{SSet}}^{\text{op}} \times \mathcal{K} & \longrightarrow & \mathcal{K} \\ \times & & A \longmapsto A^x \end{array}$$

$$\begin{array}{ccc} \mathcal{K}^{\text{op}} \times \mathcal{K} & \longrightarrow & \mathcal{K} \\ \cup & & A \longmapsto A^u \end{array}$$

COTENORS

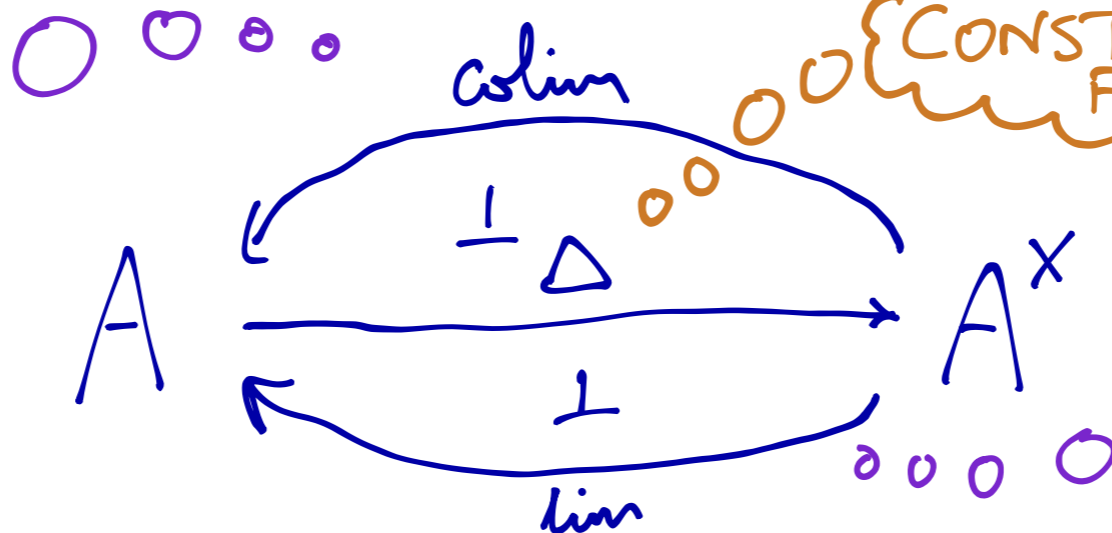
(CO)LIMITS OF SIMPLICIAL SET SHAPED DIAGRAMS

CLOSURE

(WHEN \mathcal{K} CLOSING IS CLOSED)

LIMITS OF DIAGRAMS WHOSE SHAPES ARE ∞ -CATEGORIES IN \mathcal{K}

COLIMITS OF DIAGRAMS OF SHAPE X



CONSTANT DIAGRAM FUNCTOR

LIMITS OF DIAGRAMS OF SHAPE X

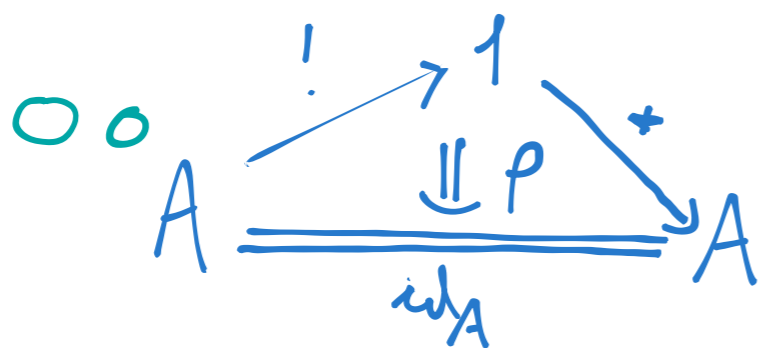
FAMILIES OF DIAGRAMS

WE MAY WISH TO ASK THAT AN ∞ -CATEGORY HAS SOME, BUT NOT ALL, LIMITS OF A GIVEN SHAPE.

EXAMPLE

- POINTED ∞ -CATEGORY HAS AN ELEMENT $*$: $1 \rightarrow A$ WHICH IS BOTH INITIAL + TERMINAL

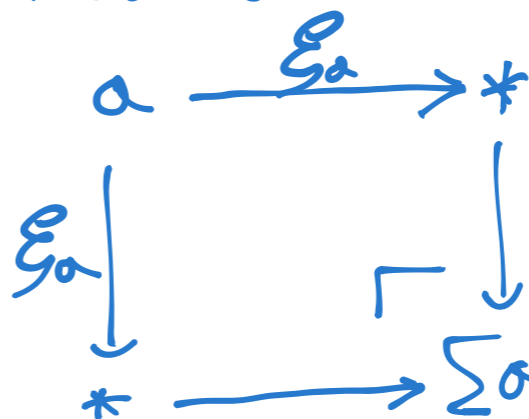
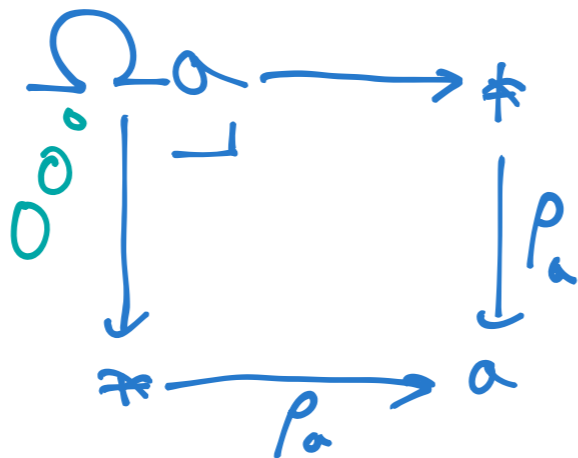
POINTS



COPOINTS

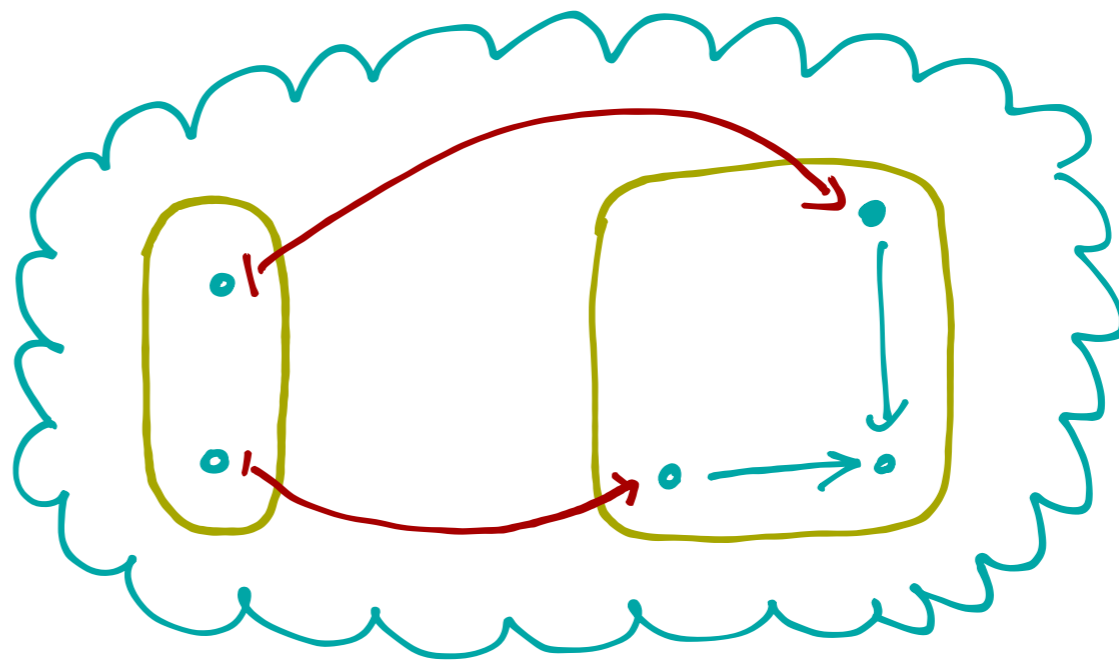
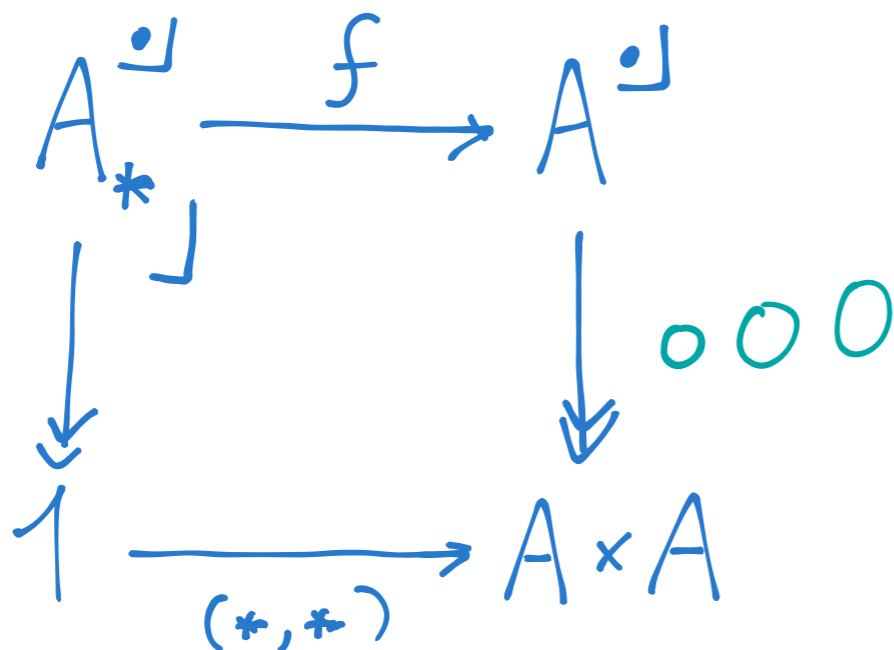
- IT IS COMMON ONLY TO ASK FOR CERTAIN PULLBACKS AND PUSHOUTS

