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Bundles in categorical geometry

by

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Abstract

This thesis further develops the theory of tangent bundles, differential bundles and principal bundles in tangent categories and restriction categories. We develop a notion of dimension in order to interpret the universality of the vertical lift, a key defining property of tangent bundles and differential bundles in category theory. This dimension provides an easy path to showing that a candidate endofunctor is not a tangent structure or that a candidate projection is not a differential bundle.

Additionally, we provide a functorial characterization of differential bundles. We show that differential bundles in a tangent category are equivalent to certain functors from the structure-category $\mathbb{N}^\bullet$ of free finite-dimensional $\mathbb{N}$-modules into that tangent category. This characterization through functors serves as a translation of differential bundles from the axiomatic language of categories to the functorial language of tangent categories as $\mathbb{W}$eil-actegories and is suitable to be applied to functor calculus.

We show that group-objects in a tangent category are well-behaved and define principal G-bundles in join restriction categories. In the combined setup of tangent join restriction categories with a group object, we show that the principal bundles previously defined fulfill some desirable properties. In particular, they can be used to construct differential bundles, generalizing the associated bundle construction from differential geometry that connects principal and differential bundles.

Preface

The main body of this manuscript-style thesis consists of three papers that have already been finished and made accessible online before this thesis was written.

Chapter 3 is based on [83]. This is a single-author paper I wrote under supervision of Kristine Bauer. I developed the theory and proofs on which Kristine Bauer gave feedback and guidance and she supervised the writing.

Chapter 4 is based on [82]. It is also a single-author paper I wrote under supervision of Kristine Bauer. Michael Ching proposed the idea of the project. Afterwards I developed the constructions and proofs, on which Kristine Bauer gave feedback and guidance and she supervised the writing.

Chapter 5 is based on [31]. This paper is a collaboration between me and my co-supervisor Robin Cockett. He gave me permission to use it as part of my thesis. The mathematical content was jointly developed, based on a suggestion by Robin Cockett. The first draft of the paper, including all the results, was written by me, based on our joint mathematical work. Robin Cockett gave feedback, supervised the writing and added historical context. Parts of the writing were also supervised by Kristine Bauer.

Chapter 6 was developed by myself under supervision of Kristine Bauer. It is not currently written as a stand-alone paper. It relates the topic of Chapter 4 and Chapter 5 in the context of this thesis.

Generative artificial intelligence (AI) was used as a writing tool in this thesis, however it did not develop the mathematics. In particular, generative AI was not used for creating theorems, lemmas, propositions, or other mathematical results, or writing or generating proofs.

Concretely, generative AI was used in the following ways.

1. Literature exploration and brainstorming: Gemini (integrated into Google Search) was used to summarize related areas and locate relevant references. Accuracy was ensured by consulting the original sources rather than relying on AI-generated summaries. This reduces the influence of the AI on the thesis content to the AI's selection of references.

2. Language and style correction: Writefull, integrated into the Overleaf LaTeX editor, was used for

grammar and style suggestions. It only proposed linguistic changes, it did not affect the mathematical content.

3. Text feedback: ChatGPT and Gemini were used to provide feedback on drafted sections to improve clarity and readability. All revisions were manually implemented to ensure that only the language and no mathematical content was altered.

In accordance with university policy, I retain full responsibility and accountability for all content in this thesis, including work produced with the assistance of AI. The language of this AI statement was also refined with the help of generative AI.

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Thank you to my girlfriend Yufang, it is so amazing that we, coming from different continents, both found each other while international students in Calgary. Having you by my side, even when thousands of kilometres apart, made my life a better life and made me feel happy despite of any struggles.

Dedicated to earth, a planet that is exceptionally good to live on, and to everyone committed to preserve it that way.

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List of Symbols, Abbreviations, and Nomenclature

Symbol	Definition
$\mathbb{N}$	The set $\{0, 1, 2, ...\}$ of natural numbers, starting with 0
$\mathbb{Z}$	The set of integers
$\mathbb{Q}$	The field of rational numbers
$\mathbb{R}$	The field of real numbers
$\mathbb{C}$	The field of complex numbers
$\mathbb{F}$	An arbitrary field
$\#X$	The cardinality of X
$\mathrm{Dim}(X)$	The (classical) dimension of the vector space / manifold X
$\mathbb{X}, \mathbb{Y}, \mathbb{Z}, \mathbb{A}, \mathbb{B}$	A category
$\mathbb{S}$tarts with blackboard font	A category
X, Y, Z, A, B, C, D	An object
1	A terminal object
$\mathrm{Ob}(\mathbb{X})$	The collection of objects in the category $\mathbb{X}$
$\mathrm{Hom}(X, Y)$	The set of morphisms from X to Y
1_X	The identity morphism on X
$\pi_0(\mathbb{X})$	The collection of isomorphism classes of objects in $\mathbb{X}$
F, G, H, T	A functor
`Typewriter` font	A functor
$g \circ f$	applicative composition $A \xrightarrow{f} B \xrightarrow{g} C$ of morphisms f and g.
fg	diagrammatic composition $A \xrightarrow{f} B \xrightarrow{g} C$ of morphisms f and g.
$f;g$	diagrammatic composition $A \xrightarrow{f} B \xrightarrow{g} C$ of morphisms f and g.

$f(x)$	application of a function f to an element x of its domain

Chapter 1

Introduction

Bundles, consisting of a total space E with a projection to a base space M, are a fundamental concept for many areas of mathematics and physics. In algebraic topology, bundles, such as covering spaces, provide a way to study topological invariants, e.g. the fundamental group. Bundles allows us to analyze the total space and use insights from the total space to make conclusions about the base space.

The utility of bundles extends to differential geometry, where we can use bundles and their sections in order to perform operations from (multi)-linear algebra locally at any point of the manifold. Consequently, many of the most essential tools in differential geometry, including vector fields, connections, differential forms and metric tensors, are all sections of bundles, offering a way to combine local calculations with global structure.

In physics, these mathematical uses of bundles are essential for modelling the natural world. Sections of vector bundles are used for the local description of physical quantities and their interaction with spacetime. Furthermore, in the study of gauge symmetries, the total space of a bundle provides a redundant description of a physical system. By analyzing these bundles, physicists can make conclusions about the behaviour of the physical system.

The main goal of this thesis is to advance the study of generalized bundles in category theory. In particular, we focus on tangent bundles, differential bundles and principal bundles in tangent categories and restriction categories. Both tangent categories and restriction categories are contexts in category theory that generalize certain aspects of the structure of smooth manifolds.

1.1 Categorical geometry

Geometry is one of the oldest and most beautiful areas of mathematics, and also one of the richest sources of ideas and intuitions. The concept of gluing local pieces together to obtain a global object, among many other geometric ideas, have incarnations outside of traditional geometry and have influenced many sciences, going beyond mathematics. Calculus, and in particular differentiation, are further tools from geometry to model dynamics in a variety of contexts, from crisis bargaining diplomacy [85] to the expansion of the universe [50]. The abstract frameworks of **tangent categories** and **restriction categories** make it possible to formally encode these geometric structures in more general settings settings which unify different approaches to geometry or which may not traditionally be considered to be part of geometry.

By modelling local properties through partial maps, maps that are only defined on a subset of their domain, restriction categories generalize local properties and gluings to a variety of setups. They can be used both to describe geometric structures like manifolds and topological spaces, but also for computational structures like Turing machines.

Complementary to that, tangent categories generalize the concept of differentiability and tangent vectors to a more general categorical setup. They reveal the geometric nature of many concepts as a form of differentiation, even non-analytic non-geometric concepts, like abelianization of groups.

1.1.1 Literature review on tangent categories

Historically, geometry has been thought about through certain desired properties instead of concretely giving concrete mathematical objects fulfilling the behavioural axioms. In Euclid's elements of geometry [78], geometric objects are defined using a list of axioms, e.g. that one can draw a line between any two points.

When Newton and Leibnitz developed modern western calculus in the 17th century (after the ideas had been around for a long time in many cultures), they specified the operations while admitting that they did not fully understand the underlying objects. They used a infinitesimal, a quantity o that whose higher powers can be omitted. While intuitively it is a very small quantity and thus its square is negligibly small, it was widely understood that this is not a well-defined mathematical concept.

Later, in the 19th century, Bolzano and Cauchy built a second approach to the underlying structure of calculus. The key to their approach was that they replaced infinitesimals with limits, building the modern foundations of analysis. For more details on the history of calculus one can, for example, read [12].

However, many still thought that the original approach using infinitesimals was still worth considering. In [65], Sophus Lie highlights that there is an "analytic" approach and a "synthetic" approach to differentiation, while admitting that the foundations of the "synthetic" approach are hard to make rigorous.

After the "analytic" approach had developed into differential geometry and the classical theory of smooth manifolds, in 1967, William Lawvere gave a series of lectures on the "synthetic" approach, which were later published as [60]. In there Lawvere rigorously defines synthetic differential geometry through a closed category and an infinitesimal object D which represents differentiation. The tangent bundle of a manifold X is then described by the object X^D. In 1981, Anders Kock wrote the textbook [57], further elaborating on this approach. Synthetic differential geometry has further grown since. Many results from classical differential geometry have been shown to have analogues in synthetic differential geometry. For example Marta Bunge and Patrice Sawyer [14] proved results about connections and Rene Lavendhomme and Hirokazu Nishimura [59] proved results about differential forms in SDG. Matthew Burke proved the correspondence between Lie groups and Lie Algebras in his thesis [15]. Burke's result in particular is similar to our results in Section 5.3.5.

In 1984 [81], Jiří Rosický developed a notion of abstract tangent bundle functors that encompass both classical differential geometry and synthetic differential geometry, finally uniting the "synthetic" and "analytic" approach to differentiation. This notion of tangent bundle functors underlies the tangent categories Robin Cockett and Geoff Cruttwell developed in 2014 [20]. They slightly generalized the definition to not assume negatives, developed substantial techniques to work with tangent categories and related them to the pre-existing notion of Cartesian differential categories. As a different formulation, Poon Leung [64] and Richard Garner [44] showed that tangent categories can also be seen as functors from a structure category of Weil-algebras. This approach is equivalent to the axiomatic definition by Cockett and Cruttwell, but allows us to view tangent categories in a functorial perspective, which is useful for certain applications like higher categories. We will revisit these equivalent characterizations in detail in Section 4.2.

The category of smooth manifolds is one example of a tangent category, but there are many more. Since tangent categories have been developed, many geometric constructions have been generalized using them. Vector bundles, one of the most fundamental and important concepts in differential geometry, were generalized as differential bundles in [29]. In his PhD thesis [67], Benjamin McAdam proved that differential bundles in the category of smooth manifolds indeed are vector bundles and thus precisely generalize vector bundles in tangent categories. Cruttwell and Marcello Lanfranchi [32] used display maps to study the existence of pullbacks, required for differential bundles, and define a notion of submersion in tangent categories.

Cruttwell and Rory Lucyshyn-Wright [35] generalized differential forms and de Rham cohomology, with their deep meaning to geometry and topology. Similarly, Cockett and Cruttwell [22] generalized connections, enabling a new perspective on curvature and torsion, independently of the explicit model of geometry. An important tool for system dynamics, differential equations, were formulated by Cockett, Cruttwell and Jean-Simon Lemay in [23]. There curve objects are introduced as the objects that parametrize the solutions of ordinary differential equations.

Since their development, tangent categories and the aforementioned concepts in them have been applied to many different areas of mathematics. Originally, Cockett and Cruttwell [20] show that tangent categories generalize the "analytic" and "synthetic" formulation of differential geometry. The category of rings is also an example of a tangent category, already mentioned by Rosický. Cruttwell and Lemay [33] further showed that schemes, the object of study in Algebraic geometry, are another example of a tangent category and studied the differential bundles therein. Thus, tangent categories do not just unite the different approaches to differential geometry, they also unite differential and algebraic geometry.

In homotopy theory and homological algebra, tangent categories have proven themselves to be a useful tool to encode approximations to certain functors. Kristine Bauer, Burke and Michael Ching use tangent categories in [6] to describe the Goodwillie functor calculus of infinity categories. In addition, work by Bauer, Brenda Johnson, Christina Osborne, Emily Riehl, and Amelia Tebbe [7] shows that the Johnson-McCarthy calculus of chain complexes fulfills a chain rule up to homotopy. This strongly suggests that a certain bicategory of chain complexes admits a tangent structure that describes the Johnson-McCarthy calculus.

In addition to all these mathematical applications, the relevance of tangent categories goes beyond pure mathematics. Machine learning has been an envisioned application since the development of Cartesian differential categories in [9]. Cockett, Cruttwell, Jonathan Gallagher, Lemay, Gordon Plotkin and Dorette Pronk [30] further developed this perspective using reverse derivatives. In his PhD thesis [45], Bruno Gavranović gave an overview of the development so far and advanced this application even further, giving explicit implementation instructions on how to apply tangent categories in a machine learning context.

Due to the size and importance of differentiation, geometry and their categorical formulations, this section is necessarily incomplete and there exist many more contributions to the field. We hope, though, that this section conveys that tangent categories are a well-established and relevant area of mathematics.

1.1.2 Literature review on restriction categories

The fundamental questions on how to relate local and global properties of geometric objects, and indeed what local properties are at all, have been around for a long time.

A key step to understand the concept of locality was the development of topological spaces. In 1914, Felix Hausdorff, in his famous book "Grundzüge der Mengenlehre" [49], proposed a list of axioms to characterize open neighbourhoods. Spaces that fulfill this list of axioms are nowadays known as Hausdorff spaces. In 1924 Paul Alexandroff und Paul Urysohn further developed our modern notion of topological spaces in [3]. For a more expanded overview on the history of topology, we point the reader towards [17].

Using these definitions, a partial, or locally defined, function on a space X can be seen as a function whose domain is some closed (or open, there is a choice here) subset of the space. In 1978, Peter Booth and Ronald Brown [11] used this topological language to study partial maps.

Another reason why researchers studied partial maps was the theory of computation. Since Alan Turing's groundbreaking work [87] it is known that programs may or may not terminate and may or may not produce an output. Motivated by both the programming and the topology, E. Robinson and G. Rosolini abstracted the notion of partiality in [80]. Fully general formulations of restriction categories were given by Hans-Jürgen Hoehnke, who called them "fractal categories" in [52], and by Marco Grandis, who called them "e-cohesive categories" [46]. Significantly for this thesis, Grandis also developed the manifold construction for join restriction categories.

The formulation of restriction categories, which we will use in this thesis, was introduced by Cockett and Steve Lack in [24], [25], and [26]. Importantly, this formulation allows for a partiality-respecting notion of limits, called restriction limits. Coming from a perspective of Boolean logic, Cockett and Ernie Manes [27] developed the theory of joins in restriction categories, which was further expanded on in Xiuzhan Guo's PhD thesis [47]. This notion of joins of partial maps allows us to combine partial maps which are only locally defined into total maps, defined globally, or on a larger subset of the domain and thereby is a key step to understand the relation between local and global properties.

While joins allow us to construct morphisms out of their local pieces, the Grandis manifold construction first developed in [46] allows us to glue together objects into an object called the "gluing". This gluing and the manifold completion, which enables gluings, are combined with tangent categories in [20] where Cockett and Cruttwell establish a notion of tangent join restriction categories and show that tangent structure is preserved by the manifold completion. Recently, Lanfranchi [58] developed a notion of differential bundles in tangent restriction categories, combining the tangent category perspective and the restriction category perspective even further.

Restriction categories have major applications outside of geometry, in particular for the theory of computation which they were motivated by. Cockett and Peter Hofstra [28] developed Turing categories based on earlier work by Longo, Moggi, Di Paolo, and Heller. Turing categories are a particular case of Cartesian restriction categories where all morphism factor through a certain object, called a "Turing object". Turing categories are closely related to other structures in computer science, in particular partial combinatory algebras in the study of lambda calculus.

For the categorical study of probability theory and Bayesian networks, Kenta Cho, Bart Jacobs [19] and Tobias Fritz [42] developed Markov categories. Both Markov categories and Cartesian restriction categories are special cases of copy-discard categories and Elena Di Lavore and Marco Román [37] have combined these two structures to form partial Markov categories, a direction of research that makes it possible to model Bayesian networks with constraints.

Like the previous section, this section does not intend to be a complete report over all work in the area. We hope though, that it showcases that the study of restriction categories is also a well-established area of research with important applications.

1.2 Key questions answered in this thesis

This thesis continues the recent developments in categorical geometry. There are many perspectives in categorical geometry and classical geometry which all aim to study similar phenomena. We will connect some of these perspectives.

1.2.1 Dimensions in tangent bundles

In the definition of a tangent category there is a somewhat unintuitive condition called the universality of the vertical lift which requires a certain diagram to be a pullback.

In the category of smooth manifolds the universality of the vertical lift enforces how "large" the tangent bundle can be. This is made precise by the dimension of the manifolds fulfilling certain equations. For general tangent categories, however, this is less understood since there was no preexisting definition of dimension for them.

In order to align the dimension interpretation of the universality of the vertical lift with general tangent categories, we invent a notion of dimension for general categories. In particular we also use it for categories with no known tangent structure. This novel categorical dimension will allow us to interpret the universality of the vertical lift in many different tangent categories and to put constraints on possible structures. Theorem 3.2.7 shows that any tangent bundle needs to respect the equation

$$\dim(T^2(X)) + 2 \cdot \dim(X) = 3 \cdot \dim(T(X))$$

about a categorical dimension dim applied to the tangent bundle $T(X)$, the tangent bundle of the tangent bundle $T^2(X)$ and the underlying object X. If the dimension is a strong tangent dimension, as defined in Definition 3.2.5, Theorem 3.2.8, shows that $T(X)$ is the same or double the dimension of X.

