Introduction to Étale Groupoids and their algebras via finiteness spaces
Part 1

Richard Blute
University Of Ottawa

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Overview

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- Groupoids capture symmetry better than groups do, as it has a local flavor.
- Interesting algebraic structure.
- Can associate convolution algebras to them. This construction can be used to construct C*-algebras and prove properties about them.
- We'll construct convolution algebras using finiteness spaces.

Groupoids

Definition (For Sensible People)

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Definition (For Functional Analysts)

A *groupoid* is a pair of sets $\mathcal{G}_1$ (arrows) and $\mathcal{G}_0$ (objects) with morphisms

- $d, r\colon \mathcal{G}_1 \to \mathcal{G}_0$
- $m\colon \mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1 \to \mathcal{G}_0$
- $u\colon \mathcal{G}_0 \to \mathcal{G}_1 \qquad i\colon \mathcal{G}_1 \to \mathcal{G}_1$

satisfying evident axioms.

$$\begin{tikzcd} \mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1 \arrow[r, "\mathrm{m}"] & \mathcal{G}_1 \arrow[loop above, "i"] \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "r"'] & \mathcal{G}_0 \arrow[l, "u"'] \end{tikzcd}$$

Examples of groupoids

- Any group is a one-object groupoid.
- Any disjoint union of groups. This is called a *group bundle*.
- The fundamental groupoid of a space.
- An equivalence relation induces a groupoid where there is precisely one arrow between two elements if they are equivalent.
- A group action induces a groupoid:
  Let G act on a set X. Let $\mathcal{G}_1 = G \times X$ and $\mathcal{G}_0 = \{e\} \times X$. Then

$$d(g,x) = x, \quad r(g,x) = gx, \quad (g,hx)\cdot(h,x) = (gh,x)$$

Semi-direct product groupoids

This last example is called the *semi-direct product* construction. It (and variants) has a number of applications.

For example, if H is a subgroup of G (not necessarily normal), then the coset space has no canonical group structure. However G acts on the coset space and so one can form the semi-direct product groupoid. One can then carry out a great deal of the usual program of obtaining subgroup theorems, etc. using this groupoid.

There are also applications in the Galois theory of commutative rings and ergodic theory.

See the survey of Ronny Brown.

More examples of groupoids

These examples illustrate the local nature of groupoids.

- Let X be a set. Objects are subsets of X. An arrow is a bijective function (so partial on X). This example can be modified to add various sorts of structure on X.
- Given a field K, define a category whose objects are natural numbers and whose arrows are invertible morphisms.
- Let V be a vector space. Define a category whose objects are subspaces of V. Morphisms are linear isomorphisms between subspaces.

Deaconu-Renault groupoids

Let X be a set, Γ an abelian group and $S \subseteq \Gamma$ a subsemigroup containing 0. Suppose S acts on X. Define a category whose objects are the elements of X. An arrow from x to y is of the form $s - t$ with $s, t \in S$ where $s \cdot x = t \cdot y$. Composition uses the group operation of Γ. Straightforward to verify that this is well-defined.

There are many variations of this, topological and otherwise.

Some category theory

- If $\mathcal{G}$ is a groupoid and x is an object of $\mathcal{G}$, then $\mathrm{Hom}_{\mathcal{G}}(x, x)$ is a group, called the *isotropy group* of x.
- A morphism of groupoids is simply a functor. The category of groupoids is cartesian closed.
- The inclusion of the category of groupoids into the category of categories has both a left and right adjoint. One adjoint is obtained by inverting all maps and the other by taking the wide subcategory of isomorphisms.
- A groupoid is *principal* if the map $\mathcal{G}_1 \to \mathcal{G}_0 \times \mathcal{G}_0$ defined by $f \mapsto (d(f), r(f))$ is injective. A groupoid is principal if and only if it is an equivalence relation.

Internal groupoids

Given a category with finite limits, one can consider groupoids internal to that category, since the definition can be expressed entirely diagrammatically. A *localic groupoid* is a groupoid internal to the category of locales.

Theorem (Joyal-Tierney)

Every Grothendieck topos is equivalent to a category of sheaves on a localic groupoid.

Theorem (Moerdijk)

The above extends to an equivalence of 2-categories.

Topological groupoids

These can be defined with various levels of generality. We'll follow A. Sims, *Hausdorff étale groupoids and their C^*-algebras*

- A *topological groupoid* is a groupoid $\mathcal{G}$ in the category of locally compact hausdorff spaces and continuous maps.