LOOP SPACE



SUSPENSION

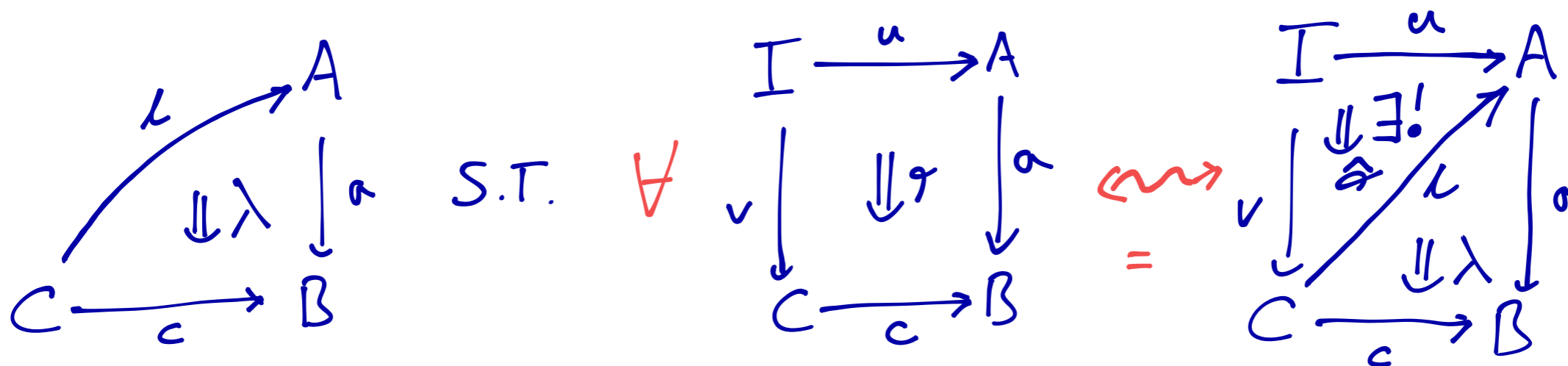
LOOP SPACES



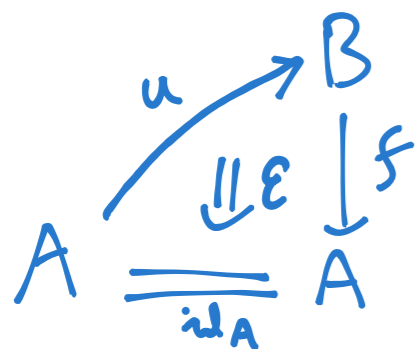
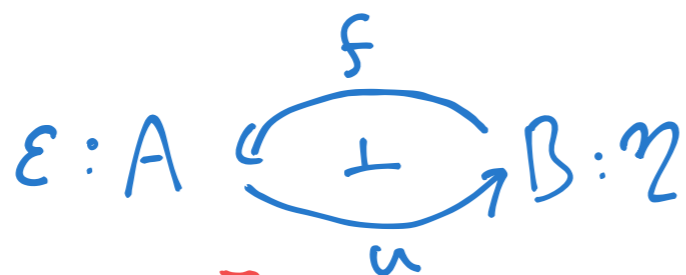
- LOOP SPACES ARE PULLBACKS, SO THEIR UNIVERSAL PROPERTY LIVES IN THE DIAGRAM ∞ -CATEGORY $\mathcal{A}^{\square} \dots$
- \dots BUT WE ONLY WANT TO ASK FOR LIMITS OF DIAGRAMS IN \mathcal{A}^{\square}
- $\mathcal{A}^{\square} \xrightarrow{f} \mathcal{A}^{\square}$ SPECIFIES A FAMILY OF DIAGRAMS IN \mathcal{A}^{\square} WHOSE LIMITS ARE LOOP SPACES.

LIMITS OF FAMILIES

ABSOLUTE RIGHT LIFTING (IN A 2-CATEGORY)



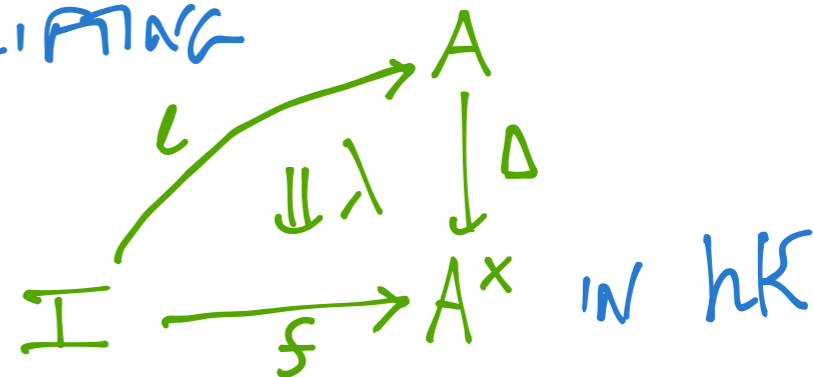
EXAMPLE



IS AN ABSOLUTE
RIGHT LIFTING

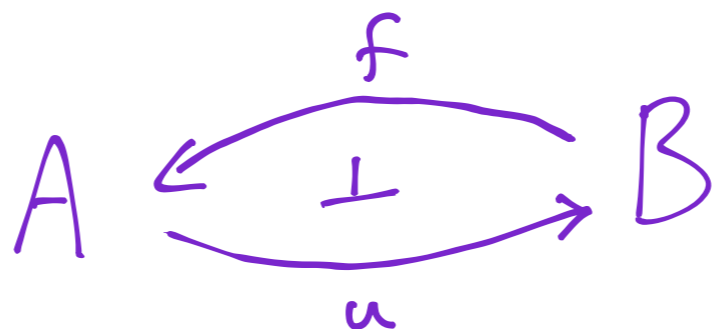
DEFN A FAMILY OF DIAGRAMS

$f: I \rightarrow A^X$ ADMITS A LIMIT
IF THERE IS AN ABSOLUTE
RIGHT LIFTING

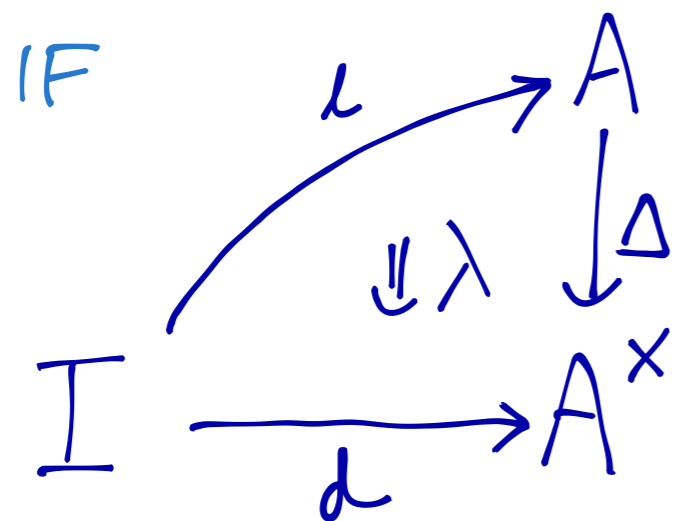


PRESERVATION

RIGHT ADJOINTS PRESERVE LIMITS OF FAMILIES

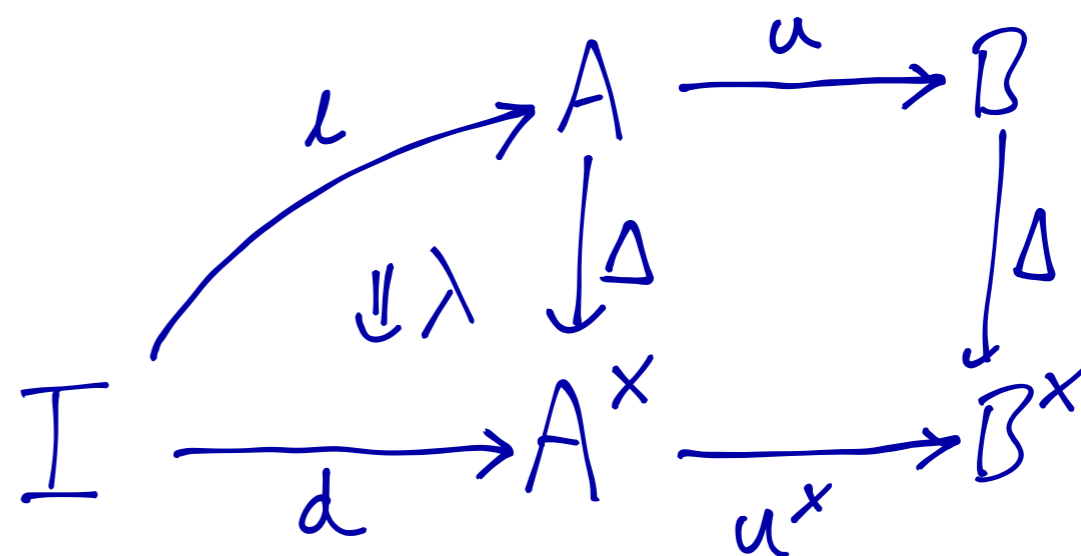


GIVEN AN ADJUNCTION



IS AN ABSOLUTE
RIGHT LIFTING

\Rightarrow



IS ALSO AN ABSOLUTE
RIGHT LIFTING.