Differential bundles are also required to respect dimension, as we show in Theorems 3.2.13 and 3.2.14. However, this condition becomes trivial if the tangent bundle endofunctor sends an object to an object of twice the dimension.

Categorical dimensions appear in many common mathematical situations. We show that dimensions are preserved by full subcategories and right-adjoints. Using this, we provide examples for sets, groups, rings, modules, algebras and topological spaces. We observe that dimensions are closely related to the concept of excision in homotopy theory. Since tangent structures need to respect dimension we can reject certain functors T as candidates for tangent structures, as showcased in Example 3.2.10. In particular, this allows us to classify all tangent bundle endofunctors on $\mathbb{S}\mathrm{et}^{\mathrm{op}}$ in Proposition 3.3.6 and $\mathbb{F}\mathrm{in}\mathbb{V}\mathrm{ect}$ in Theorem 3.6.11.

We find a categorical notion of dimension that generalizes the classical dimension of smooth manifolds and vector spaces. By providing constraints for possible structures, it helps us understand all the possible tangent bundles and differential bundles in a given category. In particular it is very easy to use dimensions to show that something is not a tangent structure or not a differential bundle. This is demonstrated in Example 3.2.10, we hope it will be helpful to exclude many more candidates for tangent structures in the future.

1.2.2 Differential bundles in the functorial perspective

As Leung showed in [64], tangent categories can be studied both from an axiomatic and from a functorial perspective. In [29], differential bundles were developed in the axiomatic perspective on tangent categories. Other constructions, in particular tangent infinity categories have been developed using the functor perspective on tangent categories. In order to connect these concepts, it is necessary to find a translation of differential bundles to the functor perspective on tangent categories.

We align Leung's functorial perspective on tangent categories with the study of differential bundles. For this purpose we study functors from a structure-category $\mathbb{N}^\bullet$. Certain functors from this structure category $\mathbb{N}^\bullet$ have been previously shown to characterize differential objects, which are more specific than differential bundles. We loosen certain requirements on the functors from $\mathbb{N}^\bullet$ in order to characterize the more general structure that are differential bundles.

We use differential functors which we introduce in Definition 4.3.3. Differential functors are functors between tangent categories that preserve certain pullbacks and come with a Cartesian natural transformation relating them to the tangent bundle endofunctor.

We show that differential functors send differential objects to differential bundles. Thus we obtain an evaluation functor `Eval` from the category of differential functors into $\mathbb{X}$ to the category of differential bundles in $\mathbb{X}$, summarized in Corollary 4.3.11. In Proposition 4.4.9 we construct for every differential bundle $E \to M$ in a category $\mathbb{X}$ a differential functor from the structure category $\mathbb{N}^\bullet$ in which all objects are differential objects. This differential functor produces the differential bundle $E \to M$ under evaluation. This leads to Theorem 4.4.10 where we define a functor `Ind` from a category of differential bundles to a category of differential functors. Indeed we show in Theorem 4.5.3 that the two functors `Eval` and `Ind` are quasi-inverse to each other and thus form an equivalence

$$\mathbb{D}\mathrm{Bun}_{\mathrm{add}}(\mathbb{X}) \cong \mathbb{D}\mathrm{iffFun}(\mathbb{N}^\bullet, \mathbb{X})$$

between the category $\mathbb{D}\mathrm{Bun}_{\mathrm{add}}(\mathbb{X})$ of differential bundles in $\mathbb{X}$ with additive morphisms and the category $\mathbb{D}\mathrm{iffFun}(\mathbb{N}^\bullet, \mathbb{X})$ of differential functors from $\mathbb{N}^\bullet$ to $\mathbb{X}$ and certain natural transformations as morphisms. Theorem 4.5.3 also includes a variant of the equivalence for linear morphisms.

Our classification of differential bundles using differential functors is a way to prove that something is a differential bundle. It allows us to express differential bundles in the functorial language developed by Leung [64] and Garner [44]. This formulation will be particularly useful for future study of higher categories, like the tangent infinity categories defined in [6]. Additionally, such a functorial characterization can be adapted more easily to categorical constructions, like Kleisli categories and opposites, helping to relate differential bundles in the opposite category to differential bundles in the original category in the future.

1.2.3 Group objects in tangent categories

Lie groups are important objects of study in classical differential geometry. Much is known about their tangent bundle. Their tangent space at the unit famously forms a Lie algebra.

In order to align the classical study of Lie Groups and principal bundles with more general categorical geometry, we first study group objects in tangent categories, and show that they enjoy similar properties to Lie groups in differential geometry.

We show in Theorem 5.3.3 that the total space $T(G)$ of the tangent bundle of a group object G in a tangent category is isomorphic to a product of a tangent space $T(G)_u$ and the group object

$$T(G) \cong G \times T(G)_u.$$

We can characterize all the structure morphisms that characterize the tangent bundle through their corresponding maps between the tangent spaces, as summarized in Table 5.1. In classical differential geometry the tangent space is known as the Lie group's Lie algebra. We show in Theorem 5.3.21 that in tangent categories, the tangent space obtains an external Lie algebra structure too.

Due to our study of group objects, we have explicit coordinates to talk about the tangent bundle of a group object. Additionally, we generalized the Lie bracket on the tangent space. Whenever we encounter a group object in a tangent category, we now have plenty of tools to study its behaviour.

1.2.4 Principal bundles in tangent categories

Symmetry and differentiation are closely related in differential geometry. Much is known about the behaviour of Lie groups, group objects in the category of smooth manifolds. Given a Lie-Group G Principal G-bundles fulfill several well-understood properties, in particular their close correspondence with vector bundles. In categorical geometry, differential bundles are the generalization of vector bundles, though neither principal bundles nor their relation to differential bundles have been generalized yet. Since principal bundles are defined using local conditions and join restriction categories are a tool to encode locality, we study principal bundles in join restriction categories. Combining these two setups we can obtain strong results in tangent restriction categories.

In order to generalize principal G-bundles, we define fibre bundles, G-bundles and principal G-bundles in join restriction categories in Definitions 5.4.1 and 5.4.14, replacing charts by partial isomorphisms. The bundles we define indeed generalize the principal bundles from differential geometry and topology, as we see in Theorem 5.4.27 and Proposition 5.4.35. We continue to study their properties. Using constructions similar to the ones in Section 5.3, we show that categorical principal bundles in tangent join restriction categories admit a total right G-action

$$r : P \times G \to P$$

as shown in Theorem 5.4.40, that their vertical bundle is trivial in the sense of Theorem 5.4.43, and they fulfill the torsor-property of Theorem 5.4.42.

Finally, we study which parts of the equivalence between principal and vector bundles from classical differential geometry hold in the more general setup of Cartesian tangent join restriction categories with gluings. Classically, for smooth manifolds, there is a correspondence between isomorphism classes of principal GL_n-bundles and isomorphism classes of vector bundles with n-dimensional fibre. We show in Theorem 6.3.12 that in any Cartesian tangent join restriction category with gluings one can construct differential bundles out of principal G-bundles and this is in fact a functor

$$\mathbb{P}\mathrm{Bun}_G^{\mathrm{ep,tot}} \times G - \mathbb{O}\mathrm{bj}^{\mathrm{lin}} \to \mathbb{D}\mathrm{Bun}^{\mathrm{lin,ep,tot}}$$

sending principal G-bundles and objects with a G-action to differential bundles. So one direction of the correspondence holds in this more general setup. However, the inverse does not, in general, exist. Example 6.3.13 includes an example of a differential bundle which can not come from a G-bundle.

By defining principal G-bundles in Cartesian tangent join restriction categories, we unify existing concepts from differential geometry, point-set topology and we conjecture it also unifies them with the notion of principal bundle from algebraic geometry. Besides these already existing notions of principal bundles, it also makes it possible to study other categorical contexts and the principal bundles therein in the future. Additionally, our construction of differential bundles from principal bundles relates these two kinds of bundles in a general context. This construction not only clarifies which parts of geometric results hold more generally, it also makes it possible to generate examples of differential bundles by constructing them out of principal bundles.

1.2.5 Overall summary

All the work in this thesis improves our understanding of bundles in category theory, aligning different perspectives from differential geometry and category theory.

This makes it possible to apply the geometric technique, which is the study of bundles, to areas of math that are not traditionally considered as part of geometry. In particular the theory of computation

(Turing categories, a special case of restriction categories), the theory of Bayesian networks (partial Markov categories, closely related to restriction categories), and machine learning (reverse derivative categories, a special case of tangent categories) all admit categorical descriptions that are closely related to the restriction categories studied here. Thus we hope that the ideas which hold in restriction categories can be applied to the setups similar to restriction categories and the results that hold in tangent categories can be applied to the setups similar to tangent categories.

1.3 Novel contributions in this thesis

While the concepts of tangent categories and differential bundles were already known before, the notion of a monoid-valued dimension and its variations (Definitions 3.2.2 and 3.2.5) are new contributions. Any results in Chapter 3 involving dimension are novel, including the classification of Cartesian tangent bundle endofunctors on $\mathbb{F}\mathrm{inSet}^{\mathrm{op}}$ and $\mathbb{F}\mathrm{inVect}$. However, almost all the tangent structures that are used in examples were previously known in the literature. The only new tangent structure is in Example 3.6.1, which is closely related to the example presented in [81, Example 1].

In Chapter 4, the concept of a (general/lax) differential functor and all the results depending on it are novel. The characterization of differential bundles is terms of (general/lax) differential functors is a new contribution. This generalizes what was known about strong differential functors. The classification of differential objects using strong differential functors was proved in [6]. This is a strong version of what we prove in Chapter 4. Theorem 4.2.18, which proves the euquivalence between tangent categories and and certain actegories, is a slight generalization of [44, Theorem 9] using a proof that is very similar to the proof in [44, Theorem 9].

In Chapter 5, the concepts introduced in Section 5.2 are a review of the theory of restriction categories, in particular restriction limits and the Grandis manifold construction. Lemma 5.2.8 is an novel result which gives an easier way to identify restriction pullbacks of total maps. Group objects in category theory are well known in the literature. The properties of their tangent space in a tangent category, described in Section 5.3 are novel in this generality.

Chapter 6 closely resembles the explicit formula for the associated bundle in classical differential geometry. Definitions 6.2.3 and 6.3.1 are taken from the literature, but the remaining results, in particular all results pertaining to principal bundles, are novel contributions.

1.4 Structure of this thesis

This is a manuscript style thesis. The main body consists of three papers that have already been made public as [31], [82] and [83]. The papers are presented in reverse chronological order. This way the technically most accessible work comes first and the work using more involved techniques comes later.

In Chapter 2 we define tangent categories and principal bundles, concepts that will be of key relevance throughout the document, providing ample examples. Chapter 3 is based on [83] and generalizes the dimension of smooth manifolds to arbitrary categories in order to provide an interpretation for the universality of the vertical lift. Chapter 4 is based on [82] and connects the axiomatic definition of differential bundles, as given in [29] with the functor perspective on tangent categories developed in [64], [44]. For this purpose we show that the category of differential bundles in a given tangent category is equivalent to a category of certain functors. Chapter 5 is based on [31] and studies group objects and principal bundles. It first studies the behaviour of group objects in tangent categories, in particular their tangent bundles and tangent spaces, developing an external Lie Algebra structure on them. Then it continues to develop the theory of fibre bundles, G-bundles and principal G-bundles in Cartesian join restriction categories and showcase their behaviour in Cartesian tangent join restriction categories. Chapter 6 relates principal bundles, as defined in Chapter 5, to differential bundles, which were studied in Chapters 3 and which were the main focus of study in Chapter 4.

Different parts of this thesis have been written for different audiences. Thus Chapters 2, 3, and 4 use applicative order of composition. Given morphisms $f : A \to B$ and $g : B \to C$, their composition is denoted $g \circ f : A \to C$ in these chapters. However, Chapters 5 and 6 use diagrammatic order of composition. Given morphisms $f : A \to B$ and $g : B \to C$, their composition is denoted $fg = f; g : A \to C$ in these chapters.

Chapter 2

Background on tangent categories

$$
\begin{tikzcd}
 & T(M) \times_M T(M) \arrow[d, "+"] & \\
M \arrow[r, bend left, "0"] & T(M) \arrow[l, bend left, "p"] \arrow[r, bend left, "\ell"] & T(T(M)) \arrow[l, bend left, "T(p)"] \arrow[loop right, "c"]
\end{tikzcd}
$$

Figure 2.1: An overview over the structures comprising a tangent category, defined formally in Definition 2.2.4.

Tangent categories and differential bundles generalizes the concept of smooth manifolds, with $T(M)$ as the tangent bundles and vector bundles. . This chapter serves to introduce these concepts by uniting the background sections of [83], [82] and [31]. Many examples which will be relevant in later chapters will be included.

This chapter uses applicative order of composition. The composition of $f : A \to B$ and $g : B \to C$ is denoted by $g \circ f : A \to C$. This is consistent with the notation used in [6], [44], [64] and [81] and inconsistent with [29], [20] and [32].

In this chapter, we will review the definitions of tangent categories and differential bundles as stated by Cockett and Cruttwell in [20] and [29]. We will also define the appropriate notion of functors between tangent categories and natural transformations between such functors.

We introduce several important examples of tangent categories: $\mathbb{S}$mMan and $\mathbb{N}^\bullet$ will be extremely important throughout this thesis. $\mathbb{S}$mMan is important because it recovers classical differential geometry. The category $\mathbb{N}^\bullet$ is important because it will be used to characterize differential bundles and objects in an arbitrary tangent category in Sections 4.4 and 4.5.

2.1 Notations for pullbacks

Notation 2.1.1. *As a convention, we will use throughout this document that, given a pullback or, in particular, a product, the projections onto the first and second components will be called π_0 and π_1, respectively. More generally, the i-th projection out of a n-fold pullback will be π_{i-1}. The morphism to the pullback of f and g induced by the universal property will be denoted by $\langle f, g\rangle$:*

$$\begin{tikzcd}
Z \arrow[dr, dashed, "\langle f,g\rangle"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "g"'] & & \\
& A \times_B C \arrow[r, "\pi_0"'] \arrow[d, "\pi_1"] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d] \\
& C \arrow[r] & B
\end{tikzcd}$$

In the case that $Z = D \times_{B'} E$ is another pullback, we will denote (for $h : D \to A$ and $k : E \to C$ over the same map $B' \to B$)

$$h \times_B k := \langle h \circ \pi_0, k \circ \pi_1\rangle : D \times_{B'} E \to A \times_B C.$$

The most relevant case happens when B and B' are the terminal object and the pullback is therefore a product: in this case, $f \times g := \langle f \circ \pi_0, g \circ \pi_1\rangle$. Note that we are using applicable order for composition here: $f \circ \pi_0$ is the composite of the projection π_0 followed by f. In the case where $f = g$, we use a subscript to denote the pullback of f along itself:

$$\begin{tikzcd}
A_2 = A \times_B A \arrow[r, "\pi_0"'] \arrow[d, "\pi_1"] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\
A \arrow[r, "f"] & B
\end{tikzcd}$$

Likewise, A_n denotes the n-fold pullback of f along itself.

2.2 Tangent categories

This section aims to recall notions from [20], namely tangent categories, lax/strong tangent functors between tangent categories and (linear) tangent transformations between tangent functors. Tangent categories were originally defined by Rosický in [81] in a slightly different way; the formulation we use is the one from [20]. We will start by noting the observation about smooth manifolds that leads to the axiomatic definition of tangent categories.

Remark 2.2.1. *Here we recall some facts about tangent spaces and tangent bundles. They are basic facts available in most differential geometry textbooks, for example in* [71, Chapter I, Sections 1.8-1.11, 6.2]*. Given a smooth manifold X and a point $x \in X$, the tangent space T_xX is, intuitively speaking, the vector space of all directional derivatives.*

A key property of the category of smooth manifolds is that for every smooth manifold X there is another smooth manifold TX which is the tangent bundle of X. The tangent bundle has as its underlying set the disjoint union of tangent spaces at all points of the manifold, though with a topology and manifold structure steming from the manifold structure of X.

- *The tangent bundle comes equipped with a* ***projection*** *$p_X : TX \to X$ that sends a tangent vector $v_x \in T_xX$ at the point $x \in X$ to the point x.*

Given a smooth map $f : X \to Y$ between smooth manifolds, there is a smooth map $Tf : TX \to TY$ which sends a tangent vector to the Jacobi matrix multiplied with the tangent vector. In fact this assignment of objects and morphisms is an endofunctor $T : \mathbb{S}\mathrm{mMan} \to \mathbb{S}\mathrm{mMan}$, which we call the ***tangent bundle endofunctor****.*

Tangent vectors at any point form a vector space. The addition and zero of this vector space give rise to certain smooth maps.

- *There is a* ***zero section*** *map $0_X : X \to TX$ that sends a point $x \in X$ to the zero tangent vector $0_x \in T_xX$ at the point $x \in X$. We call it a zero section since the composition with the projection $p_X \circ 0_X = 1_X$ is the identity.*
- *There is an* ***addition*** *map $+_X : T_2X = TX \times_X TX \to TX$ that takes two tangent vectors $v_x, w_x \in TX$ which are over the same basepoint x to their sum $v_x + w_x$.*

In addition to these well-known structures there are some less ubiquitous structures on the tangent bundle: the vertical lift and the canonical flip, both relating to $T^2M = TTM$, the tangent bundle of the tangent bundle. We emphasize them here since they are essential for the axiomatic definition of tangent categories.