- A topological groupoid is *étale* if its domain map is a local homeomorphism. (This implies the range map and multiplication are as well.)

Topological groupoids II

Lemma

If $\mathcal{G}$ is a topological groupoid, then $\mathcal{G}_0$ is closed in $\mathcal{G}$ if and only if $\mathcal{G}$ is Hausdorff.

Lemma

If $\mathcal{G}$ is an étale groupoid, then $\mathcal{G}_0$ is open in $\mathcal{G}$, and hence clopen.

Lemma

The Deaconu-Renault groupoid is étale if the action of the semigroup is by local homeomorphisms.

Topological groupoids III

- Every groupoid is a topological groupoid in the discrete topology.
- Every discrete groupoid is étale.
- If X is a locally compact Hausdorff space, and R is an equivalence relation on X, then R is a topological groupoid in the relative topology inherited from $X \times X$.
- The group action groupoid is étale if and only if the acting group is discrete (And X is locally compact hausdorff.)

Topological groupoids IV

Let $X = \Pi_0^\infty \{0,1\}$, equipped with the product topology. Define an equivalence relation R on X by $x\mathrm{R}y$ if there is a $j \in \mathbb{N}$ such that $x_k = y_k$ for all $k \geq n$.
If v and w are finite word in $\{0,1\}$, define $Z(v,w) = \{(vx, wx) | x \in X\}$.

Lemma

The sets $Z(v,w)$ form a basis for a topology. The resulting groupoid is étale.

Topological groupoids V

The following is due to Kumjian, Pask, Raeburn and Renault.

Let $G = (V, E)$ be a directed graph with V countable. We'll also assume G is row-finite, i.e. for all vertices v, $s^{-1}(v)$ is finite. Let $P(G)$ be the set of all infinite paths and $F(G)$ be the set of all finite paths. $P(G)$ can be seen as a subspace:

$$P(G) \subseteq \prod_{i=1}^{\infty} E \quad \text{with } E \text{ topologized discretely}$$

The topology can be described as follows. If $\alpha \in F(G)$, let

$$Z(\alpha) = \{x \in P(G) | x = \alpha y, \ \text{ with } y \in P(G)\}$$

Theorem (KPRR)

The sets $\{Z(\alpha) | \alpha \in F(G)\}$ form a basis for the topology on $P(G)$. The resulting topology is locally compact and totally disconnected.

Topological groupoids VI

$P(G)$ is the object part of an étale groupoid.

Definition

Suppose $x, y \in P(G)$. We say that x and y are *shift equivalent with lag* $k \in \mathbb{Z}$ if there exists $N \in \mathbb{N}$ such that $x_i = y_{i+k}$ for all $i > N$. We write $x \sim_k y$.

Lemma

We have $x \sim_0 x$ *and* $x \sim_k y \Rightarrow y \sim_{-k} x$ *and* $x \sim_k y, y \sim_l z \Rightarrow x \sim_{k+l} z$.

Define

$$\mathcal{G} = \{(x, k, y) \in P(G) \times \mathbb{Z} \times P(G) | x \sim_k y\}$$

Topological groupoids VII

Define a multiplication $\mu \colon \mathcal{G}^2 \to \mathcal{G}$

$$\mu((x,k,y_1)(y_2,l,z)) \mapsto$$
$$\begin{cases} \text{undefined} & \text{if } y_1 \neq y_2 \\ (x,k+l,z) & \text{if } y_1 = y_2 \end{cases}$$

with inverse given by $i(x,k,y) = (y,-k,x)$

Theorem

Let G be a row-finite directed graph. The sets

$$\{Z(\alpha,\beta) | \alpha,\beta \in F(G) \text{ and } r(\alpha) = r(\beta)\}$$

form a basis for a locally compact Hausdorff topology on $\mathcal{G}$. With this topology, $\mathcal{G}$ is a second countable, locally compact étale groupoid.

Associating algebras to étale groupoids

Lemma

If $\mathcal{G}$ is an étale groupoid, then for all $x \in \mathcal{G}_0$, the sets $\mathcal{G}_x = \{\gamma \in \mathcal{G} | d(\gamma) = x\}$ and $\mathcal{G}^x = \{\gamma \in \mathcal{G} | r(\gamma) = x\}$ are closed and discrete in the subspace topology.