HOM-WISE CHARACTER

OF ADJUNCTIONS

DEFN $\text{Hom}_A(f, g) \rightarrow C \times B$ IS

-) LEFT REPRESENTABLE IF THERE EXISTS A FUNCTOR $l: B \rightarrow C$ AND AN EQUIVALENCE

$$\text{Hom}_C(l, C) \simeq \text{Hom}_A(f, g) \text{ IN } \mathcal{K}/_{C \times B}$$

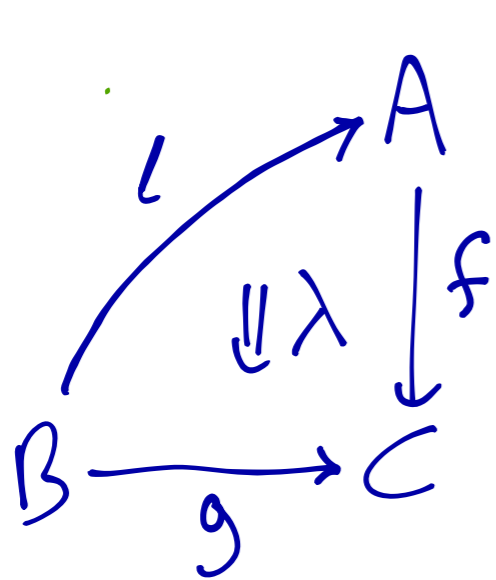
AN EQUIVALENCE
FIBRED OVER $C \times B$

-) RIGHT REPRESENTABLE IF THERE EXISTS A

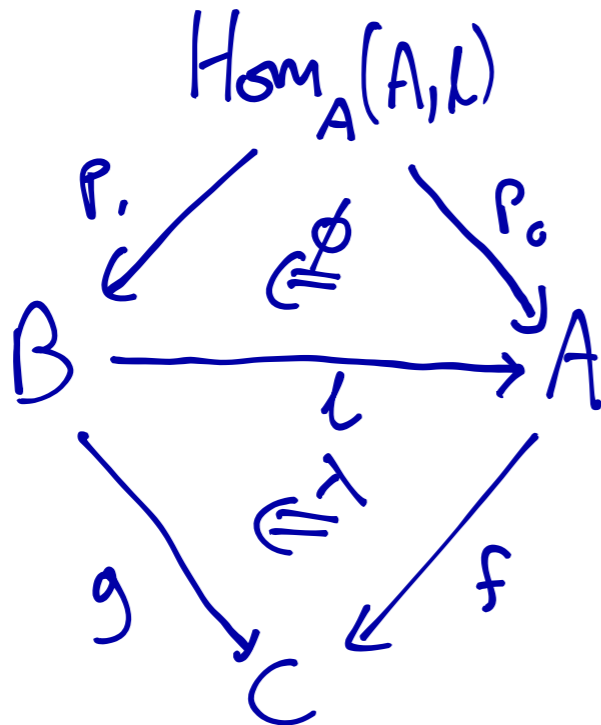
FUNCTOR $r: C \rightarrow B$ AND AN EQUIVALENCE

$$\text{Hom}_B(B, r) \simeq \text{Hom}_A(f, g) \text{ IN } \mathcal{K}/_{C \times B}$$

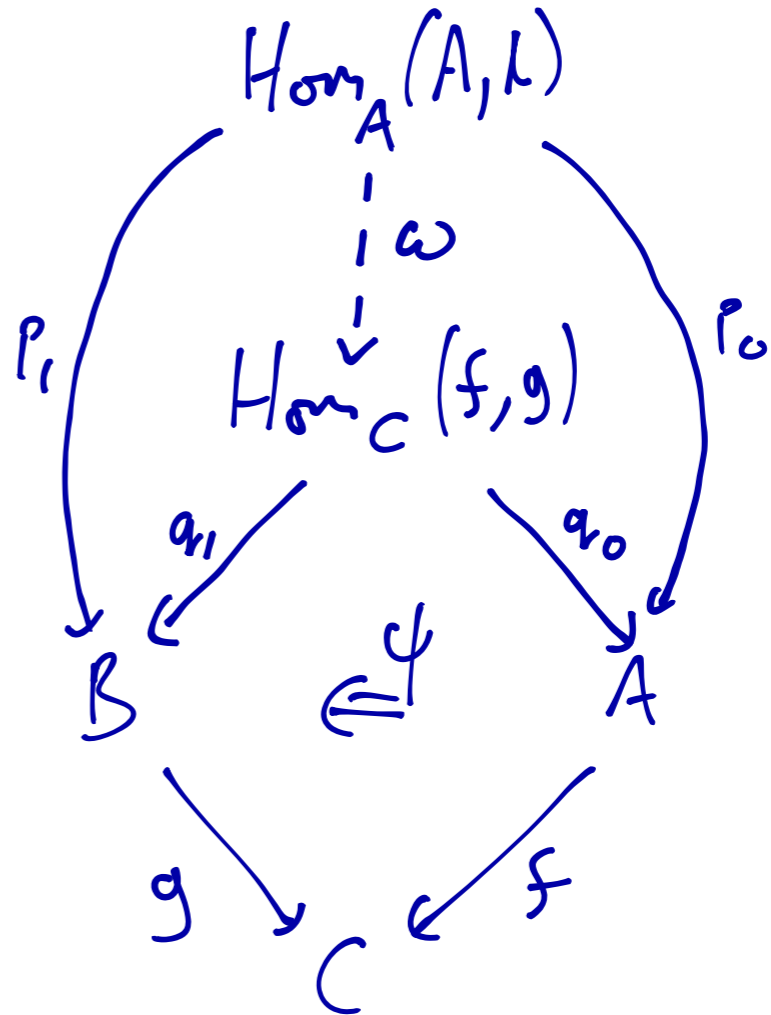
LIFTINGS MOM-WISE



LEMMA THE TRIANGLE ON THE LEFT IS AN ABSOLUTE RIGHT LIFTING IFF THE INDUCED FUNCTOR $\text{Hom}_A(A, L) \xrightarrow{\omega} \text{Hom}_C(f, g)$ IS AN EQUIVALENCE

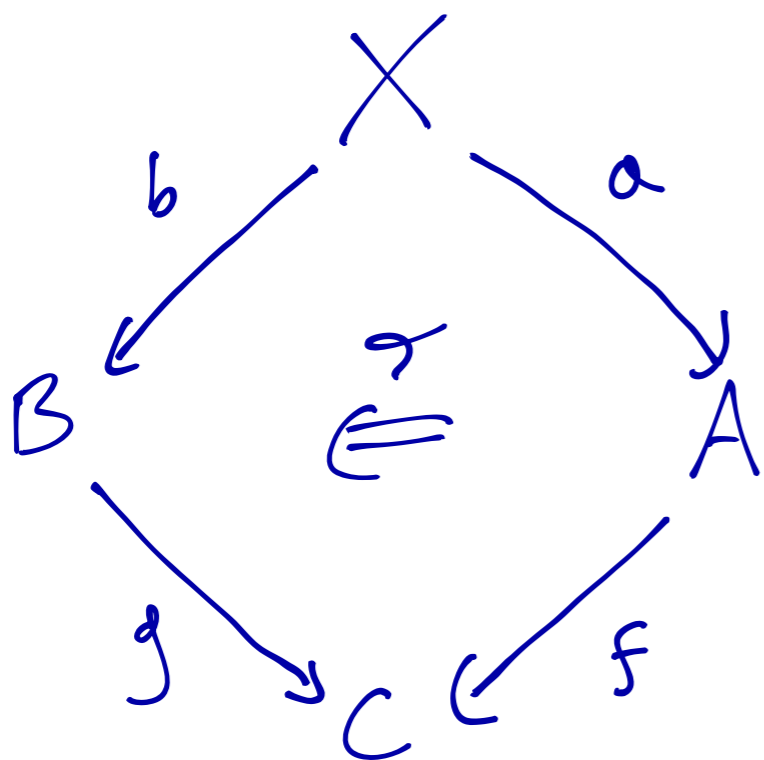


\rightsquigarrow
=



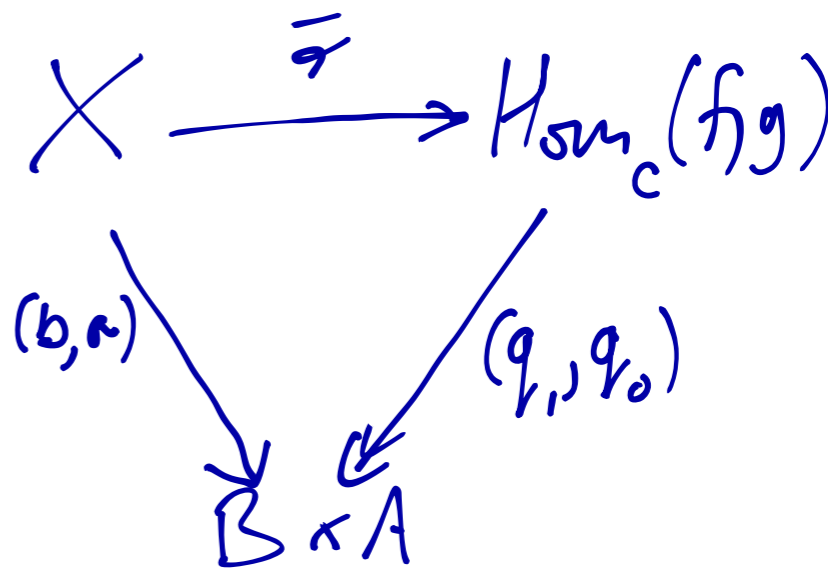
LIFTINGS MOM-WISE

KEY OBSERVATION IT IS A CONSEQUENCE OF THIS WEAK 2-UNIVERSAL PROPERTY OF $\text{Hom}_c(f, g)$ THAT THERE EXISTS A BIJECTION