- *The vertical lift $\ell : TX \to TTX$ is defined for example at the end of* [71, Section 6.12].

- *The canonical flip $c : TTX \to TTX$ is defined for example in* [71, Section 6.13].

We give more details on the vertical lift and the canonical flip of smooth manifolds in Example 2.2.7.

If we want to have an axiomatic approach to differential geometry, we need these structures.

Definition 2.2.2. [20, Definition 2.1]*Let $\mathbb{X}$ be a category and $A \in \mathrm{Ob}(\mathbb{X})$ an object. An **additive bundle** $(X, A, +, 0, p)$ **over** A consists of the following data:*

- *an object X and a morphism $p : X \to A$ such that pullback powers X_n of p exist, and*

- *morphisms $+ : X_2 \to X$ and $0 : A \to X$.*

In addition, the morphisms must satisfy the following conditions:

- *$p \circ + = p \circ \pi_0 = p \circ \pi_1$ and $p \circ 0 = 1_A$,*

- *the morphism $+$ must be associative, commutative and unital.*

That is, the following associativity, commutativity and unitality diagrams

$$\begin{tikzcd}
X_3 \arrow[r, "1 \times_A +"] \arrow[d, "+ \times_A 1"'] & X_2 \arrow[d, "+"] \\
X_2 \arrow[r, "+"] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
X_2 \arrow[d, "\langle \pi_1, \pi_0 \rangle"'] \arrow[dr, "+"] & \\
X_2 \arrow[r, "+"] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[d, "\langle 0 \circ p, 1_X \rangle"'] \arrow[dr, "1_X"] & \\
X_2 \arrow[r, "+"] & X
\end{tikzcd}$$

must commute.

A morphism of additive bundles consists of morphisms that commute with these structures, as explained in the following definition.

Definition 2.2.3. [20, Definition 2.2]*For two additive bundles $(X, A, p, +, 0)$ and $(Y, B, p', +', 0')$ an **additive bundle morphism** (f, g) is a pair of maps $f : X \to Y$ and $g : A \to B$ such that the following diagrams commute:*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "p"'] & Y \arrow[d, "p'"] \\
A \arrow[r, "g"] & B
\end{tikzcd}
\quad
\begin{tikzcd}
X_2 \arrow[r, "\langle f \circ \pi_0, f \circ \pi_1 \rangle"] \arrow[d, "+"'] & Y_2 \arrow[d, "+'"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}
\quad
\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, "0"'] & B \arrow[d, "0'"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

Additive bundles are a categorification of the basic structure underlying vector spaces. This will be helpful in defining tangent categories, since each tangent space in the classical case is a vector space. In the following definition, beware of the notation T_2 (which refers to a pullback) and T^2 (which refers to two applications of the functor T). For any category $\mathbb{X}$, let $1 : \mathbb{X} \to \mathbb{X}$ denote the identity functor.

Definition 2.2.4. [20, Definition 2.3] *A **tangent category** $(\mathbb{X}, T, p, 0, +, \ell, c)$ consists of a category $\mathbb{X}$ and the following structures:*

- *a functor $T : \mathbb{X} \to \mathbb{X}$, called the tangent bundle functor with a natural transformation $p : T \to 1$ such that pullback powers of $p_M : TM \to M$ exist for all objects $M \in \mathbb{X}$ and T preserves these pullback powers;*
- *natural transformations $+ : T_2 \to T$ and $0 : 1 \to T$ making each $(TM, M, p_M, +_M, 0_M)$ an additive bundle;*
- *a natural transformation $\ell : T \to T^2$, called the vertical lift, such that $(\ell_M, 0_M)$ is an additive bundle morphism from $(TM, M, p_M, +_M, 0_M)$ to $(T^2M, TM, Tp_M, T(+_M), T(0_M))$;*
- *a natural transformation $c : T^2 \to T^2$ with $c^2 = 1$ and $lc = l$, such that $(c_M, 1)$ is an additive bundlem morphism from $(T^2M, TM, Tp_M, T(+_M), T(0_M))$ to $(T^2M, TM, p_{TM}, +_{TM}, 0_{TM})$.*

*We will refer to the pullbacks T^nT_k as **foundational pullbacks**.*

In addition, ℓ and c are required to be compatible with themselves and with each other, meaning that the diagrams

$$
\begin{tikzcd}
TM \arrow[r, "\ell"] \arrow[d, "\ell"'] & T^2M \arrow[d, "T(\ell)"] \\
T^2M \arrow[r, "\ell_{TM}"'] & T^3M
\end{tikzcd}
\qquad
\begin{tikzcd}
T^3M \arrow[r, "T(c)"] \arrow[d, "c_{TM}"'] & T^3M \arrow[r, "c_{TM}"] & T^3M \arrow[d, "T(c)"] \\
T^3M \arrow[r, "T(c)"'] & T^3M \arrow[r, "c_{TM}"'] & T^3M
\end{tikzcd}
\qquad
\begin{tikzcd}
T^2M \arrow[r, "\ell_{TM}"] \arrow[d, "c"'] & T^3M \arrow[r, "T(c)"] & T^3M \arrow[d, "c_{TM}"] \\
T^2M \arrow[rr, "T(\ell)"'] & & T^3M
\end{tikzcd}
$$

all commute. Finally, we require that

$$
\begin{tikzcd}
T_2M \arrow[r, "\nu"] \arrow[d, "p_M \circ \pi_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & T^2M \arrow[d, "T(p_M)"] \\
M \arrow[r, "0_M"'] & TM
\end{tikzcd}
$$

*is a pullback preserved by powers of T, where ν is a shorthand notation for $\nu = T(+) \circ \langle \ell_M \circ \pi_0, 0_{TM} \circ \pi_1 \rangle$. This pullback is denoted as the **universality of the vertical lift**. The foundational pullbacks together with powers of T applied to the universality of the vertical lift will be called the **tangent pullbacks**.*

While the last condition is formulated differently in Definition 2.3 of [20], it is shown to be equivalent to the formulation here in Lemma 2.12 of [20].

Remark 2.2.5. *The last condition is called the universality of the vertical lift and it encodes the intuition that T^2M relates to T_2M in the same way that TM relates to M. For example, in the category* $\mathbb{S}$mMan *of* *manifolds and smooth maps (see Example 2.2.7), each object has a dimenion. As we will show in Chapter 3, the vertical lift ensures that* $\dim(T^2M) - \dim(T_2M) = \dim(TM) - \dim(M)$. *In particular if the dimension of the tangent bundle is a multiple by n of the dimension of the manifold, this becomes* $n^2 - (2n-1) = n-1$, *which only is the case when* $n = 1$ *or* $n = 2$. *Thus the universality of the vertical lift in* $\mathbb{S}\mathrm{mMan}$ *enforces the idea that the dimension of the tangent bundle is either the same or twice the dimension of the original manifold. We will explore the relationship between the vertical lift and the dimension of the tangent bundle in general tangent categories in Chapter 3.*

The structures involved in Definition 2.2.4 of tangent categories can be summarized as a diagram:

$$\begin{tikzcd}
 & T_2(M) \arrow[d, "+"] & \\
M \arrow[r, shift left, "0"] & T(M) \arrow[l, shift left, "p"] \arrow[r, shift left, "\ell"] & T^2(M) \arrow[l, shift left, "T(p)"] \arrow[loop above, "c"]
\end{tikzcd}$$

for every object M.

In general there can be many tangent structures for a given category $\mathbb{X}$. In particular any category with a nontrivial tangent structure $T \neq 1_{\mathbb{X}}$ has at least two tangent structures, T and the trivial tangent structure given by the identity functor.

Example 2.2.6. [20, Section 2.2] *Any category* $\mathbb{X}$ *with the identity functor and identity transformations for all of the natural transformations required by Definition 2.2.4 give an example of a tangent category,* $(\mathbb{X}, 1_{\mathbb{X}}, 1_{1_{\mathbb{X}}}, 1_{1_{\mathbb{X}}}, 1_{1_{\mathbb{X}}}, 1_{1_{\mathbb{X}}}, 1_{1_{\mathbb{X}}})$.

The main motivating example for the definition are smooth manifolds.

Example 2.2.7. [20, Section 2.2] *Let* $\mathbb{S}\mathrm{mMan}$ *be the category with smooth manifolds as objects and smooth maps between them as morphisms. Then* $\mathbb{S}\mathrm{mMan}$ *is a tangent category where the tangent functor is given by forming the usual tangent bundle from differential geometry,* $T(M) := TM$. *Elements of the tangent bundle are denoted by* (p, v_p), *where* p *is a point on the manifold and* v_p *is a vector in the tangent space* T_pM.

- *The projection* $p : TM \to M$,
- *the zero section* $0 : M \to TM$, *and*
- *the addition of tangent vectors* $+ : T_2M \to TM$

are well-known from differential geometry, for example defined in [71, Section 6.2].

Elements of $T^2(M) = T(T(M))$ *the tangent bundle of the tangent bundle are denoted by a 4-tuple* (p, v_p, u_p, w_{p,v_p}) *where* $p \in M$, $v_p \in T_pM$, $u_p \in T_pM$ *and* $w_{p,v_p} \in T_{v_p}T_pM$, *which implies* $(u_p, w_{p,v_p}) \in T_{(p,v_p)}TM$.

- *The vertical lift $\ell : TM \to TTM$ is less frequently used in differential geometry, it is defined for example at the end of* [71, Section 6.12]. *It sends an element (p, v) of the tangent bundle to $(p, 0_p, 0_p, v_{p,0_p})$. Here we use the canonical identification of the tangent space $T_{v_p}T_pM$ of the vector space T_pM with itself.*

- *The canonical flip sends an element (p, u_p, v_p, w) of TTM to the element (p, v_p, u_p, w) of TTM which is possible as both u_p and v_p are in T_pM and $T_{v_p}T_pM$ is canonically identified with $T_{u_p}T_pM$. While less ubiquitous than other structures, it can be found for example in* [71, Section 6.13].

Every manifold of dimension n is locally diffeomorphic to a subset of $\mathbb{R}^n$, see [71], *and using this equivalent formulation we can express points in the tangent bundle in local coordinates. In local coordinates, on an open neighborhood $U \subset M$, the tangent bundle $T(M)$ is isomorphic to a product of the base manifold with a vector space*

$$T(U) \cong U \times \mathbb{R}^n$$

where n is the dimension of M. Here, we use U to denote both the open neighborhood of M and its diffeomorphic image in $\mathbb{R}^n$. For mandifolds M_1, M_2 with dimensions n_1, n_2 and a smooth map $f : M_1 \to M_2$, there are open neighborhoods $U_1 \subset M_1$ and $U_2 \subset M_2$ with $f(U_1) \subset U_2$ such that $T(U_1) \cong U_1 \times \mathbb{R}^{n_1}$ and $T(U_2) \cong U_2 \times \mathbb{R}^{n_2}$. The evaluation $T(f) : T(M_1) \to T(M_2)$ of the tangent bundle functor on the morphism f can also be expressed in these local coordinates as

$$\begin{aligned} T(f)|_{T(U_1)} &: T(U_1) \to T(U_2) \\ T(U_1) \cong U_1 \times \mathbb{R}^{n_1} \ni (x, v) &\mapsto \left(f(x), \frac{\partial f(y)}{\partial y}|_{y=x} \cdot v \right) \in U_2 \times \mathbb{R}^{n_2} \cong T(U_2). \end{aligned}$$

This formulation of $T(f)$ in terms of local coordinates is standard knowledge in differential geometry and can, for example, be found in [71, Section 1.11].

Unlike classical manifolds, tangent categories do not need to be related to the real numbers. In the following example we construct a tangent category using any field $\mathbb{F}$.

Example 2.2.8 (Example 2.7 in [53]). *Let $\mathbb{F}$ be a field. Then we define a category $\mathrm{Poly}_{\mathbb{F}}$ whose objects are the finite-dimensional $\mathbb{F}$-vector spaces, $\mathbb{F}^k$. Morphisms $g : \mathbb{F}^n \to \mathbb{F}^k$ are k-tuples of of polynomials in n variables , for example,*

$$g : \mathbb{F}^3 \to \mathbb{F}^2 \qquad (x, y, z) \mapsto (x^2 z - 1, y^3 z + 2x).$$

For a morphism f let $J_f : \vec{x} \mapsto (\sum_j \frac{\partial f^i}{\partial x_j} x_j)_{1 \leq i \leq k}$ be the linear map given by the Jacobi matrix of f (where $\frac{\partial f^i}{\partial x_j}$ denotes the derivative of a polynomial defined by $\frac{\partial x_i^k}{\partial x_i} = k \cdot x_i^{k-1}$). The category $\mathrm{Poly}_{\mathbb{F}}$ has a tangent *structure, given by*

- *The tangent bundle endofunctor*

$$T(\mathbb{F}^k) := \mathbb{F}^k \times \mathbb{F}^k, \qquad T(f) := f \times J_f),$$

- *the projection* $p_{\mathbb{F}^k} := \pi_0 : \mathbb{F}^k \times \mathbb{F}^k \to \mathbb{F}^k$*,*
- *the zero section* $0_{\mathbb{F}^k} := \langle 1_{\mathbb{F}^k}, 0 \rangle : \mathbb{F}^k \to \mathbb{F}^k \times \mathbb{F}^k$*,*
- *the addition* $+_{\mathbb{F}^k} := \langle \pi_0, \mathrm{add} \circ \langle \pi_1, \pi_2 \rangle \rangle : \mathbb{F}^k \times \mathbb{F}^k \times \mathbb{F}^k \to \mathbb{F}^k \times \mathbb{F}^k$ *where* $\mathrm{add} : \mathbb{F}^k \times \mathbb{F}^k \to \mathbb{F}^k$ *is the addition of vectors,*
- *the vertical lift* $\ell_{\mathbb{F}^k} := \langle \pi_0, 0, 0, \pi_1 \rangle : \mathbb{F}^k \times \mathbb{F}^k \to \mathbb{F}^k \times \mathbb{F}^k \times \mathbb{F}^k \times \mathbb{F}^k$*, and*
- *the canonical flip* $c_{\mathbb{F}^k} := \langle \pi_0, \pi_2, \pi_1, \pi_3 \rangle : \mathbb{F}^k \times \mathbb{F}^k \times \mathbb{F}^k \times \mathbb{F}^k \to \mathbb{F}^k \times \mathbb{F}^k \times \mathbb{F}^k \times \mathbb{F}^k$*.*

This example is useful because finite fields are a well-studied area of mathematics. In Remark 2.3.7 we use this to show that not every additive morphism of differential bundles is linear.

A closely related example is the category $\mathbb{S}\mathrm{mooth}$ in which the objects are all object $\mathbb{R}^k$ and the morphisms are all smooth maps. The same structure maps defined here (with $\mathbb{F} = \mathbb{R}$) give rise to a tangent structure on $\mathbb{S}\mathrm{mooth}$.

Definition 2.2.9. [6, proof of Proposition 5.8] *Let* $\mathbb{N}^\bullet$ *be the category with objects* $\mathbb{N}^k$ *for* $k \in \mathbb{N}$ *and morphisms from* $\mathbb{N}^m$ *to* $\mathbb{N}^n$ *given by matrices* $f = (f_{ij})_{1 \le i \le n, 1 \le j \le m}$ *where* $f_{ij} \in \mathbb{N}$ *for all* i*,* j*. The composition is given by matrix multiplication and the identity matrix is the identity.*

In the following we will use Definition 2.2.9 as definition for the category $\mathbb{N}^\bullet$. Other work, like [6], defines it as the category of free finitely generated free commutative monoids, which we call $\mathbb{F}\mathrm{reeCommMon}$ here. These definitions are equivalent since the category $\mathbb{N}^\bullet$ is the skeleton of $\mathbb{F}\mathrm{reeCommMon}$, the category of free finitely generated commutative monoids, and monoid homomorphisms. The category $\mathbb{F}\mathrm{reeCommMon}$ is used in [6] to characterize infinity tangent categories.

Proposition 2.2.10. *The category* $\mathbb{N}^\bullet$ *is a skeletal subcategory of the category* $\mathbb{F}\mathrm{reeCommMon}$ *of free finitely generated commutative monoids, i.e. the inclusion functor* $\mathrm{I} : \mathbb{N}^\bullet \to \mathbb{F}\mathrm{reeCommMon}$ *is an equivalence.*

The proof is analogous to the matrix representation theorem in basic linear algebra, except that we replace $\mathbb{R}$ vector spaces with free finitely-generated commutative monoids.

Proof. Every object $\mathbb{N}^k$ in $\mathbb{N}^\bullet$ is a free finitely generated commutative monoid and every matrix describes a monoid-homomorphism, therefore $\mathbb{N}^\bullet$ is a subcategory of $\mathbb{F}$reeCommMon. Any two objects that are isomorphic in $\mathbb{N}^\bullet$ are in fact the same, therefore $\mathbb{N}^\bullet$ is skeletal.

The inclusion functor $I : \mathbb{N}^\bullet \to \mathbb{F}\text{reeCommMon}$ is essentially surjective because every free finitely-generated commutative monoid is isomorphic to $\mathbb{N}^k$ for some k.