Theorem

Let $\mathcal{C}_c(\mathcal{G}) = \{f : \mathcal{G} \to \mathbb{C} | supp(f) \text{ is compact}\}$. define $f \star g : \mathcal{G} \to \mathbb{C}$ by

$$f \star g(\gamma) = \sum_{\alpha\beta=\gamma} f(\alpha)g(\beta)$$

Then $\mathcal{C}_c(\mathcal{G})$ is a $*$-algebra with above multiplication and $f^*(\gamma) = \overline{f(\gamma^{-1})}$.

The sum is finite.

The key is showing that the sum is finite. Note that if $\alpha\beta = \gamma$ then $\alpha \in \mathcal{G}^{r(\gamma)}$ and $\beta \in \mathcal{G}_{d(\gamma)}$. So

$$\{(\alpha, \beta) \in \mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1 | \alpha\beta = \gamma \text{ and } f(\alpha)g(\beta) \neq 0\}$$

is finite, since the intersections of discrete, closed sets and compact sets are finite.

We also note that $supp(f \star g) \subseteq supp(f)supp(g)$.

Topological spaces as finiteness spaces?

Given the above, it makes sense to ask if there is a class of sufficiently nice topological spaces X such that $(X, \mathcal{U})$ is a finiteness space where $\mathcal{U}$ is the set of relatively compact subsets and $\mathcal{U}^{\perp}$ is the set of discrete, closed subspaces. (A subspace is *relatively compact* if its closure is compact in X.)

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Conjecture (The Blute Conjecture)

The above determines a finiteness space structure when X is locally compact, hausdorff.

Ehrhard's finiteness spaces I

Let X be a set and let $\mathcal{U}$ be a set of subsets of X, i.e., $\mathcal{U} \subseteq \mathcal{P}(X)$. Define $\mathcal{U}^{\perp}$ by:

$$\mathcal{U}^{\perp} = \{u' \subseteq X \mid \text{the set } u' \cap u \text{ is finite for all } u \in \mathcal{U}\}$$

Lemma

- $\mathcal{U} \subseteq \mathcal{U}^{\perp\perp}$
- $\mathcal{U} \subseteq \mathcal{V} \Rightarrow \mathcal{V}^{\perp} \subseteq \mathcal{U}^{\perp}$
- $\mathcal{U}^{\perp\perp\perp} = \mathcal{U}^{\perp}$

A *finiteness space* is a pair $\mathbb{X} = (X, \mathcal{U})$ with X a set and $\mathcal{U} \subseteq \mathcal{P}(X)$ such that $\mathcal{U}^{\perp\perp} = \mathcal{U}$. We will sometimes denote X by $|\mathbb{X}|$ and $\mathcal{U}$ by $\mathcal{F}(\mathbb{X})$. The elements of $\mathcal{U}$ are called *finitary* subsets.

Examples of finiteness spaces

Every set has two associated finiteness space structures, the minimal and maximal.

For the minimal, the only finitary subsets are the finite subsets. For the maximal, every subset is finitary.

We will see examples arising from posets and from topological spaces soon.

Finiteness spaces II: Morphisms

- A *morphism* of finiteness spaces $R\colon \mathbb{X} \to \mathbb{Y}$ is a relation $R\colon |\mathbb{X}| \to |\mathbb{Y}|$ such that the following two conditions hold:
  1. (1) For all $u \in \mathcal{F}(\mathbb{X})$, we have $uR \in \mathcal{F}(\mathbb{Y})$, where $uR = \{y \in |\mathbb{Y}| \mid \exists x \in u, xRy\}$.
  2. (2) For all $v' \in \mathcal{F}(\mathbb{Y})^\perp$, we have $Rv' \in \mathcal{F}(\mathbb{X})^\perp$.

Composition is relational and it is straightforward to verify that this is a category. We denote it FinRel.

Lemma (Ehrhard)

In the definition of morphism of finiteness spaces, condition (2) can be replaced with:

$$(2')\ \text{For all } b \in |\mathbb{Y}|, \text{ we have } R\{b\} \in \mathcal{F}(\mathbb{X})^\perp.$$

Finiteness spaces III: It's a $*$-autonomous category

Definition

A symmetric monoidal closed category is $*$-*autonomous* if there is an object $\perp$ such that the canonical natural transformation $A \longrightarrow (A \Rightarrow \perp) \Rightarrow \perp$ is a natural isomorphism.