2-CELLS

\rightsquigarrow
BIJECTION



ISO-CLASSES OF
1-CELLS OVER $B \times A$

THEOREM

A FUNCTOR $u: A \rightarrow B$ HAS A LEFT ADJOINT

IFF $\text{Hom}_B(B, u)$ IS LEFT REPRESENTABLE.

$$\text{Hom}_B(B, u) \xrightarrow{\sim} \text{Hom}_A(f, A)$$

$\searrow \quad \swarrow$
 $C \times B$

THIS IS THE USUAL
"HOM-WISE" CHARACTER
OF AN ADJUNCTION

THREE VIEWS OF ADJUNCTIONS

$$A \begin{array}{c} \xleftarrow{f} \\ \dashv \\ \xrightarrow{u} \end{array} B$$

$$\eta: \text{id}_B \Rightarrow uf$$

$$\varepsilon: fu \Rightarrow \text{id}_A$$

2-CATEGORICAL IN \mathcal{K}

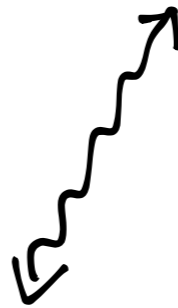
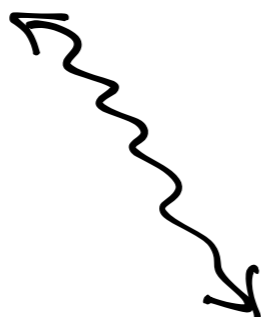


$$\text{Hom}_A(f, A) \rightarrow \text{Hom}_B(B, u)$$

$$\searrow \quad \swarrow \\ A \times B$$

"MODULES" + REPRESENTABILITY

HOMOTOPY COHERENCE?
(LECTURE 3)



$$A \xrightarrow{u} B$$

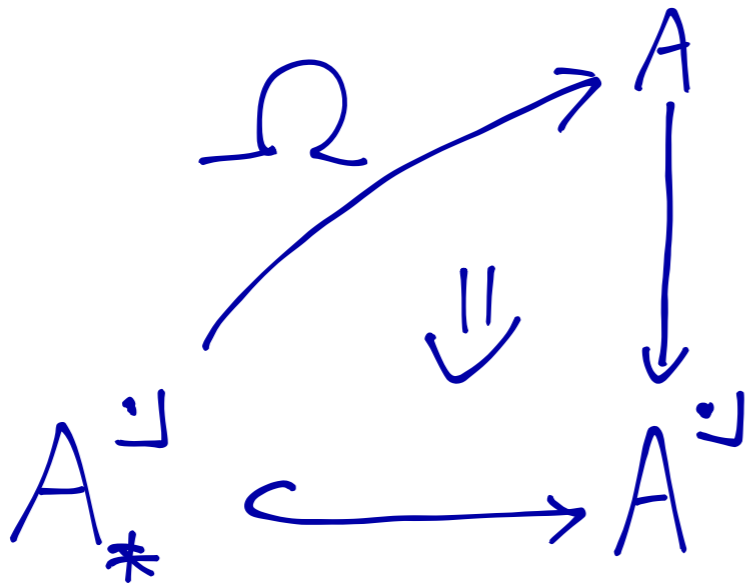
IS RIGHT ADJOINT IN \mathcal{K} IFF

$$\text{Fun}_{\mathcal{K}}(X, A) \xrightarrow{u_0} \text{Fun}_{\mathcal{K}}(X, B)$$

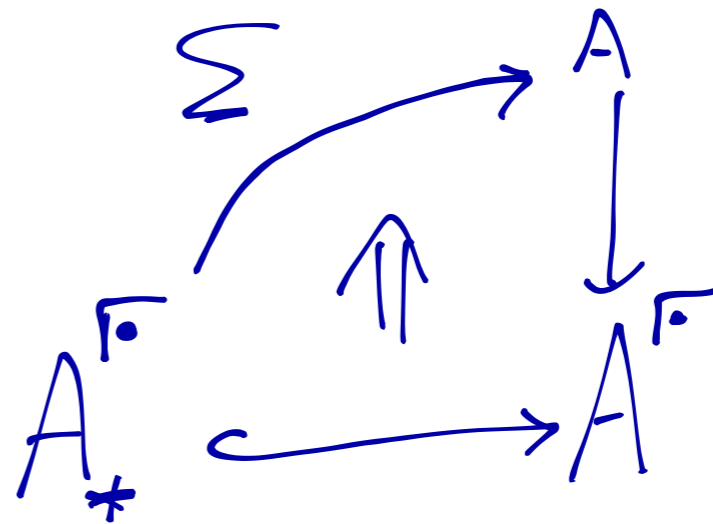
IS A RIGHT ADJOINT FUNCTOR OF QUASI-CATEGORIES FOR ALL $X \in \mathcal{K}$

RETURN TO LOOPS

WE SAY THAT A POINTED ∞ -CAT
A ADMITS LOOPS + SUSPENSIONS IF
THERE EXISTS:



ABSOLUTE
RIGHT LIFTING



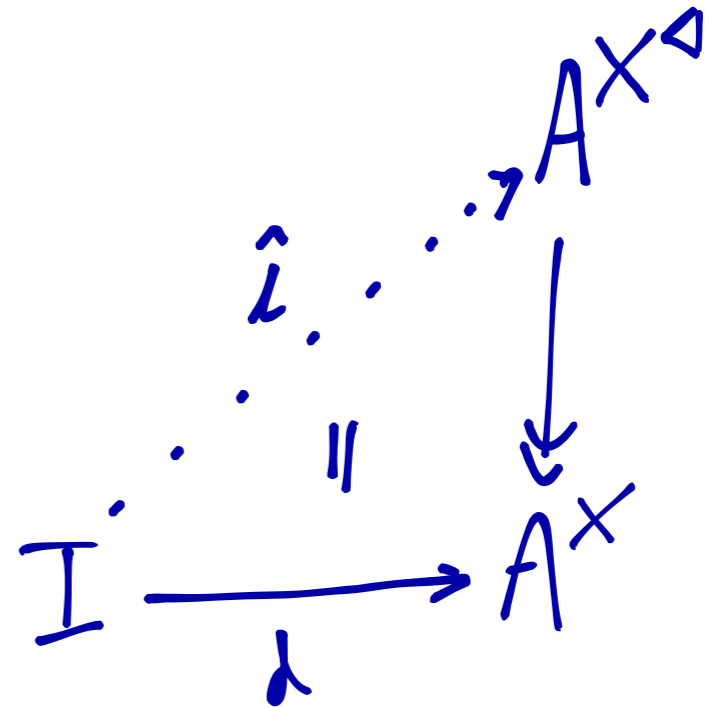
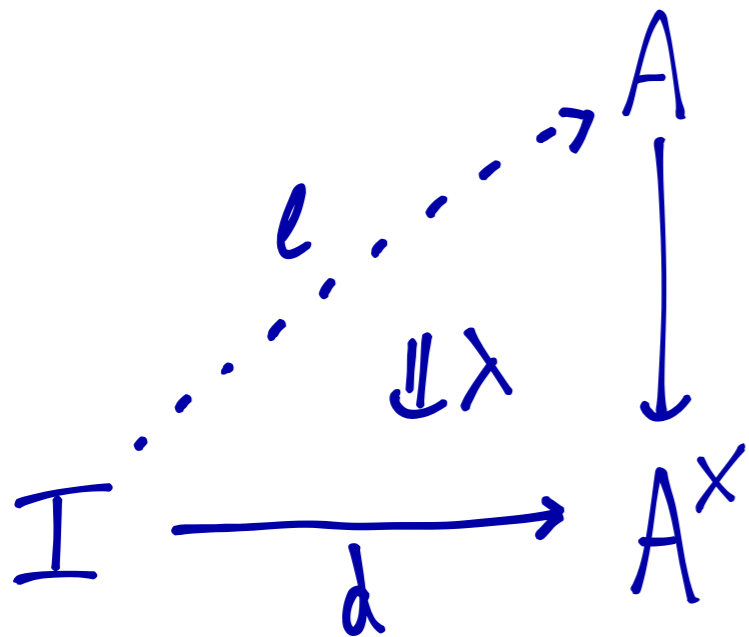
ABSOLUTE
LEFT LIFTING

(CO)LIMITS AS KAN

EXTENSIONS

× A DIAGRAM SHAPE (IN sSet)