Every monoid-morphism $f : \mathbb{N}^k \to \mathbb{N}^m$ can be split up into its components $(f_i : \mathbb{N}^k \to \mathbb{N})_{1\leq i\leq m}$ because it is a map into a product. Since each component is a monoid morphism,

$$f_i(n_1, ..., n_k) = n_1 f_i(1, 0, ..., 0) + n_2 f_i(0, 1, 0, ..., 0) + ... + n_k f_i(0, ..., 0, 1).$$

We now define the number $f_{ij} := f_i(0, ..., 0, 1, 0, ..., 0)$ where the j-th component is the unique non-zero component of the vector. We see that $f_i(n_1, ..., n_k) = \sum_{1\leq j\leq k} f_{ij} n_j$. Thus every monoid-morphism is exactly given by a matrix and therefore $I : \mathbb{N}^\bullet \to \mathbb{F}\text{reeCommMon}$ is fully faithful. This shows it is an equivalence. $\square$

Example 2.2.11. *The category $\mathbb{N}^\bullet$ has products and the product structure is given by $\mathbb{N}^k \times \mathbb{N}^l = \mathbb{N}^{k+l}$. There is an addition map $\mathrm{add} : \mathbb{N}^k \times \mathbb{N}^k \to \mathbb{N}^k$ given by $(\mathbf{1}_{k\times k}|\mathbf{1}_{k\times k})$, where $\mathbf{1}_{k\times k}$ is the $k \times k$ identity matrix. There is a tangent structure on $\mathbb{N}^\bullet$ with a tangent functor $D : \mathbb{N}^\bullet \to \mathbb{N}^\bullet$ and structures defined by*

(i) the tangent bundle endofunctor $D(\mathbb{N}^k) = \mathbb{N}^k \times \mathbb{N}^k = \mathbb{N}^{2k}, D(f) = f \times f$ doubling the dimension,

(ii) the projection $p := \pi_0 : \mathbb{N}^k \times \mathbb{N}^k \to \mathbb{N}^k$ projecting to the first component, which is the component holding the base point,

(iii) the zero section $0 := \langle 1^k_{\mathbb{N}}, 0\rangle : \mathbb{N}^k \to \mathbb{N}^k \times \mathbb{N}^k$ inserting the base point into the first component and zero into the second,

(iv) the addition $+ := \langle \pi_0, \mathrm{add} \circ \langle \pi_1, \pi_2\rangle\rangle : \mathbb{N}^k \times \mathbb{N}^k \times \mathbb{N}^k \to \mathbb{N}^k \times \mathbb{N}^k$ sending $(\vec{u}, \vec{v}, \vec{w}) \mapsto (\vec{u}, \vec{v} + \vec{w})$,

(v) the vertical lift $\ell := \langle \pi_0, 0, 0, \pi_1\rangle : (\mathbb{N}^k)^2 \to (\mathbb{N}^k)^4$ inserting zeros in the two middle components, i.e. $(\vec{v}, \vec{w}) \mapsto (\vec{v}, 0, 0, \vec{w})$, and

(vi) the canonical flip $c := \langle \pi_0, \pi_2, \pi_1, \pi_3\rangle : (\mathbb{N}^k)^4 \to (\mathbb{N}^k)^4$ switching the two middle components, i.e. $(\vec{t}, \vec{u}, \vec{v}, \vec{w}) \mapsto (\vec{t}, \vec{v}, \vec{u}, \vec{w})$.

Note that in these definitions, $D^2(A) = A \times A \times A \times A$ and $D_2(A) = A \times A \times A$. It was proven in Proposition 5.18 of [6] *that $(\mathbb{N}^\bullet, D)$ is a tangent category.*

Example 2.2.12. *A more involved – but important – example is the category of **Weil-algebras**,* $\mathbb{W}$eil. *The key building block of this category is the polynomial algebra* $W := \mathbb{N}[x]/\langle x^2\rangle$ *called the dual numbers. We define product powers* $W^n := \mathbb{N}[x_1, ..., x_n]/\langle x_i x_j | i, j \in \{1, ..., n\}\rangle$ *(with* $W^0 := \mathbb{N}$*) and coproducts* $W^n \otimes W^k := \mathbb{N}[x_1, ..., x_n, y_1, ..., y_k]/\langle x_i^2, x_i x_j, y_i^2, y_i y_j\rangle$*. Of particular significance is the category* $\mathbb{W}$eil *whose objects are coproducts of product-powers of* W*. Morphisms are algebra-maps that preserve the constant terms (or augmentation) . More details on* $\mathbb{W}$eil *can be found in* [64]*, where it is called* $\mathbb{W}\mathrm{eil}_1$*.*

The category $\mathbb{W}$eil *has a tangent structure given by the following:*

(i) $T(A) := A \otimes W, T(f) := f \otimes 1_W$*, the tangent structure is given by tensoring with the dual numbers.*

(ii) $p := 1_A \otimes \hat{p} : A \otimes W \to A \otimes \mathbb{N} = A$ *is given by the identity of* A *and the projection* $\hat{p} : W \to N, a + bx \mapsto a$ *that is the projection to the constant term of a polynomial.*

(iii) $0 := 1_A \otimes \hat{0} : A \to$ *is given by the identity of* A *and the projection* $\hat{0}$ *where* $\hat{0} : \mathbb{N} \to W$ *sends the natural number* n *to the polynomial* $n + 0x$*.*

(iv) $\ell := 1_A \otimes \hat{\ell} : A \otimes W \to A \otimes W \otimes W$ *is given by the identity on* A *and the map* $\hat{\ell} : W \to W \otimes W; a + bx \mapsto a + bxy$ *(where* $W \otimes W = \mathbb{N}[x, y]/\langle x^2, y^2\rangle$*).*

(v) $c := 1_A \otimes \hat{c} : A \otimes W \otimes W \to A \otimes W \otimes W$ *is given by the identity of* A *and the map* $\hat{c} : W \otimes W \mapsto W \otimes W; a + bx + cy + dxy \mapsto a + by + cx + dxy$*.*

In [64] it is shown that a tangent category can equivalently be defined as a category $\mathbb{X}$ with a strong monoidal functor $\hat{T} : \mathbb{W}\mathrm{eil} \to \mathrm{End}(\mathbb{X})$ preserving certain pullbacks. We will revisit the details of this correspondence in Section 4.2.

Other important tangent structures are the known tangent structures on $\mathbb{C}\mathrm{Ring}_u$ (unital rings) and on $\mathbb{C}\mathrm{Ring}_u^{\mathrm{op}}$.

Examples 2.2.13. *Let* $\mathbb{C}\mathrm{Ring}_u$ *be the category of commutative unital rings with unit-preserving ring-homomorphisms.*

(i) As outlined in [81, Example 2]*, there is a tangent structure on* $\mathbb{C}\mathrm{Ring}_u$*, which is given by the following data.*

- *The tangent bundle endofunctor* T *sends a ring* R *to* $\frac{R[x]}{\langle x^2\rangle}$*.*
- *The projection* $p_R : \frac{R[x]}{\langle x^2\rangle} \to R$ *is the evaluation of* x *as zero.*
- *The zero section is the inclusion* $0_R : R \to \frac{R[x]}{\langle x^2\rangle}$*.*
- *The addition* $+_R : \frac{R[x,y]}{\langle x^2, y^2, xy\rangle} \to \frac{R[x]}{\langle x^2\rangle}$ *sends* y *to* x*.*

- *The vertical lift* $\ell_R : \frac{R[x]}{\langle x^2 \rangle} \to \frac{R[x,y]}{\langle x^2, y^2 \rangle}$ *sends* x *to* xy.
- *The canonical flip* $c_R : \frac{R[x,y]}{\langle x^2, y^2 \rangle} \to \frac{R[x,y]}{\langle x^2, y^2 \rangle}$ *sends* x *to* y *and* y *to* x.

(ii) As shown in [34], *there is also a tangent structure on* $\mathbb{C}\mathrm{Ring}_u^{\mathrm{op}}$, *which is given by the following data.*

- *The tangent bundle endofunctor sends a commutative ring* R *to the free symmetric* R*-algebra over its modules of Kähler differentials. Explicitly that means it is the free algebra generated by* $\{d(a) | a \in R\}$ *modulo the equations*

$$d(1) = 0, \qquad d(a+b) = d(a) + d(b), \text{ and} \qquad d(ab) = ad(b) + bd(a).$$

It sends a ring morphism $f : R \to S$ *is to the ring morphism* $T(f) : T(R) \to T(S)$ *defined as on generators as*

$$T(f)(a) = f(a) \qquad T(f)(d(a)) = d(f(a)).$$

- *The projection* $p_R : T(R) \to R$ *in* $\mathbb{C}\mathrm{Ring}_u^{\mathrm{op}}$ *is defined as the inclusion of* R *into* $T(R)$ *in* $\mathbb{C}\mathrm{Ring}_u$.
- *The zero section* $0_R : R \to T(R)$ *in* $\mathbb{C}\mathrm{Ring}_u^{\mathrm{op}}$ *is the ring morphism* $T(R) \to R$ *defined on generators as follows:*

$$0_R(a) = a, \qquad 0_R(d(a)) = 0.$$

- *The addition* $+_R : T_2(R) \to T(R)$ *in* $\mathbb{C}\mathrm{Ring}_u^{\mathrm{op}}$ *is the ring morphism* $T(R) \to T_2(R) = T(R) \otimes_R T(R)$, *defined on generators as follows:*

$$+_R(a) = a \otimes_R 1 = 1 \otimes_R a, \qquad +_R(d(a)) = d(a) \otimes_R 1 + 1 \otimes_R d(a).$$

- *The vertical lift* $\ell_R : T(R) \to T^2(R)$ *in* $\mathbb{C}\mathrm{Ring}_u^{\mathrm{op}}$ *is the ring morphism* $T^2(R) \to T(R)$, *defined on generators as follows:*

$$\ell_R(a) = a, \qquad \ell_R(d(a)) = 0, \qquad \ell_R(d'(a)) = 0, \qquad \ell_R(d'd(a)) = d(a).$$

- *The canonical flip* $c_R : T^2(R) \to T^2(R)$ *in* $\mathbb{C}\mathrm{Ring}_u^{\mathrm{op}}$ *is the ring morphism* $T^2(R) \to T^2(R)$, *defined on generators as follows:*

$$\ell_R(a) = a, \qquad \ell_R(d(a)) = d'(a), \qquad \ell_R(d'(a)) = d(a), \qquad \ell_R(d'd(a)) = d'd(a).$$

More details on this tangent structures are explained in [34, Section 4].

The second example is particularly relevant to algebraic geometry since the category of affine schemes is equivalent to $\mathbb{C}\mathrm{Ring}_u^{\mathrm{op}}$.

We now turn to establishing a way to compare tangent categories by defining tangent functors. The main issue in defining tangent functors is how strongly a functor between tangent categories may preserve the tangent bundle functor. Whether or not the tangent bundle functor is preserved up to natural isomorphism or only preserved up to a natural transformation gives us two different possible defintions for tangent functors, as in the next definition.

Definition 2.2.14. [20, Definition 2.7] *Suppose that* $(\mathbb{X}, T, p, 0, +, \ell, c)$ *and* $(\mathbb{X}', T', p', 0', +', \ell', c')$ *are tangent categories.*

1. *A* ***lax tangent functor*** *(called a morphism of tangent categories in* [20, Definition 2.7]*)* $(F, \alpha) : (\mathbb{X}, T) \to (\mathbb{X}', T')$ *is a functor* $F : \mathbb{X} \to \mathbb{X}'$ *and a natural transformation* $\alpha : F \circ T \Rightarrow T' \circ F$*, called the* ***distributor****, such that the following diagrams commute:*

$$\begin{tikzcd} F \circ T \arrow[r, "\alpha"] \arrow[dr, "F(p)"'] & T' \circ F \arrow[d, "p'_F"] \\ & F \end{tikzcd} \qquad \begin{tikzcd} F \arrow[d, "F(0)"'] \arrow[dr, "0'_F"] & \\ F \circ T \arrow[r, "\alpha"'] & T' \circ F \end{tikzcd} \qquad \begin{tikzcd} F \circ T_2 \arrow[r, "\alpha_2"] \arrow[d, "F(+)"'] & T'_2 \circ F \arrow[d, "+'_F"] \\ F \circ T \arrow[r, "\alpha"'] & T' \circ F \end{tikzcd}$$

$$\begin{tikzcd} F \circ T \arrow[r, "\alpha"] \arrow[d, "F(\ell)"'] & T' \circ F \arrow[d, "\ell'_F"] \\ F \circ T^2 \arrow[r, "T(\alpha)\circ\alpha_T"'] & T'^2 \circ F \end{tikzcd} \qquad \begin{tikzcd} F \circ T^2 \arrow[r, "T(\alpha)\circ\alpha_T"] \arrow[d, "F(c)"'] & T'^2 \circ F \arrow[d, "c'_F"'] \\ F \circ T^2 \arrow[r, "T(\alpha)\circ\alpha_T"'] & T'^2 \circ F \end{tikzcd}$$

2. *A* ***strong tangent functor*** *is a lax tangent functor such that the distributor* α *is a natural isomorphism and* F *preserves the tangent pullbacks of Definition 2.2.4, i.e. the foundational pullbacks and the universality of the vertical lift.*

One example of a tangent functor, that will be an example of our classification of differential bundles in Theorem 4.5.3, is given by the following functor.

Example 2.2.15. *There is a tangent functor* $F : \mathbb{N}^\bullet \to \mathbb{S}\mathrm{mooth}$ *given by sending an object* $\mathbb{N}^k$ *to* $\mathbb{R}^k$ *and each* $\mathbb{N}$*-linear map* f *represented by a matrix* (f_{ij}) *to the matrix* (f_{ij})*, considered as an* $\mathbb{R}$*-linear map using the embedding of* $\mathbb{N}$ *into* $\mathbb{R}$*. This linear map is smooth, and it is apparent that this assignment preserves composition and the identity. The natural isomorphism* $\alpha : F \circ D \to T_{\mathbb{S}\mathrm{mooth}} \circ F$ *is given by*

$$\alpha_{\mathbb{N}^k} := 1_{\mathbb{R}^{2k}} : F(D(\mathbb{N}^k)) = \mathbb{R}^{2k} \to \mathbb{R}^{2k} = T_{\mathbb{S}\mathrm{mooth}}(F(\mathbb{N}^k)).$$

Definition 2.2.16. *[20], Definition 2.8] A* ***Cartesian tangent category*** *is a tangent category that has all finite products (including a final object), where the product functor $- \times - : \mathbb{X} \times \mathbb{X} \to \mathbb{X}, (X, Y) \mapsto X \times Y$ is equipped with a natural isomorphism $\alpha : T(X) \times T(Y) \to T(X \times Y)$ such that the pair $(- \times -, \alpha)$ forms a strong tangent functor.*

In particular this means there is a natural isomorphism $\alpha : T(X)\times T(Y) \to T(X\times Y)$ which is compatible with the natural transformations in the tangents structure, in the sense that the diagrams

$$\begin{tikzcd} T(X)\times T(Y) \arrow[r, "\alpha"] \arrow[d, "p_X\times p_Y"'] & T(X\times Y) \arrow[dl, "p_{X\times Y}"] \\ X\times Y & \end{tikzcd} \qquad \begin{tikzcd} & X\times Y \arrow[dl, "0_X\times 0_Y"'] \arrow[d, "0_{X\times Y}"] \\ T(X)\times T(Y) \arrow[r, "\alpha"'] & T(X\times Y) \end{tikzcd} \qquad \begin{tikzcd} T_2(X)\times T_2(Y) \arrow[r, "\alpha_2"] \arrow[d, "+_X\times +_Y"'] & T_2(X\times Y) \arrow[d, "+_{X\times Y}"] \\ T(X)\times T(Y) \arrow[r, "\alpha"'] & T(X\times Y) \end{tikzcd}$$

$$\begin{tikzcd} T(X)\times T(Y) \arrow[r, "\alpha"] \arrow[d, "\ell_X\times \ell_Y"'] & T(X\times Y) \arrow[d, "\ell_{X\times Y}"] \\ T^2(X)\times T^2(Y) \arrow[r, "T(\alpha)\circ\alpha"'] & T^2(X\times Y) \end{tikzcd} \qquad \begin{tikzcd} T^2(X)\times T^2(Y) \arrow[r, "T(\alpha)\circ\alpha"] \arrow[d, "c_X\times c_Y"'] & T^2(X\times Y) \arrow[d, "c_{X\times Y}"] \\ T^2(X)\times T^2(Y) \arrow[r, "T(\alpha)\circ\alpha"'] & T^2(X\times Y) \end{tikzcd}$$

commute. The identity natural transformation makes the trivial tangent structure on any category $\mathbb{X}$ into a Cartesian tangent category. In fact, we also have seen nontrivial examples of Cartesian tangent categories. The tangent structures on $\mathbb{S}\text{mMan}$, $\mathbb{P}\text{oly}_{\mathbb{F}}$ and $\mathbb{N}^{\bullet}$ are all Cartesian tangent categories. $\mathbb{W}\text{eil}$ is not a Cartesian tangent category because not all binary products exist in $\mathbb{W}\text{eil}$. For example, the product of $W \otimes W$ and W does not exist in $\mathbb{W}\text{eil}$.

In order to obtain a 2-category of tangent categories and tangent functors, we need to specify the 2-cells. We will introduce two different choices of 2-cells which will correspond to different notions of morphisms of differential bundles in Chapter 4. Linear tangent transformations will correspond to linear morphisms of differential bundles.