Theorem

FinRel *is a $*$-autonomous category. The tensor*

$$\mathbb{X} \otimes \mathbb{Y} = (|\mathbb{X} \otimes \mathbb{Y}|, \mathcal{F}(\mathbb{X} \otimes \mathbb{Y}))$$

is given by setting $|\mathbb{X} \otimes \mathbb{Y}| = |\mathbb{X}| \times |\mathbb{Y}|$ *and*

$$\begin{aligned} \mathcal{F}(\mathbb{X} \otimes \mathbb{Y}) &= \{u \times v \mid u \in \mathcal{F}(\mathbb{X}), v \in \mathcal{F}(\mathbb{Y})\}^{\perp\perp} \\ &= \{w \mid \exists u \in \mathcal{F}(\mathbb{X}), \exists v \in \mathcal{F}(\mathbb{Y}), w \subseteq u \times v\}. \end{aligned}$$

Finiteness spaces IV: It's a model of linear logic

We note that it also has sufficient structure to model the rest of the connectives of linear logic, i.e. it has a comonad ! (and monad ? by duality) satisfying all of the necessary equations to be a *Seely model*, i.e. provides a sound interpretation of the sequent calculus, the key being the isomorphism

$$!(A \times B) \cong !A \otimes !B$$

If $\mathbb{X} = (X, \mathcal{U})$ is a finiteness space, then the underlying set of $!\mathbb{X}$ will be $Mult_{fin}(X)$, the finite multisets of X and finitary subsets are defined by

$$\mathcal{F}(!\mathbb{X}) = \{Mult_{fin}(u) | u \in \mathcal{F}(\mathbb{X})\}^{\perp\perp}$$

Finiteness spaces V: Other choices of morphism

Ehrhard was motivated by linear logic to construct a $*$-autonomous category and hence chose relations as morphisms. But the choice has issues. Much like the usual category of relations, FinRel is lacking many limits and colimits. Other choices are possible:

Definition

We define the category FinF*. Objects are finiteness spaces and a morphism* $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ *is a function satisfying the same conditions as above.*

Proposition

The category FinF *is a symmetric monoidal but not closed, it doesn't have a terminal object.*

Finiteness spaces VI: Other choices of morphism

Definition

We define the category FinPf*. Objects are finiteness spaces and a morphism* $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ *is a partial function satisfying the same conditions as above.*

Proposition

The category FinPf *is a symmetric monoidal closed, complete and cocomplete category.*

This fits perfectly with our intention of considering étale groupoids as special finiteness spaces, since the multiplication is a partial map.

Finiteness spaces VII: Calculating equalizers in FinPf.

Given two parallel morphisms

$$(X,\mathcal{U}) \underset{g}{\overset{f}{\rightrightarrows}} (Y,\mathcal{V})$$

in FinPf, let

$$\begin{aligned} E &= \{x \in X \mid f(\{x\}) = g(\{x\})\} \\ &= \{x \in X \mid \text{either both } f(x) \text{ and } g(x) \text{ are undefined} \\ &\qquad\qquad \text{or they are both defined and } f(x) = g(x)\}. \end{aligned}$$

Let $\mathcal{W} \subseteq \mathcal{P}(E)$ be $\mathcal{W} = \{u \in \mathcal{U} \mid u \subseteq E\}$. Then it is routine to show that

$$\mathcal{W}^{\perp} = \{u' \in \mathcal{U}^{\perp} \mid u' \subseteq E\},$$

$(E,\mathcal{W})$ is a finiteness space and the inclusion $(E,\mathcal{W}) \hookrightarrow (X,\mathcal{U})$ is the equalizer of f and g in FinPf.

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Theorem

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Theorem

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But a smaller class of spaces does work.

Definition

- X *is* σ*-compact if it can be covered by a countable family of compact subsets.*
- X *is* σ*-locally compact if it is both* σ*-compact and locally compact.*

Topological spaces as finiteness spaces II

Theorem (B-F,D,D)

- *Let X be a σ-locally compact hausdorff space. Then it is a finiteness space.*
- *The converse is false. Let X be an uncountable discrete space. Then X is locally compact and hausdorff, but not σ-compact. But X is a finiteness space.*

Nonetheless, the class of σ-locally compact Hausdorff spaces is quite large. Indeed, it contains every (paracompact second-countable Hausdorff). It also contains every CW-complex with countably many cells, because it is the union of countably many images of disks.

Some of our étale groupoids have underlying spaces which are σ-locally compact Hausdorff. We will use this fact to give a new approach to constructing algebras for them.

Homework for Ernie

Improve the students' results. In particular, find a necessary and sufficient condition to ensure that a locally compact hausdorff space becomes a finiteness space under this construction.