×[▷] OBTAINED BY ADDING INITIAL OBJECT

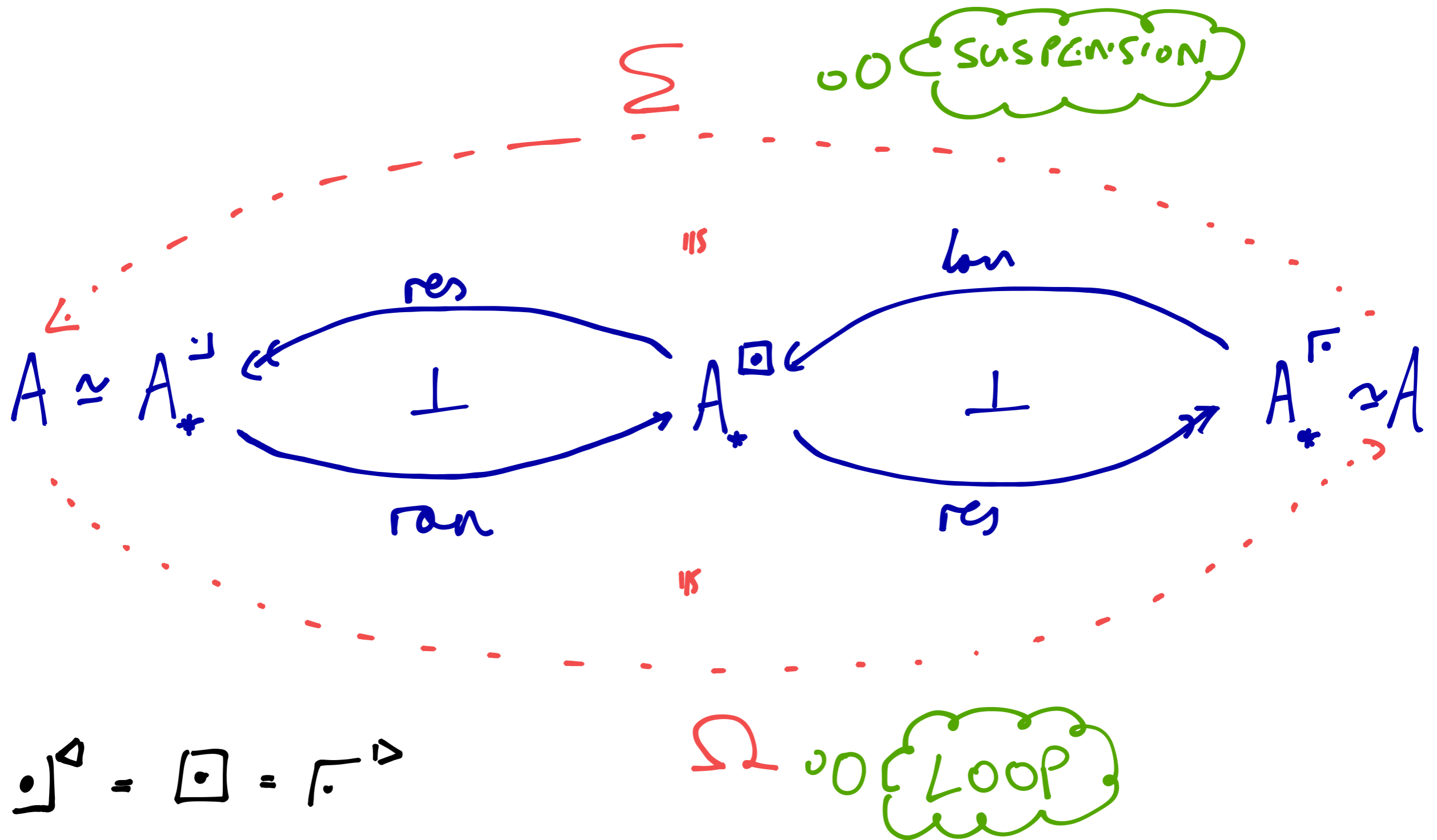


ADMITTS AN
ABSOLUTE RIGHT LIFTING



ADMITTS AN ABSOLUTE
RIGHT LIFT WHOSS 2-CELL
IS THE IDENTITY.

LOOP - SUSPENSION ADJUNCTION



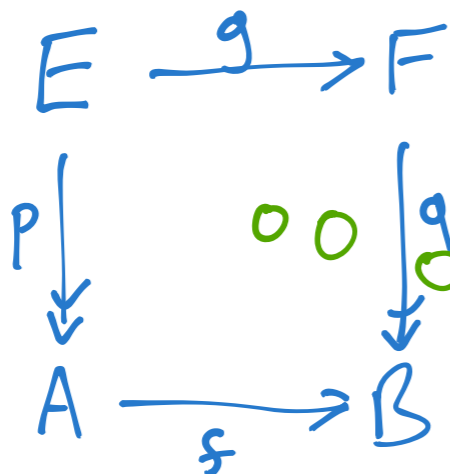
NEW ∞ -COSMOS FROM OLD.

IMPORTANT: MANY (2-) CATEGORICALLY INSPIRED CONSTRUCTIONS APPLY TO ∞ -COSMOS GIVING NEW STRUCTURES WHICH ARE AGAIN ∞ -COSMOS.