Definition 2.2.17. *A* ***tangent transformation*** *between (lax/strong) tangent functors $(F, \alpha) \Rightarrow (F', \alpha')$ is a natural transformation of the underlying functors $\varphi : F \Rightarrow F'$. It is called a* ***linear tangent transformation*** *if*

$$\begin{tikzcd} F\circ T \arrow[r, "\varphi_T"] \arrow[d, "\alpha"] & F'\circ T \arrow[d, "\alpha'"] \\ T'\circ F \arrow[r, "T'(\varphi)"] & T'\circ F' \end{tikzcd} \tag{2.1}$$

commutes.

2.3 Differential bundles

Many constructions relevant in differential geometry and mathematical physics, such as metric tensors, curvature, symplectic structures, spinors and differential forms are given by sections of certain vector bundles. The study of vector bundles is also the central theme of K-theory. Thus a central object of study in tangent categories are differential bundles, the categorical generalization of the notion of vector bundles from differential geometry. We start by recalling the definition of differential bundles from [29]:

Definition 2.3.1. [29, Definition 2.3] *Given a tangent category* $(\mathbb{X}, T)$*, a* ***differential bundle*** $(E, M, q, \zeta, \sigma, \lambda)$ *consists of*

1. *an object* E *(the total space),*
2. *an object* M *(the base space),*
3. *a morphism* $q : E \to M$ *(the projection), admitting finite pullback powers* $\{E_n\}_{n\in\mathbb{N}}$ *over itself that are preserved by powers of the tangent functor* T^k*,*
4. *a morphism* $\zeta : M \to E$ *(the zero section) and a morphism* $\sigma : E_2 \to E$ *(fiberwise addition) such that* (q, ζ, σ) *is an additive bundle, and*
5. *a morphism* $\lambda : E \to T(E)$ *(the lift) such that*

$$(\lambda, 0) : (E, M, q, \sigma, \zeta) \to (T(E), T(M), T(q)), T(\sigma), T(\zeta))$$

and

$$(\lambda, \zeta) : (E, M, q, \sigma, \zeta) \to (T(E), E, p, +, 0)$$

are additive bundle morphisms.

In addition the diagram

$$\begin{tikzcd} E_2 \arrow[r, "T(\sigma)\circ\langle\lambda\circ\pi_0, 0\circ\pi_1\rangle"] \arrow[d, "q\circ\pi_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & TE \arrow[d, "T(q)"] \\ M \arrow[r, "0"] & TM \end{tikzcd} \tag{2.2}$$

is a pullback and the morphisms $\ell_E \circ \lambda, T(\lambda) \circ \lambda : E \to T(T(E))$ *are equal. In the special case when* M *is the terminal object, the differential bundle is called a* ***differential object****.*

The maps ζ, σ and λ will be suppressed from the notation when they are not needed. Then the differential bundle $(E, M, q, \zeta, \sigma, \lambda)$ will be denoted by $E \xrightarrow{q} M$ or (E, M, q). Due to the commutativity of the addition, it is equivalent to ask for Diagram 2.2 or

$$\begin{tikzcd} E_2 \arrow[r, "T(\sigma)\circ\langle 0\circ\pi_0, \lambda\circ\pi_1\rangle"] \arrow[d, "q\circ\pi_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & TE \arrow[d, "T(q)"] \\ M \arrow[r, "0"] & TM \end{tikzcd}$$

to be a pullback. Since they are equivalent, we use both conditions, depending on what is more convenient in a given situation.

Remark 2.3.2. *For a differential object $(V, *, q, \zeta, \sigma, \lambda)$, the morphism $q : V \to *$ to the terminal object must be the unique morphism $! : V \to *$ to the terminal object.*[1] *Thus we denote a differential object by $(V, \sigma, \zeta, \lambda)$. If the morphisms σ, ζ, λ are not needed they are suppressed and the differential object is just denoted by V.*

Example 2.3.3. *A Cartesian tangent category is a tangent category in which finite products exist and the product projections are compatible with the tangent structure, see* [20, Section 2.4]. *Given a differential object $(V, \sigma, \zeta, \lambda)$ and an object M in a Cartesian tangent category, there is a differential bundle with:*

- *total space $M \times V$,*
- *base space M,*
- *projection $\pi_0 : M \times V \to M$*
- *addition $1 \times \sigma : M \times V \times V \to M \times V$*
- *zero section $1 \times \zeta : M \cong M \times * \to M \times V$*
- *vertical lift $0_M \times \lambda : M \times V \to T(M) \times T(V) \cong T(M \times V)$*

We call bundles of this form trivial differential bundles.

Classically defined vector bundles from differential geometry were the motivating example of differential bundles. Theorem 2.5.1 of [67] shows that differential bundles in $\mathbb{S}$mMan are exactly vector bundles. In particular, differential objects in $\mathbb{S}$mMan are simply vector spaces. The subcategory Smooth in $\mathbb{S}$mMan is the full subcategory consisting of the differential objects, i.e. the vector spaces.

Example 2.3.4.

[1] It is also true that the universality of the vertical lift shows that $\sigma \circ (\lambda \times 0) : V^2 \to T(V)$ is an isomorphism. We will not need this, but it is an important property of differential objects.

(a) Every object in $\mathbb{N}^\bullet$ *is a differential object, i.e. every* $\mathbb{N}^k$ *is a differential bundle over* $\mathbb{N}^0$ *with the structure given by the following maps:*

$$\zeta := 0 : \mathbb{N}^0 \to \mathbb{N}^k, \textit{ the inclusion of zero in } \mathbb{N}^k;$$
$$\sigma := add : \mathbb{N}^{2k} \to \mathbb{N}^k, \textit{ the usual addition map;}$$
$$\lambda := \langle 0, 1_{\mathbb{N}^k} \rangle : \mathbb{N}^k \to \mathbb{N}^{2k}.$$

It is an easy exercise to check that these maps fulfill all the conditions required for a differential bundle.

(b) Replacing $\mathbb{N}$ *with* $\mathbb{R}$ *in the previous example shows that every object of* $\mathbb{S}\mathrm{mooth}$ *is a differential object. This is because the tangent functor* $(F, \alpha) : \mathbb{N}^\bullet \to \mathbb{S}\mathrm{mooth}$ *from Example 2.2.15 preserves the differential object structure. Given a differential object in* $\mathbb{N}^\bullet$ *with structure maps* ζ, σ *and* λ*, the maps* $F(\zeta), F(\sigma)$ *and* $F(\lambda)$ *are the structure maps of a differential object because* F *leaves these maps essentially unchanged. Every object of* $\mathbb{S}\mathrm{mooth}$ *is of this type because* F *is essentially surjective. This, plus the fact that* α *is the identity transformation in this case, make it trivial to verify that the resulting structures fulfill Definition 2.3.1.*

(c) Replacing $\mathbb{N}$ *with any field* $\mathbb{F}$ *will show that every objection of* $\mathbb{P}\mathrm{oly}_{\mathbb{F}}$ *is a differential object. This works essentially the same as the argument in (b).*

The differential object structure on $\mathbb{N}^1$ is a key property of the category $\mathbb{N}^\bullet$ that will allow us to classify differential bundles in Theorem 4.5.3.

Definition 2.3.5. [29, Definition 2.3] *A* ***morphism of differential bundles***

$$(f, g) : (E, M, q, \sigma, \zeta, \lambda) \to (E', M', q', \sigma', \zeta', \lambda')$$

is a pair of maps $f : E \to E', g : M \to M'$ *such that the diagram*

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "q"'] & E' \arrow[d, "q'"] \\ M \arrow[r, "g"'] & M' \end{tikzcd}$$

commutes. A morphism of differential bundles is called ***linear*** *if*

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "\lambda"'] & E' \arrow[d, "\lambda'"] \\ TE \arrow[r, "T(f)"'] & TE' \end{tikzcd} \tag{2.3}$$

commutes and it is called ***additive*** *if the diagrams*

$$\begin{array}{ccccccc} E_2 & \xrightarrow{f_2} & E_2' & \quad & M & \xrightarrow{g} & M' \\ \sigma\downarrow & & \downarrow\sigma' & & \zeta\downarrow & & \downarrow\zeta' \\ E & \xrightarrow[f]{} & E' & & E & \xrightarrow[f]{} & E' \end{array} \tag{2.4}$$

commute.

In the special case of differential objects, a morphism of differential bundles between differential objects is just a morphism in the underlying tangent category.

It was shown in [29] that every linear morphism of differential bundles is additive.

Linearity means that $(f, T(f))$ preserves the lift, and additivity means that (f, g) preserves the addition and zero morphisms. In order to understand linearity from an intuitive perspective, we consider the example of trivial vector bundles $M \times V \xrightarrow{\pi_0} M$ in $\mathbb{S}$mMan.

Example 2.3.6 (Linear morphisms in $\mathbb{S}$mMan)**.** *Let $(f, g) : (M \times V, M, \pi_0) \to (M' \times V', M', \pi_0)$ be a morphism between two trivial differential bundles as in Example 2.3.3 in* $\mathbb{S}$mMan*. Since this is a morphism of differential bundles, $g \circ \pi_0 = \pi_0 \circ f$. This means the map $f : M \times V \to M' \times V'$ has to be of the form $f = \langle g \circ \pi_0, f_1 \rangle$ for some map $f_1 : M \to M' \times V'$. We would like to know when (f, g) is a linear morphism. The linearity condition from Definition 2.3.5, Diagram 2.3, says that $\lambda \circ f = T(f) \circ \lambda : M \times V \to T(M \times V)$. In the local coordinates from Example 2.2.7 the tangent bundle is a product, $T(U) \cong U \times \mathbb{R}^n$ for some open $U \subset M$. For $x \in U \subset M$ and $v \in T_xM \cong \mathbb{R}^n$ and a sufficiently small neighbourhood U' around $f(x)$, the composition*

$$\lambda \circ f : U \times V \to T(U' \times V') \cong U' \times T_{f(x)}U' \times V' \times V'$$

sends (x, v) to

$$\lambda \circ f(x, v) = \lambda(g(x), f_1(x, v)) = (g(x), 0, 0, f_1(x, v)).$$

If (f, g) is linear, this must be equal to

$$\begin{aligned} T(f) \circ \lambda(x, v) &= T(\langle g \circ \pi_0, f_1 \rangle)(x, 0, 0, v) \\ &= \left(g(x), f_1(x, 0), \frac{\partial g(y)}{\partial y}|_{y=x} \cdot 0, \frac{\partial f_1(x, w)}{\partial w}|_{w=0} \cdot v \right) \\ &= \left(g(x), f_1(x, 0), 0, \frac{\partial f_1(x, w)}{\partial w}|_{w=0} \cdot v \right) \end{aligned}$$

which we expanded using the local coordinate expression from Example 2.2.7. We conclude that

$$\left(g(x), f_1(x,0), 0, \frac{\partial f_1(x,w)}{\partial w}|_{w=0} \cdot v\right) = (g(x), 0, 0, f_1(x,v))\,.$$

Therefore (f,g) *is linear precisely when* $f_1(x,v) = \frac{\partial f(x,w)}{\partial w}|_{w=0} \cdot v$ *for all* $v \in \mathbb{R}^n$*, including* $v = 0$.

Remark 2.3.7. *It is possible for bundle morphisms to be additive but not linear. Consider the category* $\mathbb{P}\mathrm{oly}_{\mathbb{Z}/p}$ *of polynomials over the field with* p *elements, as introduced in Example 2.2.8. Let* $\mathbb{Z}/p$ *be the differential object with the zero map as* ζ*, the addition of field elements as* σ *and* $\langle 0, 1_{\mathbb{Z}/p}\rangle$ *as* λ *as in Example 2.3.4(b). Define a morphism* $f : \mathbb{Z}/p \to \mathbb{Z}/p$ *by* $x \mapsto x^p$*. This is a morphism of differential bundles because* $\mathbb{Z}/p$ *is a differential object and thus every morphism between them is a differential bundle morphism. Since we are working in* $\mathbb{Z}/p$*, the identities* $x^p + y^p = (x+y)^p$ *and* $0^p = 0$ *hold, making Diagram (2) from Definition 2.3.5 commute. Thus* f *is an additive morphism.*

From Example 2.2.8 and Example 2.2.7, $T(f)$ *is given by the Jacobi matrix. However, the Jacobi matrix (i.e. derivative) of* f *is* $f'(x) = px^{p-1} = 0$*. Therefore,* $T(f)$ *is zero. Then* $T(f) \circ \lambda(1) = T(f)(0,1) = (f(0), f'(0)) = (0,0)$ *while* $\lambda \circ f(1) = \lambda(1^p) = (0,1)$*. Consequently Diagram (1) in Definition 2.3.5 fails to commute.*

The composition of additive/linear maps is additive/linear, thus we can define three different categories of differential bundles: we can include any differential bundle morphisms, additive differential bundle morphisms or linear differential bundle morphisms.

Definition 2.3.8. *Let* $\mathbb{X}$ *be a tangent category.*

(a) Let $\mathbb{D}\mathrm{Bun}(\mathbb{X})$ *be the category of differential bundles and morphisms of differential bundles in* $\mathbb{X}$*.*

(b) The categories of differential bundles with additive/linear morphisms are denoted by $\mathbb{D}\mathrm{Bun}_{\mathrm{add}}(\mathbb{X})$ *and* $\mathbb{D}\mathrm{Bun}_{\mathrm{lin}}(\mathbb{X})$ *respectively.*

(c) The full subcategories of $\mathbb{D}\mathrm{Bun}(\mathbb{X})$*,* $\mathbb{D}\mathrm{Bun}_{\mathrm{add}}(\mathbb{X})$ *and* $\mathbb{D}\mathrm{Bun}_{\mathrm{lin}}(\mathbb{X})$ *consisting of the differential objects (differential bundles over the terminal object) are denoted by* $\mathbb{D}\mathrm{Obj}(\mathbb{X})$*,* $\mathbb{D}\mathrm{Obj}_{\mathrm{add}}(\mathbb{X})$ *and* $\mathbb{D}\mathrm{Obj}_{\mathrm{lin}}(\mathbb{X})$*, respectively.*

If there is no terminal object we use the convention that each of $\mathbb{D}\mathrm{Obj}(\mathbb{X})$*,* $\mathbb{D}\mathrm{Obj}_{\mathrm{add}}(\mathbb{X})$ *and* $\mathbb{D}\mathrm{Obj}_{\mathrm{lin}}(\mathbb{X})$ *are the empty category.*

In Theorem 4.5.3, we will see an equivalent characterization of these categories of differential bundles.

Chapter 3

The dimension of the tangent bundle and the universality of the vertical lift

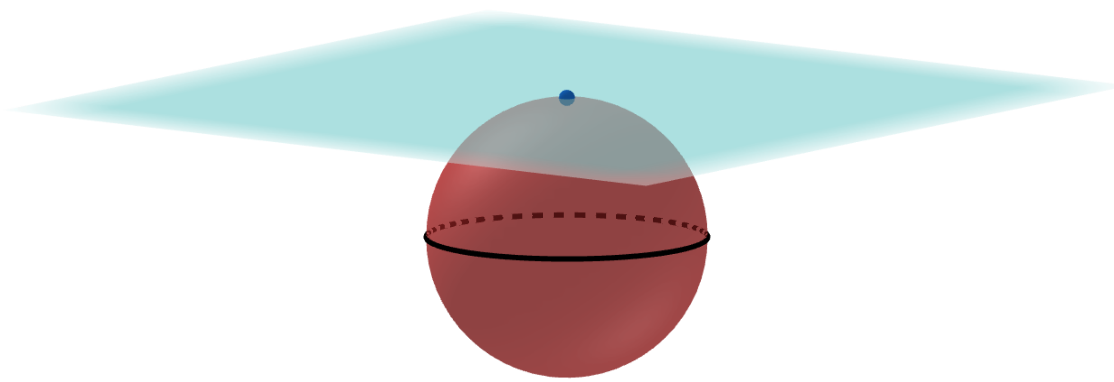

Figure 3.1: Classically, the tangent space has the same dimension as the base space and thus the dimension of the tangent bundle is twice the dimension of the base space. Here the dimension of the base manifold is 2, thus the dimension of the tangent bundle is 4 [2]

This chapter is the categorification of the dimension of smooth manifolds and in particular the dimension of the tangent bundle of smooth manifolds. We replace the classical intrinsic notion of dimension with assignments that send isomorphism classes to elements in a monoid fulfilling certain additivity conditions. This dimension provides a new perspective on the universality of the vertical lift in tangent categories. Many examples of dimension functions are provided to demonstrate the utility of the definition. In particular, we show that this means that there are no non-trivial Cartesian tangent structures on sets, as an example.

This chapter is based on the preprint [83], with small corrections and without the background section (since this background was already given in Chapter 2). Therefore, like in [83], we use applicative composition order, i,e, we denote the composition of $f : A \to B$ and $g : B \to C$ as $g \circ f : A \to C$.

[2] Figure created using GeoGebra 3D

3.1 Introduction

The dimension is an essential part of the classical definition of smooth manifolds. Most definitions, like in [71, Section I.1.1], specifically define a n-dimensional manifold for a certain $n \in \mathbb{N}$ as a separable metrizable space M which is locally homeomorphic to $\mathbb{R}^n$.

While the dimension does not uniquely characterize a manifold, it is a number that is invariant under diffeomorphisms and hence it effectively differentiates manifolds whose dimensions are not equal. The dimension fulfills many desirable properties beyond diffeomorphism invariance: it is additive under products and it is doubled when forming the tangent bundle. In this work, we generalize the concept of a dimension and investigate the consequences in tangent categories.