SLICES \mathcal{K}/A

OBJECTS ARE ISOFIBRATIONS $p: E \rightarrow A$

COSMOS OF ISOFIBRATIONS \mathcal{K}^2



OBJECTS ARE ISOFIBRATIONS

ARROWS ARE COMMUTATIVE SQUARES

(CO) COMPLETE ∞ -CATS

$\text{Compl}(\mathcal{K})$

OBJECTS: THOSE OF \mathcal{K} ADMITTING SOME CLASS OF LIMITS

ARROWS: FUNCTORS IN \mathcal{K} THAT PRESERVE THOSE LIMITS

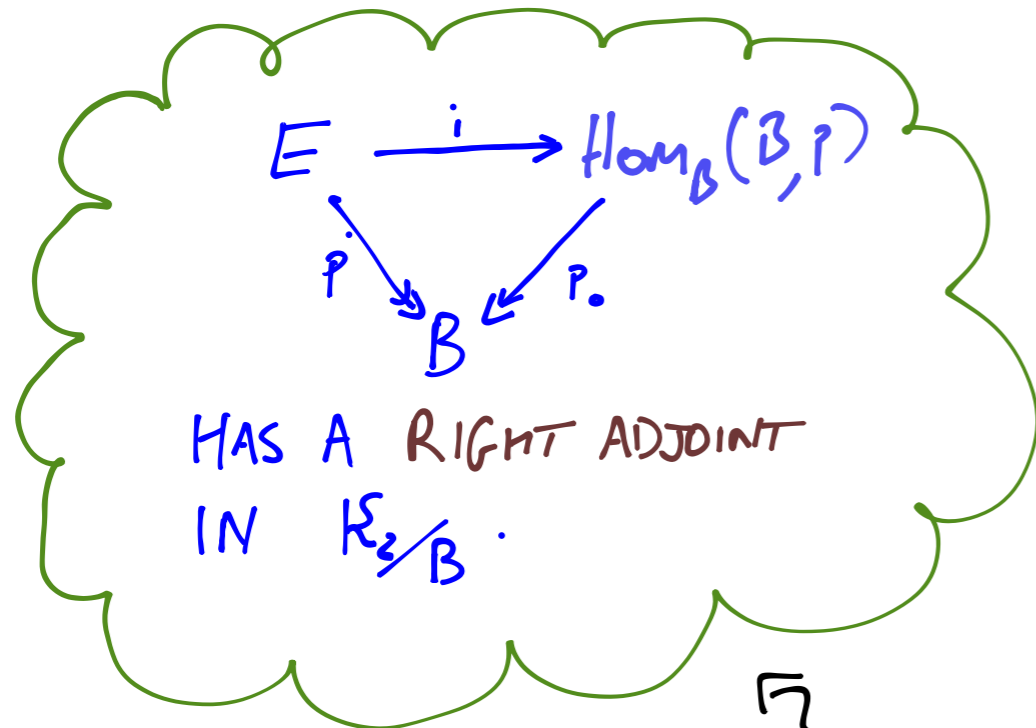
POINTED ∞ -CATS

\mathcal{K}_*

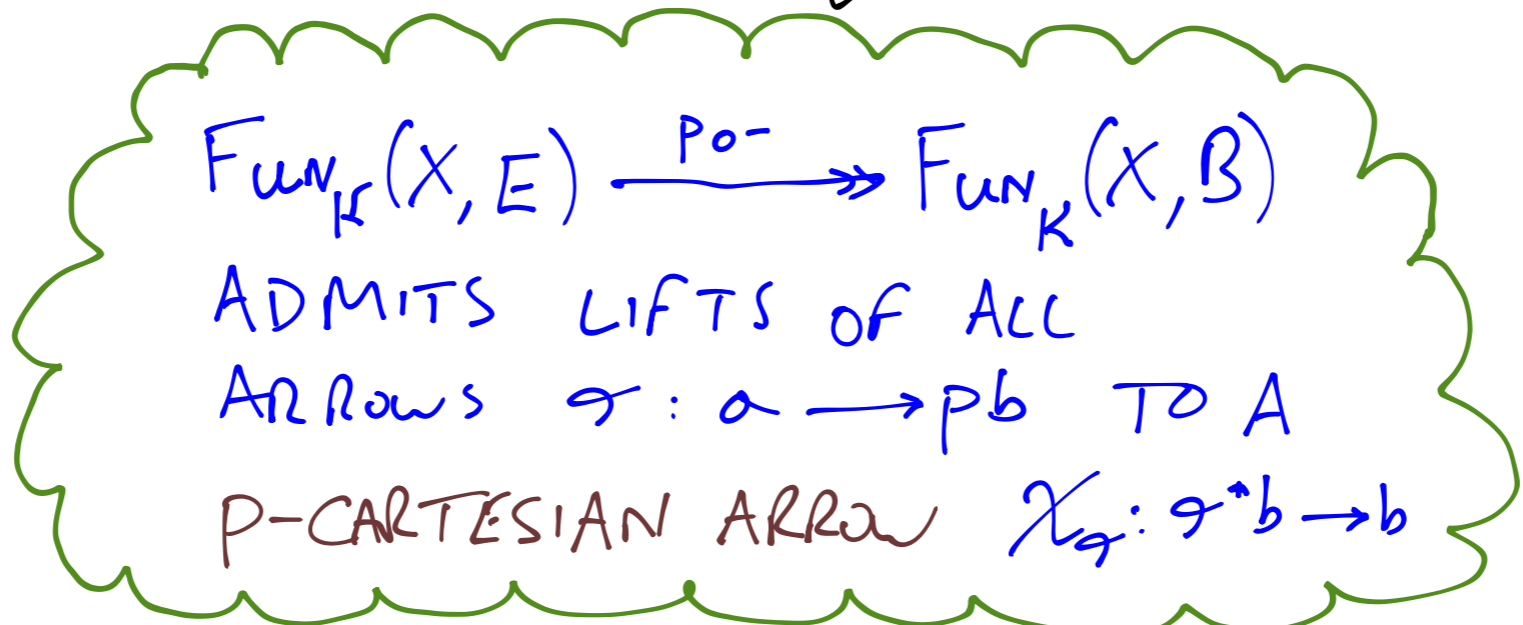
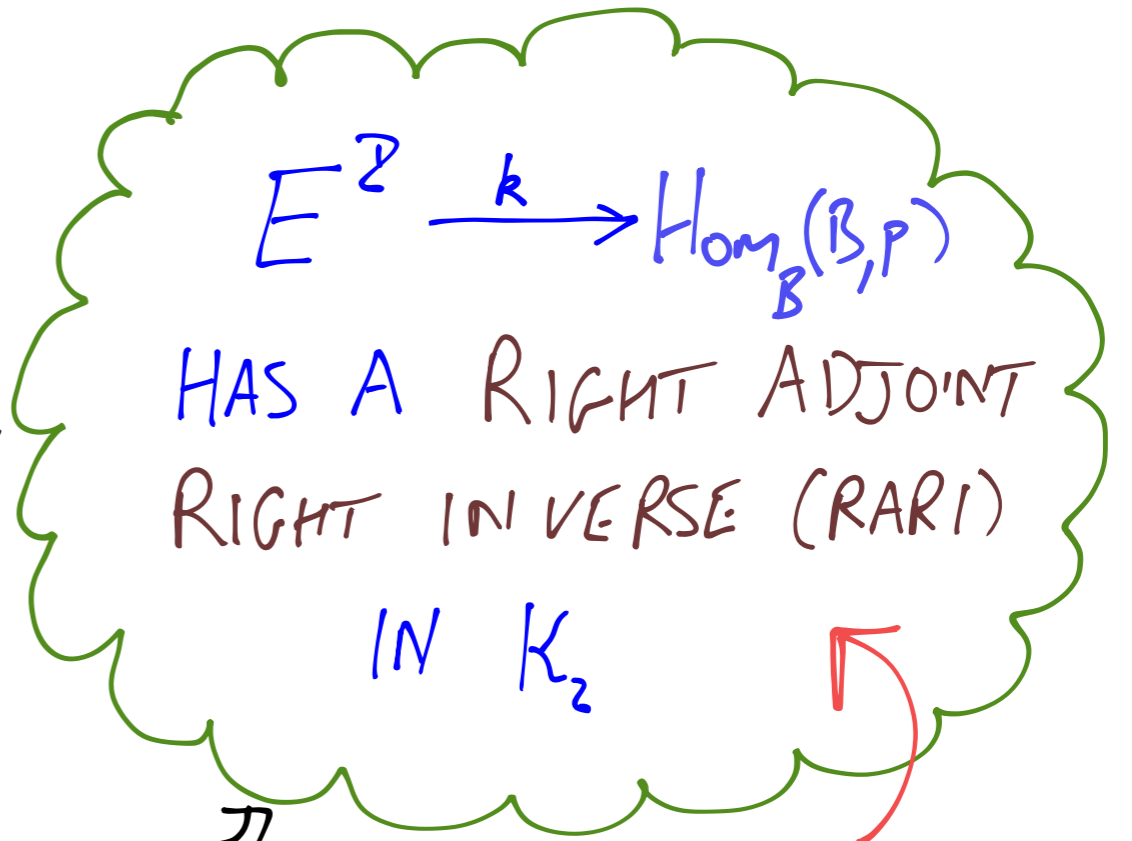
POINTED OBJECTS OF \mathcal{K} + POINT PRESERVING FUNCTORS

THREE VIEWS OF GROTHENDIECK FIBRATIONS

$p: E \rightarrow B$ AN ISOFIBRATION IN \mathcal{K} .



\iff



MORE ∞ -COSMOI

CARTESIAN FIBRATIONS

$\text{Cort}(K) / A \circ \circ \circ \subseteq K / A$ OVER A FIXED BASE

CARTESIAN FIBRATIONS WITH BASE A + CARTESIAN FUNCTORS BETWEEN THEM

2-SIDED CARTESIAN FIBRATIONS

$\text{Fib}(K) / B \circ \circ \circ \subseteq K / A \times B$
 $\text{Cocort}(\text{Cort}(K) / B) / A \times B \xrightarrow{\pi} B$

ALL CARTESIAN FIBRATIONS

$\text{Cort}(K) \circ \circ \circ \subseteq K^2$

$$E \xrightarrow{\gamma} F$$

$$\begin{array}{ccc} E & \xrightarrow{\gamma} & F \\ \downarrow p & & \downarrow q \\ A & \xrightarrow{\gamma} & B \end{array}$$

CARTESIAN FUNCTOR:
PRESERVES CARTESIAN 1-ARROWS

CARTESIAN FIBRATION

GROUPOIDAL ∞ -CATS

$\text{Gpd}(K) \circ \circ \circ \subseteq K$

$A \in \text{Gpd}(K) \text{ IFF } \forall X \in K$


$\text{Fun}_K(X, A)$

IS A KAN COMPLEX.

MODULES

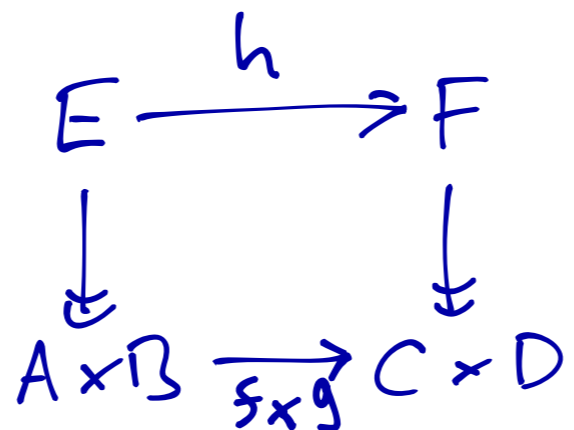
AKA PROFUNCTORS

$$A \backslash \text{Mod}(K) / B \quad \doteq \quad \text{Gpd} \left(A \backslash \text{Fib}(K) / B \right)$$

GIVEN $A \xrightarrow{f} C \xleftarrow{g} B$
 $\text{Hom}_C(g, f)$ IS A MODULE


ALL MODULES

$\text{Mod}(K) \circlearrowright + \text{MAPS}$

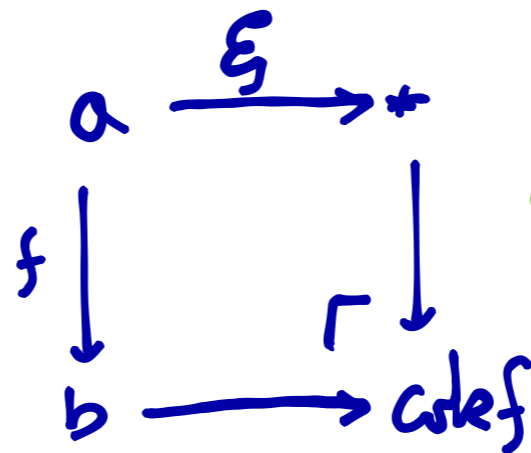
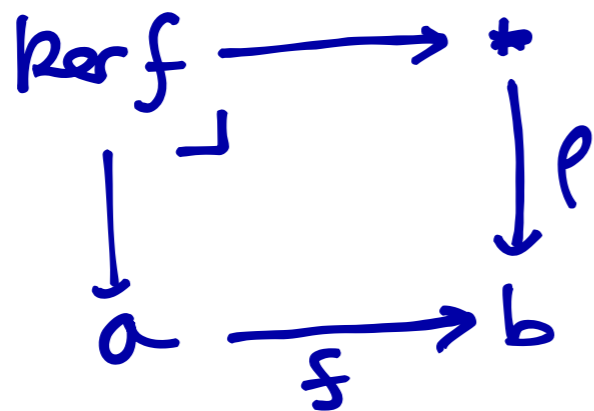


THIS IS AGAIN!
 AN ∞ -COSMOS.

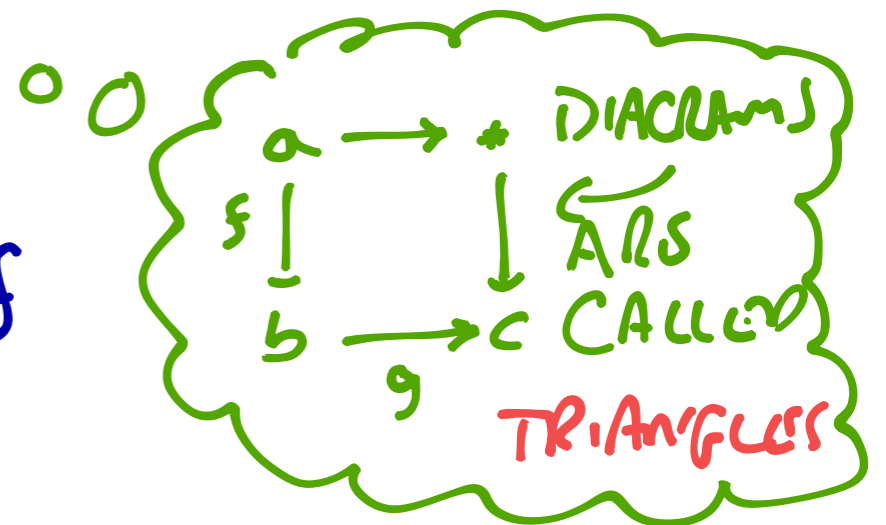
STABLE ∞ -CATEGORIES

AN ∞ -CATEGORY \mathcal{A} IS SAID TO BE STABLE IF

- \mathcal{A} HAS A ZERO OBJECT $*$: $1 \rightarrow \mathcal{A}$
- EVERY X -INDEXED FAMILY $f: a \rightarrow b$ OF ARROWS IN \mathcal{A} ADMITS KERNELS + COKERNELS

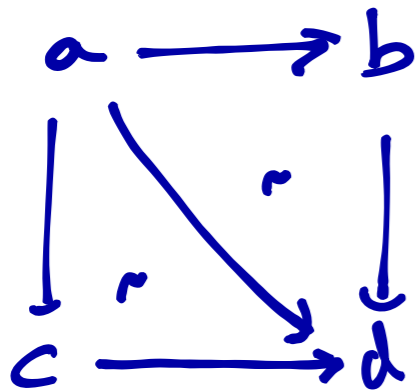


$$f: X \rightarrow A^2$$



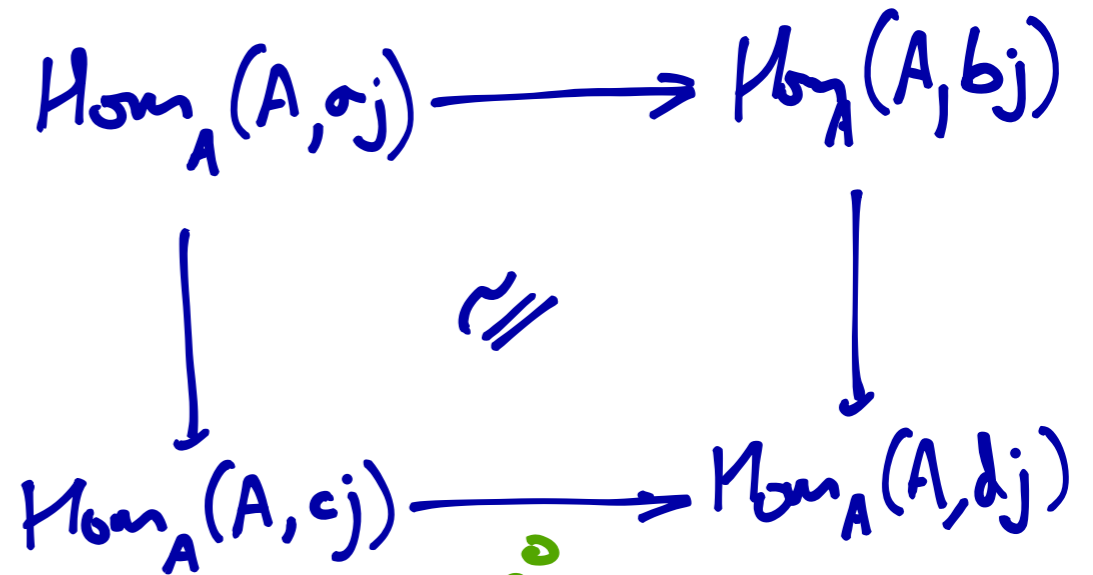
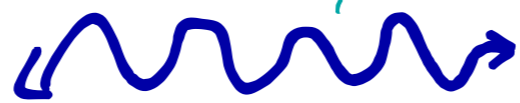
- A TRIANGLE IS A PUSHOUT (COKERNEL) IFF IT IS A PULLBACK (KERNEL).

A HOM-WISE CHARACTERISATION OF PULLBACKS



I-INDEXED COMMUTATIVE SQUARES IN A

THIS SQUARE IS AN I-INDEXED FAMILY OF PULLBACKS



ISO-SQUARES IN 2-CAT $\text{Mod}_K(J, A)$

THIS ISO-SQUARE IS A WEAK ISO-COMMA FOR ALL $j: J \rightarrow I$.



APPLICATION: THE COMPOSITION + CANCELLATION LAWS HOLD FOR PULLBACKS IN ANY ω -CAT.

ELEMENTARY RESULTS ...

- COMPOSITION + CANCELLATION OF PULLBACK / PULLBACK SQUARES \Rightarrow STABLE ∞ -CATS ADMIT ALL PULLBACKS / PUSHOUTS:

$$\begin{array}{ccccc}
 \ker kg & \longrightarrow & a & \longrightarrow & * \\
 \downarrow & \lrcorner & \downarrow f & \lrcorner & \downarrow \\
 c & \xrightarrow{g} & b & \xrightarrow{k} & \text{coker } f
 \end{array}$$

- THE LOOP-SUSPENSION ADJUNCTION OF A STABLE ∞ -CAT IS AN EQUIVALENCE.
- A POINTED ∞ -CATEGORY IS STABLE IFF IT ADMITS KERNELS + ITS LOOP FUNCTOR IS AN EQUIVALENCE.

STABILISATION

THE SUBCATEGORY OF

-) POINTED ∞ -CATEGORIES ADMITTING KERNELS
-) ∞ -FUNCTORS THAT PRESERVE POINTS AND KERNELS.