An important but subtle property that tangent categories are required to fulfill, according to their definition, is the universality of the vertical lift which states that

$$\begin{tikzcd} T_2M \arrow[r, "\nu"] \arrow[d, "p_M \circ \pi_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & T^2M \arrow[d, "T(p_M)"] \\ M \arrow[r, "0_M"'] & TM \end{tikzcd}$$

is a pullback diagram. In [20] Cockett and Cruttwell say that "while being perhaps the least intuitive aspect of tangent structure, [the universality of the vertical lift] is also perhaps the most important".

The lack of intuition for the universality of the vertical lift makes it hard to construct additional examples of tangent categories or check if a certain endofunctor is part of a tangent structure.

However, in the category of smooth manifolds for a pullback along a submersion, adding the dimensions along both diagonals gives the same number. It follows from a short argument that the universality of the vertical lift restricts the dimension of the tangent bundle $\dim(T(M))$ to be either $\dim(M)$ or $2\dim(M)$.

Analogously, in Definition 3.2.2 we define categorical dimensions on other categories and interpret the universality of the vertical lift in terms of them.

The point is not to generalize existing notions of dimension in many different categories (though it generalizes the dimension of manifolds and the rank of modules), but to interpret the universality of the vertical lift and put restrictions on possible tangent structures.

The main result is Theorem 3.2.7 which states that given a dimension dim on a tangent category $(\mathbb{X}, T)$, for any object X, the equation

$$\dim(T^2(X)) + 2 \cdot \dim(X) = 3 \cdot \dim(T(X))$$

holds. The second main result, Theorem 3.2.8, states that under certain circumstances $\dim(T(X)) = \dim(X)$ or $\dim(T(X)) = 2 \cdot \dim(X)$ has to hold. Both of them are consequences of the universality of the vertical lift in the presence of a dimension.

We will define a generalized notion of dimension in Section 3.2. One key aspect of this generalization is that the dimension can be an element of any commutative monoid, not just the natural numbers with addition. In section 3.2.2, we will prove Theorems 3.2.7 and 3.2.8, the main results of this chapter.

These results allow us to narrow down the search for tangent structures on a given category (with dimension). For different underlying categories $\mathbb{X}$ we obtain results of the form "any tangent structure on $\mathbb{X}$ fulfills this equation".

We give several examples of categorical dimensions that go beyond the motivating example of smooth manifolds. In Section 3.3 we observe that the cardinality is a categorical dimension on $\mathbb{S}\mathrm{et}^{\mathrm{op}}$ and use this to show that the trivial tangent structure is the only Cartesian tangent structure on $\mathbb{S}\mathrm{et}^{\mathrm{op}}$. In Section 3.4, we observe that the cardinality is a categorical dimension on $\mathbb{G}\mathrm{rp}$. However, instead of being valued in the monoid given by addition, it is valued in the monoid given by multiplication, demonstrating the versatility of our use of monoids in the definition.

In Section 3.5 we observe that, while the Krull dimension is not a categorical dimension for rings, the characteristic of rings is a categorical dimension. Interestingly, the monoid this dimension is valued in is given neither by addition nor multiplication, it is the least common multiple (lcm).

In Section 3.6 we observe that the rank of a module over a ring is a categorical dimension. Using the example of the free forgetful adjunction between modules and algebras, we showcase the compatibility of dimensions under adjunctions. In addition, in the case of vector spaces the rank is the classical dimension. The dimension can be used to prove that there are only two Cartesian tangent structures on the category $\mathbb{V}\mathrm{ect}_F$ of vector spaces over a field F.

In Section 3.7 we observe that the dimension of CW complexes is not a categorical dimension. However, the Betti numbers are a dimension on the opposite category of CW complexes $\mathbb{C}\mathrm{W}^{\mathrm{op}}$. In Proposition 3.7.8 we show that, under certain circumstances, dimensions can be transported along excisive functors. Since excisive functors can be thought of as a generalization of homology, this hints at a deep connection between categorical dimensions and homotopy theory.

We include many non-examples which stem from concepts that may intuitively evoke the idea of dimension, but which turn out not to be categorical dimensions. This highlights the main purpose of the concept that we are introducing in this chapter: the dimension is meant to provide context for the universality of the vertical lift.

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3.2 Categorical dimensions of tangent bundles

In this section, we will present a definition of dimension in categories which generalizes the definition of dimension for smooth manifolds. The dimension of manifolds has many desirable properties, some of which we will list here.

Remark 3.2.1. *Let X, X' and X'' be smooth manifolds. The dimension of manifolds has certain properties which we would consider desirable for generalizing dimensions.*

(a) ***(valued in $\mathbb{N}$)*** *The dimension assigns manifolds to natural numbers* $\mathrm{Dim}(X) \in \mathbb{N} = \{0, 1, 2...\}$.

(b) ***(additive w.r.t. pullbacks)*** *The dimension is additive under certain pullbacks: Given a cospan $X \to X' \leftarrow X''$ of transversal maps,* $\mathrm{Dim}(X \times_{X'} X'') + \mathrm{Dim}(X') = \mathrm{Dim}(X) + \mathrm{Dim}(X'')$

(b') ***(additive w.r.t. products)*** *The dimension is additive under Cartesian products:* $\mathrm{Dim}(X \times X') = \mathrm{Dim}(X) + \mathrm{Dim}(X')$

(c) ***(isomorphism invariant)*** *The dimension is invariant under isomorphism: Given $X \cong X'$, then* $\mathrm{Dim}(X) = \mathrm{Dim}(X')$.

The topology and tangent structure of smooth manifolds also has implications for the dimension.

(d) ***(embeddings to inequalities)*** *The dimension sends embeddings to inequalities: Given an embedding $X \hookrightarrow X'$,* $\mathrm{Dim}(X) \leq \mathrm{Dim}(X')$

(d') ***(the same as locally)*** *The dimension is local: Given an open subset $U \hookrightarrow X$,* $\mathrm{Dim}(U) = \mathrm{Dim}(X)$.

(e) ***(tangent bundle doubles dim)*** *Applying the tangent bundle endofunctor doubles the dimension:* $\mathrm{Dim}(TX) = 2 \cdot \mathrm{Dim}(X)$.

(f) ***(bundle fiber arbitrary dim)*** *A vector bundle can have a fiber of any dimension: For any $n \in \mathbb{N}$, there is a vector bundle $E \to X$ such that* $\dim(E) = \dim(X) + n$.

We will take property (b) (additive w.r.t. pullbacks) as the definition of categorical dimensions. The reason we choose this condition is that we want to apply the properties of a dimension to the pullback diagram

$$\begin{tikzcd} T_2X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & T^2X \arrow[d, "T(p_X)"] \\ X \arrow[r, "0_X"'] & TX \end{tikzcd}$$

encoding the universality of the vertical lift. We will replace the notion of transversality (which inherently depends on the tangent structure) with a more general notion, in order to define a dimension on a category without knowing the tangent structure. While we will not assume desired Property (b') (additive w.r.t. products), however if the dimension of the terminal object is the neutral element $\dim(*) = 0$, it follows from Property (b) (additive w.r.t. pullbacks).

We generalize Property (a) (valued in $\mathbb{N}$) by asking the target of the dimension to be a commutative monoid. This is partially motivated by existing generalizations of dimension (such as infinite dimension and fractional dimension). In Sections 3.3–3.7, we will provide examples of this categorical dimension which satisfies the definition and is not an integer.

Since none of our results rely on it, we also do not require Property (d) (embeddings to inequalities). We also choose to not require property (d')(the same as locally) since we want to avoid using any notion of locality.

Our definition of dimension will not be unique. For example, the constant assignment $\dim(X) = 0, \forall X$ will be a dimension and so is any other constant assignment $\dim(X) = 42, \forall X$.

As a convention, we denote the classical notion of dimension of manifolds and vector spaces as Dim with a capital D, whereas we denote the generalized categorical dimension as dim with lowercase d.

3.2.1 Monoid valued dimensions

In this section we will define our categorical notion of dimension. The key will be how the dimension behaves under the pullbacks

$$\begin{tikzcd} T_2M \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & TM \arrow[d, "p"] \\ TM \arrow[r, "p"'] & M \end{tikzcd} \qquad \begin{tikzcd} T_2M \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & T^2M \arrow[d, "T(p)"] \\ M \arrow[r, "0"'] & TM \end{tikzcd}$$

which are the definition of T_2M and the universality of the vertical lift. Thus we specify the behaviour for pullbacks of this form.

Definition 3.2.2. *Let M be a commutative monoid and $\mathbb{X}$ be a category. Then an M**-valued dimension** on $\mathbb{X}$ is a function* $\dim : \pi_0(\mathbb{X}) \to M$ *from the set of isomorphism classes of objects of $\mathbb{X}$ to the monoid M,* *satisfying the following property:*

For any pullback

$$\begin{tikzcd} A \times_B C \arrow[r] \arrow[d] & A \arrow[d, "r"] \\ C \arrow[r, "f"'] & B \end{tikzcd}$$

of f along a retraction r,

- *if f is a retraction, or*
- *if f is a section,*

the equation

$$\dim(A \times_B C) + \dim(B) = \dim(A) + \dim(C)$$

holds.

While we chose to loosen desired Property (a) (valued in $\mathbb{N}$) of Remark 3.2.1, Property (c) (isomorphism invariant) and a variation of Property (b) (additive w.r.t. pullbacks) hold by definition.

In a category with products, the morphism to the terminal object, $*$, is a retraction and thus any pullback over the terminal object is one of the pullbacks in Definition 3.2.2. If the dimension of the terminal object satisfies $\dim(*) = 0$, we obtain $\dim(X \times Y) = \dim(X) + \dim(Y)$. That is, if $\dim(*) = 0$ property (b') (additive w.r.t. products) of Remark 3.2.1 holds.

We actively rejected to generalize property (d')(the same as locally) of Remark 3.2.1 in order to disregard notions of locality. Since we, in general, do not have an order on the monoid M, property (d) (embeddings to inequalities) does not hold in general. However it holds for most of the examples we have listed since we have ordered monoids there.

Proving properties similar to (e)(tangent bundle doubles dim) and (f)(bundle fiber arbitrary dim) of Remark 3.2.1 will be the topic of Section 3.2.2.

In the following proposition we see that dimensions interact well with functors and in particular adjunctions.

Proposition 3.2.3. *Let $\mathbb{X}$ and $\mathbb{Y}$ be categories, let $\mathbb{X}$ have an R-valued dimension* $\dim_{\mathbb{X}} : \pi_0(\mathbb{X}) \to R$*. Let $F : \mathbb{Y} \to \mathbb{X}$ be a functor that preserves finite limits. Then the assignment that sends $Y \in \pi_0(\mathbb{Y})$ to*

$$Y \mapsto \dim_{\mathbb{X}}(F(Y))$$

is an R-valued dimension on $\mathbb{Y}$.

Proof. Functors preserve composition and identity, thus F preserves sections and retractions. Since the functor F preserves limits, it preserves in particular pullbacks. Thus for any pullback in $\mathbb{Y}$ of a section or retraction along a retraction, the dimension formula still holds. $\square$

Two important classes of examples of functors that preserve finite limits are full subcategories and right-adjoint functors.

Corollary 3.2.4. *Let $\mathbb{X}$ and $\mathbb{Y}$ be categories, let $\mathbb{X}$ have an R-valued dimension $\dim_{\mathbb{X}} : \pi_0(\mathbb{X}) \to R$.*

(a) Let $F : \mathbb{Y} \to \mathbb{X}$ be a functor which is a right adjoint. Then the assignment that sends $Y \in \pi_0(\mathbb{Y})$ to

$$Y \mapsto \dim_{\mathbb{X}}(F(Y))$$

is an R-valued dimension on $\mathbb{Y}$.

(b) Let $\mathbb{Y}$ be a full subcategory of $\mathbb{X}$. Let $\dim : \pi_0(\mathbb{X}) \to R$ be a R-valued dimension on $\mathbb{X}$. Then $\dim|_{\mathrm{Obj}(\mathbb{Y})}$ is an R-valued dimension on $\mathbb{Y}$.

Since many adjunctions are well known from the literature, this inconspicuous corollary is a useful way to construct new dimensions from existing ones and will be frequently used throughout the chapter. One major application of Corollary 3.2.4 will be in Section 3.6.2. There we will show that the forgetful functor from algebras over a ring R to modules over R has a left-adjoint and thus a dimension on the category of modules will induce a dimension on the category of Algebras.

Recall that in a tangent category $(\mathbb{X}, T)$, a pullback is called a T-pullback if it is preserved by all powers of T. T-pullbacks were first developed by McAdam in [66, Definition 1.1.3] and later analyzed by Marcello Lanfranchi and Cruttwell in [32]. We will use the definition of a T-pullback to define a slightly weaker version of dimension and a slightly stronger version of dimension. These definitions will be the setup of Theorem 3.2.7 and Theorem 3.2.8.

Definition 3.2.5. *Let M be a commutative monoid and $(\mathbb{X}, T)$ be a tangent category. Then an M**-valued tangent dimension** on $(\mathbb{X}, T)$ is a function $\dim : \pi_0(\mathbb{X}) \to M$ satisfying the following property:*
For any T-pullback

$$\begin{tikzcd} A \times_B C \arrow[r] \arrow[d] & A \arrow[d, "r"] \\ C \arrow[r, "f"'] & B \end{tikzcd}$$

along a retraction r,

- *if f is a retraction, or*

- *if f is a section,*

the equation

$$\dim(A \times_B C) + \dim(B) = \dim(A) + \dim(C)$$

holds. An M-valued tangent dimension is ***strong*** *if M is a commutative rig and there is an element $a \in \mathbb{M}$ such that for every object $X \in \mathrm{Ob}(\mathbb{X})$ the equation*

$$\dim(T(X)) = a \cdot \dim(X)$$

holds.

Since every T-pullback is a pullback, it is immediate that every M-valued dimension is an M-valued tangent dimension (regardless of the tangent structure) but not vice versa.

Again, M-valued tangent dimensions satisfy Properties (b) (additive w.r.t. pullbacks) and (c) (isomorphism invariant) of Remark 3.2.1. We will explore in the next sections what the properties corresponding to (e) (tangent bundle doubles dim) and (f) (bundle fiber arbitrary dim) are.

Example 3.2.6. *Given any tangent category $(\mathbb{X}, T, p, 0, +, \ell, c)$ and any commutative monoid M, let $m \in M$ be any element of M. Then the constant assignment $\dim(X) = m$ is a M-valued dimension and a strong M-valued tangent dimension. All the equations become trivial in this situation. We will call dimensions of this form* ***trivial dimensions*** *on $\mathbb{X}$.*

We explore the properties of M-valued tangent dimensions in Theorems 3.7 and 3.8.

3.2.2 The tangent structure in the presence of a dimension

If we have a tangent dimension on a category, it interacts with the tangent structure since the tangent structure requires certain diagrams to be pullbacks. We see in Theorem 3.2.8 that in certain situations the dimension of the tangent bundle behaves in a way similar to desirable Property (e)(tangent bundle doubles dim) of Remark 3.2.1.

Theorem 3.2.7. *Let M be a commutative monoid. Let $(\mathbb{X}, T)$ be a tangent category. Let $\dim : \mathrm{Ob}(\mathbb{X}) \to M$ be a M-valued tangent dimension. Then for every object $X \in \mathrm{Ob}(\mathbb{X})$*

$$\dim(T^2(X)) + 2 \cdot \dim(X) = 3 \cdot \dim(T(X)).$$

Proof. We consider the following two pullback diagrams:

$$\begin{tikzcd} T_2X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & TX \arrow[d, "p"] \\ TX \arrow[r, "p"'] & X \end{tikzcd} \qquad \begin{tikzcd} T_2X \arrow[r, "\nu"] \arrow[d, "\pi_0 p"'] \arrow[dr, phantom, "\lrcorner", very near start] & T^2X \arrow[d, "T(p)"] \\ X \arrow[r, "0"'] & TX \end{tikzcd}$$

where ν is a shorthand notation for $\nu = T(+) \circ \langle \ell_M \circ \pi_0, 0_{TM} \circ \pi_1 \rangle$. The first diagram is a pullback since this is the definition of T_2M. The second diagram is a pullback, because this is an equivalent formulation for the universality of the vertical lift.

Since $0 : X \to T(X)$ and $p : T(X) \to X$ form a section retraction pair, the pullbacks above fulfill the requirements in Definition 3.2.2 we obtain that

$$\dim(T_2(X)) + \dim(X) = 2 \cdot \dim(T(X)) \text{ and } \dim(T_2(X)) + \dim(T(X)) = \dim(T^2(X)) + \dim(X).$$

Therefore we obtain

$$2 \cdot \dim(T(X)) + \dim(T(X)) = \dim(T_2(X)) + \dim(X) + \dim(T(X)) = \dim(T^2(X)) + \dim(X) + \dim(X).$$

□

Theorem 3.2.8. *Let R be an integral domain (unital commutative ring without zero divisors). Let $(\mathbb{X}, T)$ be a tangent category. Let $\dim : \mathrm{Ob}(\mathbb{X}) \to R$ be a strong R-valued tangent dimension on $(\mathbb{X}, T)$. Then for any object X, $\dim(T(X)) = \dim(X)$ or $\dim(T(X)) = 2 \cdot \dim(X)$ holds.*

If there exists an object with $\dim(X) \neq 0$ this means $a = 1$ or $a = 2$, where a is the constant from Definition 3.2.5 fulfilling $\dim(T(X)) = a \cdot \dim(X)$.

Proof. We consider the following two pullback diagrams:

$$\begin{tikzcd} T_2X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & TX \arrow[d, "p"] \\ TX \arrow[r, "p"'] & X \end{tikzcd} \qquad \begin{tikzcd} T_2X \arrow[r, "\nu"] \arrow[d, "\pi_0 p"'] \arrow[dr, phantom, "\lrcorner", very near start] & T^2X \arrow[d, "T(p)"] \\ X \arrow[r, "0"'] & TX \end{tikzcd}$$

where ν is a shorthand notation for $\nu = T(+) \circ \langle \ell_M \circ \pi_0, 0_{TX} \circ \pi_1 \rangle$.

The first diagram is a pullback since this is the definition of T_2M. The second diagram is a pullback, because this is an equivalent formulation for the universality of the vertical lift. Therefore we obtain that

$$\dim(T(X)) + \dim(T(X)) - \dim(X) = \dim(T_2X) = \dim(T^2X) + \dim(X) - \dim(TX).$$

This can be reformulated into

$$\dim(T^2X) - 3\dim(TX) + 2\dim(X) = 0.$$

Using that $\dim(TX) = a \cdot \dim(X)$, we obtain the equation

$$\begin{aligned}0 &= a^2 \cdot \dim(X) - 3a \cdot \dim(X) + 2\dim(X)\\ 0 &= (a-1)\cdot(a-2)\cdot\dim(X)\end{aligned}$$

This implies $a = 1$, $a = 2$ or $\dim(X) = 0$. If $a = 1$, then $\dim(TX) = \dim(X)$. If $a = 2$, then $\dim(TX) = 2 \cdot \dim(X)$. If $\dim(X) = 0$, both hold. □

Theorem 3.2.8 is the categorical analogue to Property (g)

Example 3.2.9. *In the tangent category of smooth manifolds, the classical dimension* $\dim(M) = \mathrm{Dim}(M)$ *is an example for a* $\mathbb{Z}$*-valued strong tangent dimension. If the retraction* $r : A \to B$ *is a submersion, it is transversal to any smooth map* $C \to B$*. Thus, by* [13, Theorem 15.2]*, the pullback fulfills* $\dim(A \times_B C) = \dim(A) + \dim(C) - \dim(B)$*. Therefore Theorem 3.2.8 shows that for any tangent structure on SmMan with* $\dim(TM) = a \cdot \dim(M)$*, we obtain that* $a = 1$ *or* $a = 2$*. We prove the general statement that the classical dimension* Dim *fulfills Definition 3.2.2 in Proposition 3.2.12, where we do not assume that r is a submersion.*

One important application of Theorem 3.2.7 and Theorem 3.2.8 is that they provide a method to prove that some functor can not be part of a tangent structure. If a category has a good notion of dimension, but the proposed tangent functor fails to satisfy the conditions of Theorem 3.2.7 and Theorem 3.2.8, then the functor can not be a tangent functor. The following example, discovered by G. Vooys [89], showcases this application.

Example 3.2.10. *Let* SmMan *be the category of smooth real manifolds and smooth maps. Given the functor* $\tilde{T} : \mathrm{SmMan} \to \mathrm{SmMan}$ *that sends an object* M *to the object* $M \times \mathbb{R}$ *and a morphism* f *to* $f \times 1_{\mathbb{R}}$*. Then there is no tangent structure* $(\mathrm{SmMan}, \hat{T}, p, +, 0, \ell)$ *for which the tangent functor is* $\tilde{T}$*. The reason is that according to Theorem 3.2.7 the equation*

$$\dim(M \times I \times I) + 2 \cdot \dim(M) = 3 \cdot \dim(M \times I)$$

would need to hold. However, this would mean that

$$3 \cdot \dim(M) + 2 = 3 \cdot \dim(M) + 3$$

and thereby $3 = 2$ *proving by contradiction that there is no such tangent structure.*

In fact, for this endofunctor $\tilde{T}$, it is possible to define natural transformations $p, 0, +, \ell, c$ that fulfill all the properties of a tangent structure, except the universality of the vertical lift.

As we promised in Example 3.2.9, below we prove that the dimension of manifolds fulfills the pullback formula of Definition 3.2.2. We begin with a lemma turning a general retraction into a submersion.

Lemma 3.2.11. *Let* $r : A \to B$ *be a retraction with section* $s : B \to A$*. Then there is an open subset* $U \subseteq A$ *such that* $\mathrm{Im}(s) \subset U$ *and* $r|_U$ *is submersion.*

The fact that $\mathrm{Im}(s) \subset U$ simultaneously shows that U is non-empty and also that s is a section of $r|_U$.

Proof. Let $U \subset A$ be the set of regular points in A. Due to [61, Proposition 4.1], U is open. By its definition, all points in U are regular, therefore $r|_U$ is a submersion.

It only remains to show that $\mathrm{Im}(s) \subset U$. Let $a = s(b)$ be in $\mathrm{Im}(s) \subset A$. In order to show that a is a regular point, let $v \in T_b(B)$ be an arbitrary tangent vector. Then $T_b s(v) \in$ is a tangent vector at a fulfilling that

$$T_a r \circ T_b s(v) = T_a(r \circ s)(v) = v$$

Since v is arbitrary, a is a regular point. Since $a \in \mathrm{Im}(s)$ was arbitrary, this shows that $\mathrm{Im}(s) \subset U$, concluding the proof. ☐

Proposition 3.2.12. *Let* $r : A \to B$ *be a retraction of smooth manifolds. Suppose the pullback*

$$\begin{tikzcd}
A \times_B C \arrow[r, "\pi_0"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "r"] \\
C \arrow[r, "f"'] & B
\end{tikzcd}$$

exists. Then the equation

$$\dim(A \times_B C) = \dim(A) + \dim(C) - \dim(B)$$

holds.

Proof. Due to Lemma 3.2.11, there is an open subset $U \subset A$ containing $\mathrm{Im}(s)$ such that $r|_U$ is a submersion. Since the embedding $U \hookrightarrow A$ is a submersion, the pullback

$$\begin{tikzcd}
\pi_0^{-1}(U) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & U \arrow[d, hook] \\
A \times_B C \arrow[r, "\pi_0"'] & A
\end{tikzcd}$$

exists and euqals the preimage $\pi_0^{-1}(U)$. As the preimage of an open set, $\pi_0^{-1}(U)$ is an open subset of $A \times_B C$.

Both inner squares in

$$
\begin{tikzcd}
\pi_0^{-1}(U) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & U \arrow[d, hook] \\
A \times_B C \arrow[r, "\pi_0"'] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "r"] \\
C \arrow[r, "f"'] & B
\end{tikzcd}
$$

are pullbacks. Therefore the outer square is the pullback of f along $r \circ i$. Since $r \circ i = r|_U$ is a submersion, the dimension formula

$$\dim(\pi_0^{-1}(U)) = \dim(U) + \dim(C) - \dim(B)$$

holds. Since the dimension of an open nonempty subset equals the dimension of the surrounding manifold, $\dim(\pi_0^{-1}(U)) = \dim(A \times_B C)$ and $\dim(U) = \dim(A)$. Therefore we obtain

$$\dim(A \times_B C) = \dim(\pi_0^{-1}(U)) = \dim(U) + \dim(C) - \dim(B) = \dim(A) + \dim(C) - \dim(B)$$

which concludes the proof. □

3.2.3 Differential bundles in the presence of a dimension

For tangent structures, the dimension helps to give context to the universality of the vertical lift. Similarly, the pullback in Diagram 2.2 is the universality of the vertical lift for differential bundles. In this section, we prove the analogues of Theorems 3.2.7 and 3.2.8 for differential bundles.

In the classical case of smooth manifolds, desirable Property (f)(bundle fiber arbitrary dim) of Remark 3.2.1 says that there is no constraint on the dimension of a differential bundle. However, in certain non-classical circumstances there is. This will lead to a somewhat strange genaralization of desirable Property (f)(bundle fiber arbitrary dim) of Remark 3.2.1 in Theorem 3.2.14.

Theorem 3.2.13. *Let M be a commutative monoid and let* $\dim : \pi_0(X) \to M$ *be a tangent dimension on a tangent category* $(\mathbb{X}, T, p, 0, +, \ell, c)$. *Let* $(E, M, q, \zeta, \sigma, \lambda)$ *be a differential bundle. Then the equation*

$$\dim(T(E)) + 2 \cdot \dim(M) = \dim(T(M)) + 2 \cdot \dim(E)$$

holds.

Proof. The diagrams

$$\begin{tikzcd} E_2 \arrow[r, "\pi_0"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "q"] \\ E \arrow[r, "q"'] & M \end{tikzcd} \qquad \begin{tikzcd} E_2 \arrow[r, "T(\sigma)\circ\langle\lambda\circ\pi_0, 0\circ\pi_1\rangle"] \arrow[d, "q\circ\pi_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & T(E) \arrow[d, "T(q)"] \\ M \arrow[r, "0"'] & T(M) \end{tikzcd}$$

are pullbacks. Since q is a retraction, 0 is a section and $T(q)$ is a retraction, a dimension in the sense of Definition 3.2.2 will fulfill the equations

$$\dim(E_2) + \dim(M) = \dim(E) + \dim(E)$$
$$\dim(E_2) + \dim(T(M)) = \dim(M) + \dim(T(E)).$$

Thus we obtain that

$$\dim(T(E)) + 2 \cdot \dim(M) = \dim(E_2) + \dim(M) + \dim(T(M)) = \dim(T(M)) + 2 \cdot \dim(E).$$

□

Theorem 3.2.14. *Let R be a ring and let $\dim : \pi_0(X) \to M$ be a strong tangent dimension on a tangent category $(\mathbb{X}, T, p, 0, +, \ell, c)$ with $a \in R$ such that $\dim(T(X)) = a \cdot \dim(X)$. Then for any differential bundle $(E, M, q, \zeta, \sigma, \lambda)$ the equation*

$$(a - 2)\dim(E) = (a - 2)\dim(M)$$

holds

Proof. Due to Theorem 3.2.13 the euqation

$$\dim(T(E)) + 2 \cdot \dim(M) = \dim(T(M)) + 2 \cdot \dim(E)$$

has to hold. Substituting in $\dim(T(E)) = a \cdot \dim(E)$ and $\dim(T(M)) = a \cdot \dim(M)$, we obtain

$$a \cdot \dim(E) + 2 \cdot \dim(M) = a \cdot \dim(M) + 2 \cdot \dim(E).$$

Subtracting $2 \cdot \dim(M) + 2 \cdot \dim(E)$ on both sides leads to the desired result that

$$(a - 2)\dim(E) = (a - 2)\dim(M).$$

□

Due to Theorem 3.2.8, we know that a is either 1 or 2. If $a = 2$, then there is no condition on the dimension of a differential bundle. If $a = 1$, then $\dim(E) = \dim(M)$, which can be interpreted as differential bundles having 0-dimensional fibre. Thus desired Property (f)(bundle fiber arbitrary dim) may only hold in tangent categories with a strong tangent dimension $a = 2$. In the tangent category of smooth manifolds this is the case.

In the following sections 3.3-3.7 we list some examples of dimensions on well-known categories and the consequences stemming from the dimension. In case it is a strong dimension, the consequences for differential bundles are either no condition (if $a = 2$) or that $\dim(E) = \dim(M)$ (if $a = 1$). Thus we will only mention differential bundles in case we have an example of a tangent category with a tangent dimension that is not strong.

3.3 Cardinality as a non-trivial dimension structure on $\mathbb{S}\mathrm{et}^{\mathrm{op}}$

In this section we will show that $\mathbb{S}\mathrm{et}^{\mathrm{op}}$, the opposite category of sets, has a categorical dimension, given by the cardinality. We are using Definition 3.2.2, which does not require a tangent structure. There is no known nontrivial tangent structure on $\mathbb{S}\mathrm{et}^{\mathrm{op}}$ and in fact we will prove that there is no nontrivial Cartesian tangent structure.

All mathematical objects are constructed from sets. In Corollary 3.2.4, we saw that precomposition with right-adjoint functors preserves dimensions. Most algebraic structures have a free functor from sets that is left-adjoint to the forgetful functor to sets. Thus, by precomposition with the forgetful functor, a dimension on sets would give rise to many dimensions in many different categories.

Since sets are discrete, there is no obvious structure in sets from which we could build tangents. Thus, we do not expect the category of sets to have a non-trivial tangent structure.

However, that does not mean that there can not be a dimension on the category of sets. In fact, as we see in Proposition 3.3.6, a dimension exists on $\mathbb{S}\mathrm{et}^{\mathrm{op}}$ and this dimension can be used to show that there is no nontrivial tangent structure.

The dimension of a vector space (and therefore the dimension of a manifold) is defined by the cardinality of a basis. Like the dimension for vector spaces, the cardinality is the main quantity describing the size of a set.

The cardinality fulfills desirable Properties (a) (valued in $\mathbb{N}$), (c) (isomorphism invariant) of Remark 3.2.1, since it is valued in the natural numbers and preserved by isomorphisms. When choosing multiplication as the monoid structure, it also fulfills (b') (additive w.r.t. products), since $\#(A \times B) = \#A \cdot \#B$. However the cardinality is not a categorical dimension in the sense of Definition 3.2.2 on the category $\mathbb{S}\mathrm{et}$. Additionally, we will prove that it is a dimension on $\mathbb{S}\mathrm{et}^{\mathrm{op}}$, the opposite category of sets.

As a consequence we will be able to describe potential tangent structures on the category $\mathbb{S}\mathrm{et}^{\mathrm{op}}$. In particular we will use the dimension to prove that there is no nontrivial Cartesian tangent structure on $\mathbb{S}\mathrm{et}^{\mathrm{op}}$.

Definition 3.3.1. *Let $\mathbb{N} = \{0, 1, 2, ...\}$ be the set of natural numbers. Let $\mathbb{N}_\infty$ be the set $\mathbb{N} \cup \{\infty\}$. Then we define the following three commutative monoid structures on $\mathbb{N}_\infty$.*

(a) The addition $+ : \mathbb{N}_\infty \times \mathbb{N}_\infty \to \mathbb{N}_\infty$ is defined to be the usual addition $a + b$ on natural numbers and to give ∞ if any of the two inputs is ∞. Together with the neutral element $0 \in \mathbb{N} \subset \mathbb{N}_\infty$ this is a commutative monoid.

(b) The multiplication $\cdot : \mathbb{N}_\infty \times \mathbb{N}_\infty \to \mathbb{N}_\infty$ is defined to be the usual multiplication on natural numbers and to give ∞ if any of the two inputs is ∞ (in particular $0 \cdot \infty = \infty$). Together with the neutral element $1 \in \mathbb{N} \subset \mathbb{N}_\infty$ this is a commutative monoid.

(c) The maximum $\max : \mathbb{N}_\infty \times \mathbb{N}_\infty \to \mathbb{N}_\infty$ is defined to output the lowest number that is greater or equal to both inputs and to give ∞ if any of the two inputs is ∞. Together with the neutral element $0 \in \mathbb{N} \subset \mathbb{N}_\infty$ this is a commutative monoid.

Remark 3.3.2. *The cardinality of a set is not an $\mathbb{N}_\infty$-valued dimension (for either of the monoid structures in Definition 3.3.1) on the category $\mathbb{S}\mathrm{et}$ of all sets and functions. The reason for this is that coproducts (disjoint unions) of sets are preserved by pullbacks and introduce some non-trivial combinatorics to the cardinality of pullbacks. A counterexample can be easily constructed by taking the disjoint union of any pullback diagram with a singleton.*

However, on the opposite category $\mathbb{S}\mathrm{et}^{\mathrm{op}}$, the cardinality is a dimension.

Theorem 3.3.3. *For the opposite category of sets, $\mathbb{S}\mathrm{et}^{\mathrm{op}}$ the cardinality $\dim(X) = \#X$ is an $(\mathbb{N}, +)$-valued dimension on $\mathbb{S}\mathrm{et}^{\mathrm{op}}$.*

Proof. Since we are in the opposite category of sets, the retractions in $\mathbb{S}\mathrm{et}^{\mathrm{op}}$ are exactly the injections of sets. For every pushout along an injection i

$$\begin{tikzcd} C \sqcup_A B & B \arrow[l] \\ C \arrow[u] & A \arrow[l, "f"'] \arrow[u, hook, "i"'] \end{tikzcd}$$

the set $C \sqcup_A B$ can be written as a disjoint union of C with $B - A$. Thus

$$\#(C \sqcup_A B) + \#A = \#C + \#(B - A) + \#A = \#C + \#B$$

and thereby the condition of Definition 3.2.2 is fulfilled. $\square$

Remark 3.3.4. *All the results about* $\mathbb{S}\mathrm{et}$ *in this section also hold for* $\mathbb{F}\mathrm{inSet}$*, the category of finite sets. Since* $\mathbb{F}\mathrm{inSet}$ *is a full subcategory of* $\mathbb{S}\mathrm{et}$*, this follows from Corollary 3.2.4 and the fact that we never used infinite sets in any proof.*

In particular the cardinality is an $(\mathbb{N}, +)$*-valued dimension on* $\mathbb{F}\mathrm{inSet}^{\mathrm{op}}$*.*

Below, we will apply Theorems 3.2.7 and 3.2.8 to the cardinality as a dimension. A particular situation are Cartesian tangent categories. Cartesian tangent categories are defined in 2.2.16 as tangent categories that admit finite products and where the products are compatible with the tangent structure.