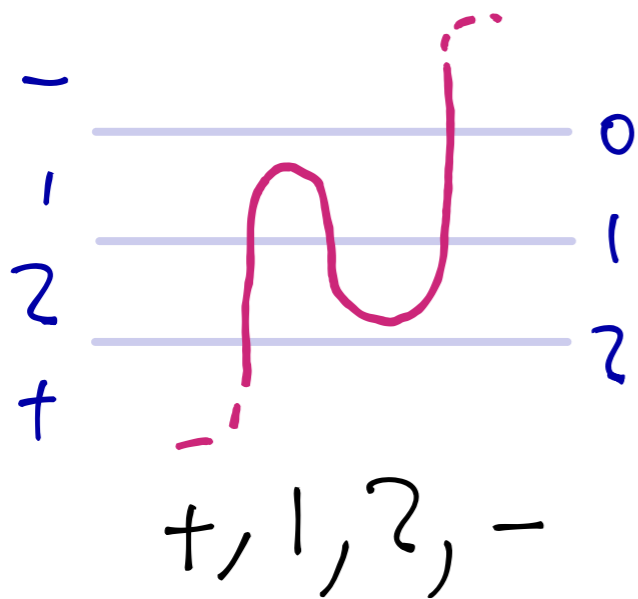
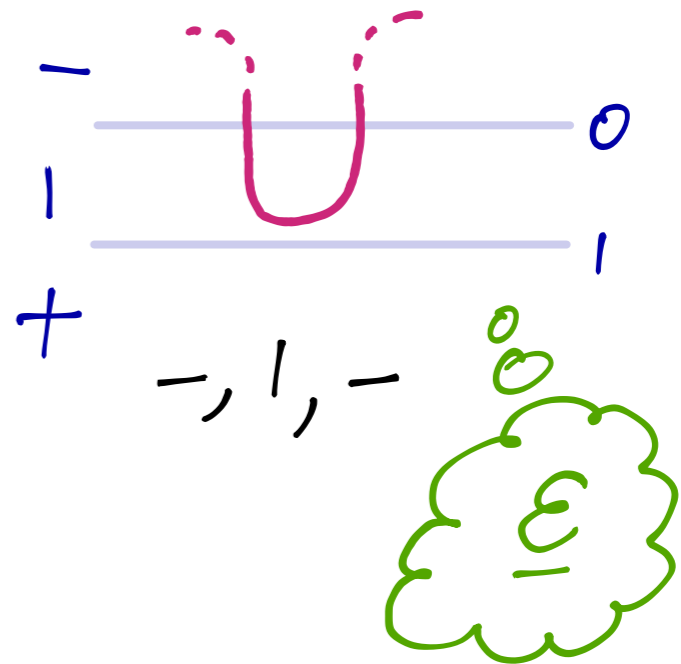
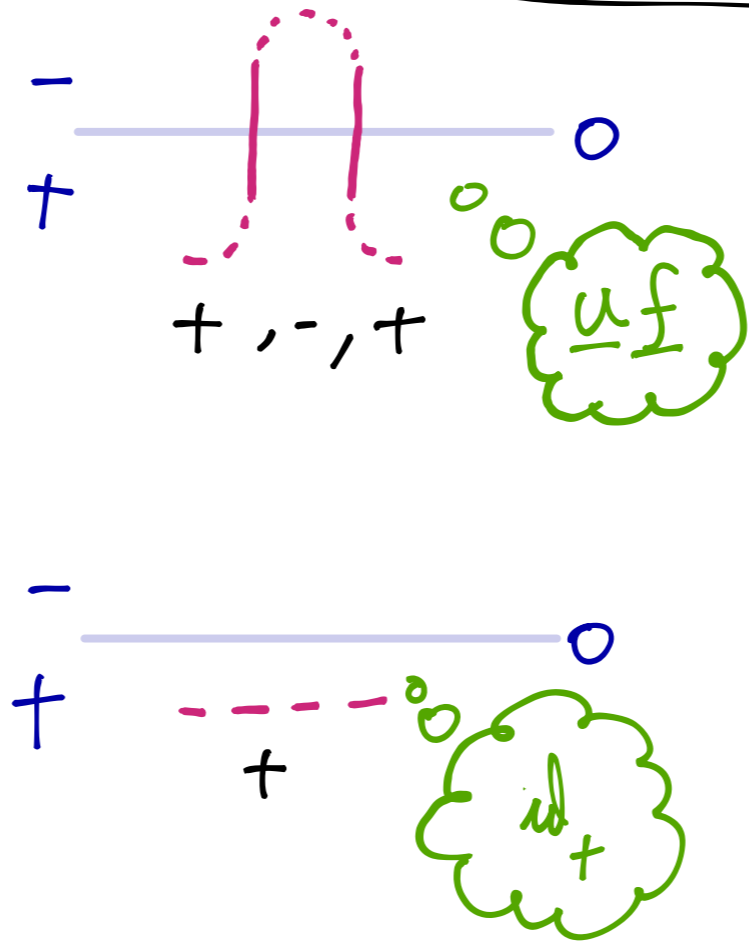
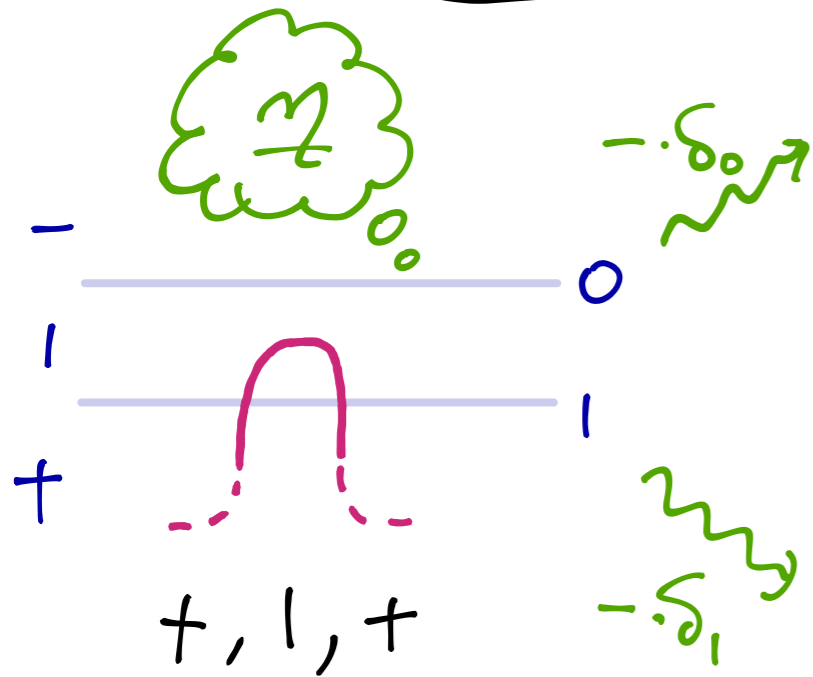
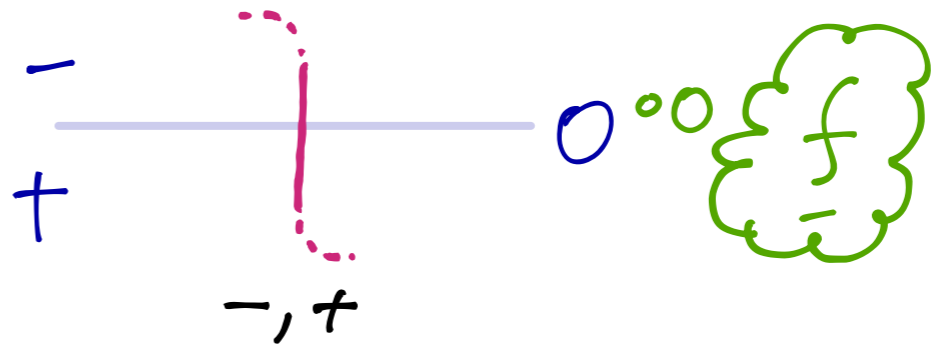
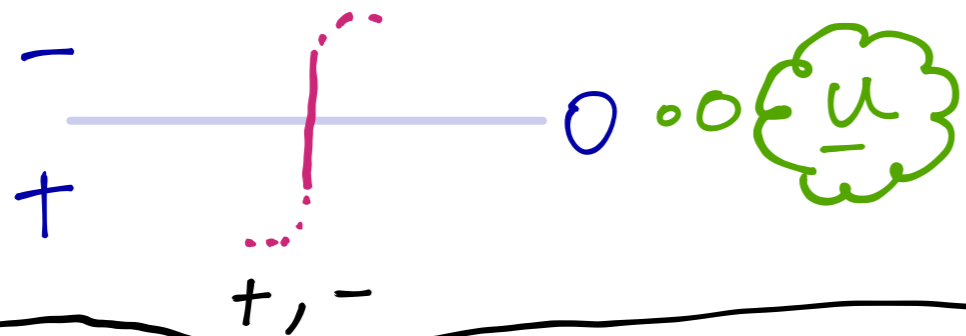
IS A **SUB- ∞ -COSMOS** SO IT ADMITS A HOMOTOPY LIMIT OF EACH CHAIN:

$$\dots \xrightarrow{\Omega} A \xrightarrow{\Omega} A \xrightarrow{\Omega} A \xrightarrow{\Omega} A$$

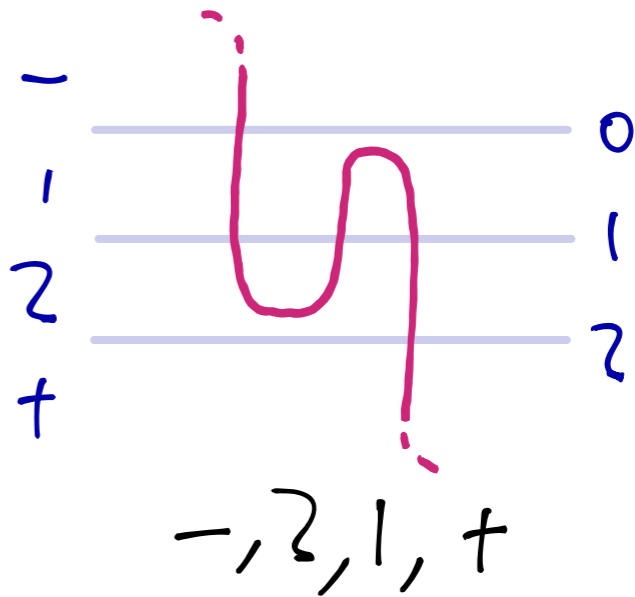
PROP LOOPS FUNCTOR ON THIS LIMITING ∞ -CATEGORY IS AN EQUIVALENCE.

\Rightarrow THIS ∞ -CATEGORY $\text{STAB}(A)$ IS STABLE.

SPECTRA
LIVE
HERE!

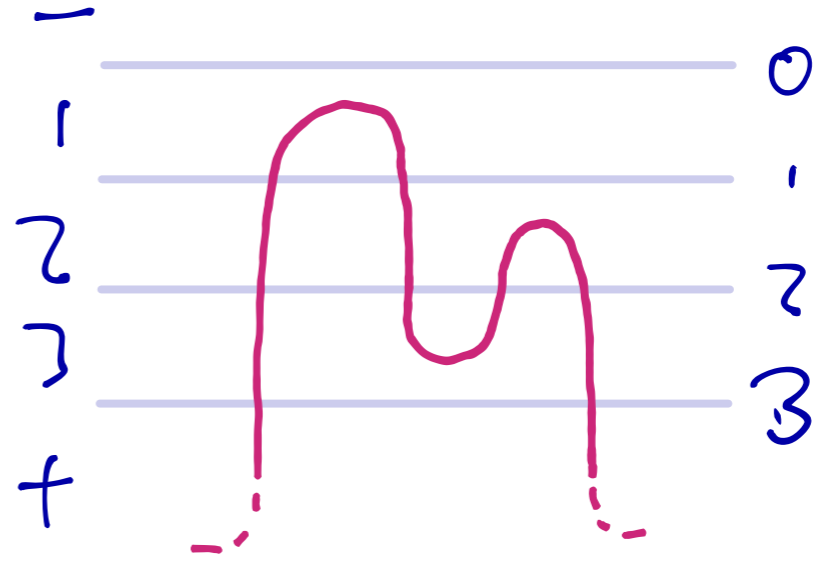


WITNESSES
OF THE
TRIANGLE
IDENTITIES





t, 2, 3, 1, t



t, 1, 3, 2, t

SOME HIGHER
COHERENCES BETWEEN
TRIANGLE IDENTITIES

ADJUNCTIONS ARE COHERENT

THEOREM: AN ADJUNCTION $\mathcal{F} \dashv \mathcal{U}$ IN $\mathcal{H}\mathcal{K}$ GIVES RISE TO A SIMPLICIAL FUNCTOR $A: \text{Adj} \rightarrow \mathcal{K}$ WHICH CARRIES THE CANONICAL ADJUNCTION $\underline{\mathcal{F}} \dashv \underline{\mathcal{U}}$ IN Adj TO THE ADJUNCTION $\mathcal{F} \dashv \mathcal{U}$ IN $\mathcal{H}\mathcal{K}$. THE SPACE OF FUNCTORS THAT EXTEND $\mathcal{F} \dashv \mathcal{U}$ IN THIS WAY IS CONTRACTIBLE.