Corollary 3.3.5.

(a) Any tangent structure on $\mathbb{S}\mathrm{et}^{\mathrm{op}}$ *will fulfill that*

$$\#T^2(X) + 2 \cdot \#X = 3 \cdot \#T(X).$$

(b) Every Cartesian tangent structure on $\mathbb{F}\mathrm{inSet}^{\mathrm{op}}$ *fulfills* $\#T(X) = \#T(\{*\}) \cdot \#X$ *and* $\#T(\{*\})$ *is either* $\{*\}$ *or* $\{*, *\}$*.*

Proof. Part (a) is a direct application of Theorem 3.2.7 to the cardinality.

For part (b), observe that every Cartesian tangent structure on $\mathbb{F}\mathrm{inSet}^{\mathrm{op}}$ will preserve disjoint unions (coproducts) and therefore $T(X) \cong T(\bigsqcup_{x \in X} \{*\}) \cong \bigsqcup_{x \in X} T(\{*\})$ which means $\#T(X) = \#T(\{*\}) \cdot \#X$. Then the hypothesis of Theorem 3.2.8 is fulfilled and therefore $\#T(X) = \#X$ or $\#T(X) = 2 \cdot \#X$, which shows that $\#T(\{*\})$ is either $\{*\}$ or $\{*, *\}$. $\square$

We can further specify the result of Corollary 3.3.5.b by showing that only one of these cases is possible. As a result we can show that $\mathbb{F}\mathrm{inSet}^{\mathrm{op}}$ has no Cartesian tangent structure beyond the trivial one.

Proposition 3.3.6. *Let* $(\mathbb{F}\mathrm{inSet}^{\mathrm{op}}, T, p, +, 0, \ell, c)$ *be a Cartesian tangent structure on* $\mathbb{F}\mathrm{inSet}^{\mathrm{op}}$*. Then it is in fact the trivial tangent structure i.e.* $T = 1_{\mathbb{F}\mathrm{inSet}^{\mathrm{op}}}$*,* $p = + = 0 = \ell = c = 1$*.*

Proof. According to Corollary 3.3.5, there are two cases, either the tangent functor sends the one point set to two points $T(\{*\}) = \{0, 1\}$ or $T(\{*\}) = \{*\}$.

Case 1: Suppose that $T(\{*\}) = \{0,1\}$. We will show that this is actually impossible and it leads to a contradiction. For this we look at the following diagrams

$$
\begin{tikzcd}
T(M) \arrow[r, "\langle 0\circ p, 1_{T(M)}\rangle"] \arrow[dr, "1_{T(M)}"'] & T_2(M) \arrow[d, "+"] \\
& T(M)
\end{tikzcd}
\qquad
\begin{tikzcd}
T(M) \arrow[r, "\langle 1_{T(M)}, 0\circ p\rangle"] \arrow[dr, "1_{T(M)}"'] & T_2 M \arrow[d, "+"] \\
& T(M)
\end{tikzcd}
\tag{3.1}
$$

which have to commute in any tangent category In $\mathbb{F}\mathrm{inSet}^{\mathrm{op}}$ with $T(\{*\}) = \{0,1\}$, the pullback of T along itself needs to have 3 elements (because cardinality is a dimension), we denote it as $T_2(*) = \{a,b,c\}$. The projection $p : T(\{*\}) \to \{*\}$ in $\mathbb{F}\mathrm{inSet}^{\mathrm{op}}$ corresponds to a function $\{*\} \to \{0,1\}$ in $\mathbb{F}\mathrm{inSet}$. Without loss of generality we assume it sends $*$ to 0, i.e. the projection morphism is the inclusion of zero function $p = i_0$. The zero section $0 : \{*\} \to T(\{*\})$ starts at the initial object of $\mathbb{F}\mathrm{inSet}^{\mathrm{op}}$ and thus is the unique such morphism. It corresponds to the unique function $\{0,1\} \to \{*\}$. Using these insights we now write the diagrams 3.1 as

$$
\begin{tikzcd}
\{0,1\} & \{a,b,c\} \arrow[l, "a\mapsto 0, b\mapsto 0, c\mapsto 1"'] \\
& \{0,1\} \arrow[u, "+"'] \arrow[ul, "1_{\{0,1\}}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\{0,1\} & \{a,b,c\} \arrow[l, "a\mapsto 0, b\mapsto 1, c\mapsto 0"'] \\
& \{0,1\} \arrow[u, "+"'] \arrow[ul, "1_{\{0,1\}}"]
\end{tikzcd}
$$

in $\mathbb{F}\mathrm{inSet}$. The maps $\{a,b,c\} \to \{0,1\}$ are obtained because the unique morphisms into pullbacks correspond to unique morphisms out of pushouts. The first diagram only commutes if $+(1) = c$ and the second diagram only commutes if $+(1) = b$. Thus they can not both commute, a tangent structure with $T(\{*\}) = \{0,1\}$ is impossible

Case 2: Suppose that $T(\{*\}) = \{*\}$. Then $p_{\{*\}}$, $+_{\{*\}}$, $0_{\{*\}}$, $\ell_{\{*\}}$ and $c_{\{*\}}$ are morphisms from $\{*\}$ to $\{*\}$. Since $\{*\}$ is initial in $\mathbb{S}\mathrm{et}^{\mathrm{op}}$, the only such morphism is the identity. Now, since $(\mathbb{F}\mathrm{inSet}^{\mathrm{op}}, T, p, +, 0, \ell, c)$ is Cartesian and every object is a product power (disjoint union) of $\{*\}$, the functor and structure morphisms also need to be the identity on every other object.

□

As a consequence of the cardinality as a dimension on $\mathbb{S}\mathrm{et}^{\mathrm{op}}$, the vertex set cardinality is also a dimension on $\mathbb{G}\mathrm{rph}^{\mathrm{op}}$, the opposite category of looped graphs.

The category $\mathbb{G}\mathrm{rph}$ of looped graphs has a forgetful functor $U : \mathbb{G}\mathrm{rph} \to \mathbb{S}\mathrm{et}$ given by taking the vertex set. This forgetful functor has a right-adjoint $R : \mathbb{S}\mathrm{et} \to \mathbb{G}\mathrm{rph}$ that sends a set to the complete graph on the set.

Proposition 3.3.7.

(a) *The assignment* $v : \pi_0(\mathbb{G}\mathrm{rph}) \to \mathbb{N}_\infty$ *that sends a graph to its number of vertices is an* $(\mathbb{N}_\infty, +)$*-valued dimension on* $\mathbb{G}\mathrm{rph}^{\mathrm{op}}$.

(b) *Let* $(\mathbb{G}\mathrm{rph}^{\mathrm{op}}, T, p, 0, +, \ell, c)$ *be a tangent structure on* $\mathbb{G}\mathrm{rph}^{\mathrm{op}}$*. Then for any graph* G*, the equation*

$$v(T^2(G)) + 2 \cdot v(G) = 3 \cdot v(T(G))$$

holds.

(c) *Let* $(\mathbb{G}\mathrm{rph}^{\mathrm{op}}, T, p, 0, +, \ell, c)$ *be a tangent structure on* $\mathbb{G}\mathrm{rph}^{\mathrm{op}}$ *and let* $a \in \mathbb{N}$ *be such that for every graph* G*,* $v(T(G)) = a \cdot v(G)$*. Then* $a = 1$ *or* $a = 2$*.*

Proof. (a) Since $U : \mathbb{G}\mathrm{rph} \to \mathbb{S}\mathrm{et}$ is left-adjoint, $U^{\mathrm{op}} : \mathbb{G}\mathrm{rph}^{\mathrm{op}} \to \mathbb{S}\mathrm{et}^{\mathrm{op}}$ is right-adjoint. Now Corollary 3.2.4 and Theorem 3.3.3 produce the dimension on the opposite category of looped graphs described above.

(b) This is a direct application of Theorem 3.2.7.

(c) This is a direct application of Theorem 3.2.8.

□

An important extension of the opposite category of sets is the category of polynomial functors. Thus one could hope that the cardinality can be extended to a dimension on the category of polynomial functors.

Example 3.3.8. *The category* $\mathbb{P}\mathrm{oly}$ *of polynomial functors and natural transformations has been extensively analyzed in the literature, for example in* [76]*. For the precise definitions and constructions in* $\mathbb{P}\mathrm{oly}$ *we ask the reader to read* [76]*. One of its key properties is that it contains* $\mathbb{S}\mathrm{et}^{\mathrm{op}}$ *as a full subcategory, concretely as the full subcategory of monomials. The degree of a monomial corresponds to the cardinality of a set.*

Since $\mathbb{P}\mathrm{oly}$ *is an extension of* $\mathbb{S}\mathrm{et}^{\mathrm{op}}$ *it seems promising to define the degree of a polynomial functor as the dimension. However, we will present an example of a pullback where the degree of polynomials does not add up like in Definition 3.2.2.*

Pullbacks in $\mathbb{P}\mathrm{oly}$ *are explained in* [76, Example 5.38]*. They are given by taking the pullback of monomial-sets and the pushout of exponents. Due to this characterization of pullbacks, the following square is a pullback in* $\mathbb{P}\mathrm{oly}$*.*

$$\begin{tikzcd}
y^{\{1,2,3\}} + y^{\{a,b,c\}} \arrow[r] \arrow[d] & y^{\{3\}} + y^{\{a,b\}} \arrow[d, "y^0+y^0"] \\
y^{\{1,2\}} + y^{\{c\}} \arrow[r, "y^0+y^0"'] & y^{\emptyset} + y^{\emptyset}
\end{tikzcd}$$

where for any set X, the map $0 : \emptyset \to X$ is the unique element of $\mathrm{Hom}_{\mathbb{S}\mathrm{et}}(\emptyset, X)$ *and* $y : \mathbb{S}\mathrm{et}^{\mathrm{op}} \to \mathrm{Fun}(\mathbb{S}\mathrm{et}, \mathbb{S}\mathrm{et})$ *is the Yoneda embedding. Since $0 : \emptyset \to X$ is a section in* $\mathbb{S}\mathrm{et}$, *the natural transformations y^0 and $y^0 + y^0$ are retractions in* $\mathbb{P}\mathrm{oly}$. *Then*

$$\deg(y^{\{3\}} + y^{\{a,b\}}) + \deg(y^{\{1,2\}} + y^{\{c\}}) = 2 + 2 = 4$$

while

$$\deg(y^{\{1,2,3\}} + y^{\{a,b,c\}}) + \deg(y^0 + y^0) = 3 + 0 = 3.$$

Thus

$$\deg(y^{\{3\}} + y^{\{a,b\}}) + \deg(y^{\{1,2\}} + y^{\{c\}}) \neq \deg(y^{\{1,2,3\}} + y^{\{a,b,c\}})$$

showing that the polynomial functor degree is not a categorical dimension on $\mathbb{P}\mathrm{oly}$ *in the sense of Definition 3.2.2.*

3.4 Cardinality of groups as a dimension valued in a non-standard monoid

In this section we will show that $\mathbb{G}\mathrm{rp}$, the category of groups, has a categorical dimension given by the cardinality. A tangent structure on $\mathbb{G}\mathrm{rp}$ can be found in [54]. The tangent bundle endofunctor is given by $T(G) = G \times \mathrm{Ab}(G)$ where $\mathrm{Ab}(G)$ is the abelianization of G. Since every categorical dimension on a tangent category is a tangent dimension, the cardinality will be a tangent dimension.

One of the most important algebraic structures on a set is a group. Like for sets the cardinality is the most important quantity to describe the size of a group. Therefore we will check below, if it is a dimension on $\mathbb{G}\mathrm{rp}$ or $\mathbb{G}\mathrm{rp}^{\mathrm{op}}$.

When thinking about tangent structures on groups, one may first think of the underlying manifold structure of Lie groups, which is a trivial tangent structure on discrete groups. However, in [54], Ikonicoff, Lemay and Van der Linden showed that there is a different tangent structure on the category of groups that is based purely on algebraic properties (in particular the abelianization), not topological properties of a group. We will explore the consequence of dimensions on this tangent structure in Example 3.4.4.

An interesting aspect of the dimension on $\mathbb{G}\mathrm{rp}$ is that it is valued in $(\mathbb{N}_\infty, \cdot)$ and not in $(\mathbb{N}_\infty, +)$. Unlike in all previous examples, the commutative monoid structure it is valued in is given by the multiplication, not the addition. This showcases the use coming from being valued in any commutative monoid.

When considering the cardinality as a candidate for a dimension on $\mathbb{G}\mathrm{rp}$, one needs to choose multiplication as the monoid operation to obtain an analogue for desired Property (b') (additive w.r.t. products) of Remark 3.2.1 as $\#(G \times G') = \#G \cdot \#G'$.

Lemma 3.4.1. *Let $A \xrightarrow{f} C \xleftarrow{g} B$ be a cospan in $\mathbb{G}\mathrm{rp}$, the category of groups, where f is an epimorphism. Then the pullback $A \times_C B$ exists and*

$$\#(A \times_C B) \cdot \#C = \#A \cdot \#B.$$

Proof. The pullback exists since the category of groups is complete. Since the forgetful functor from groups to sets preserves limits, the underlying set of the pullback can explicitly be described by the formula

$$A \times_C B = \{(a, b) \in A \times B \mid f(a) = g(b)\}.$$

We now rewrite this formula as

$$A \times_C B = \bigsqcup_{c \in \mathrm{Im}(g)} g^{-1}(c) \times f^{-1}(c),$$

where $\mathrm{Im}(g)$ denotes the image and $g^{-1}(c)$ and $f^{-1}(c)$ denote pre-images. If C and thus $\mathrm{Im}(g)$ is finite, it follows from the first isomorphism theorem [4, Corollary 8.2] and the surjectivity of f that for any $c \in \mathrm{Im}(g)$, this means that $\#g^{-1}(c) = \frac{\#B}{\#\mathrm{Im}(g)}$ and $\#f^{-1}(c) = \frac{\#A}{\#C}$. Therefore we conclude that

$$\#(A \times_C B) = \#\mathrm{Im}(g) \cdot \frac{\#B}{\#\mathrm{Im}(g)} \cdot \frac{\#A}{\#C} = \frac{\#B \cdot \#A}{\#C}.$$

Multiplying both sides of the equation with $\#C$ concludes the proof for the case that C is finite.

If C is infinite, the surjectivity of f implies that A is also infinite. Thus

$$\#A \cdot \#B = \infty = \#C \cdot \#(A \times_B C).$$

Here we did not encounter any multiplications of the form $0 \cdot \infty$, since a group always has at least one element. □

We observe that this formula looks very like Definition 3.2.2 with multiplication instead of addition. Therefore we will now consider the commutative monoid $(\mathbb{N}_\infty, \cdot)$, as defined in Definition 3.3.1.

Theorem 3.4.2. *The cardinality is*

(a) an $(\mathbb{N}_\infty, \cdot)$-valued dimension on $\mathbb{G}\mathrm{rp}$ and

(b) an $(\mathbb{N}, \cdot)$-valued dimension on $\mathbb{F}\mathrm{in}\mathbb{G}\mathrm{rp}$.

Proof.

(a) This is follows from Lemma 3.4.1.

(b) This follows from Corollary 3.2.4 and the fact that $\mathbb{F}\mathrm{in}\mathbb{G}\mathrm{rp}$ is a full subcategory of $\mathbb{G}\mathrm{rp}$.

$\square$

Corollary 3.4.3. *Let X be a group.*

(a) For any tangent structure $(T, p, 0, +, \ell, c)$ on $\mathbb{G}\mathrm{rp}$ or $\mathbb{F}\mathrm{in}\mathbb{G}\mathrm{rp}$, the equation

$$\#T^2(X) \cdot (\#X)^2 = (\#T(X))^3$$

has to hold.

(b) For any tangent structure $(T, p, 0, +, \ell, c)$ on $\mathbb{F}\mathrm{in}\mathbb{G}\mathrm{rp}$ with $\#T(X) = (\#X)^n$, the constant n can only be $n = 1$ or $n = 2$.

(c) The only tangent structure $(T, p, 0, +, \ell, c)$ on $\mathbb{F}\mathrm{in}\mathbb{G}\mathrm{rp}$ with $\#T(X) = n \cdot \#X$ fulfills $T(X) \cong X$.

Proof.

(a) This follows from Theorems 3.4.2 and 3.2.7. When formulated for the commutative monoid $(\mathbb{N}, \cdot)$ the commutative monoid operation that is denoted as "$+$" in Theorem 3.2.7 becomes the multiplication "$\cdot$".

(b) Due to Part (a), the equation

$$((\#X)^n)^n \cdot (\#X)^2 = (\#X)^{3n}$$

has to hold. For brevity, we denote $a := \#X$ and simplify the equation to

$$a^{n^2+2} = a^{3n}.$$

Since this has to hold for any finite group X, it has to hold for any $a \in \mathbb{N}$ and this implies that $n^2 + 2 = 3n$ which only holds for $n = 1$ or $n = 2$.

(c) Due to Part (a), the equation

$$n^2 \cdot \#X \cdot (\#X)^2 = (n \cdot \#X)^3$$

has to hold. This implies $n^2 = n^3$, which is only true for $n = 0$ and $n = 1$. However, $n = 0$, since every group has at least one element and thus $\#T(X) = 0$ is impossible. Thus $\#X = \#T(X)$. As a consequence of the first isomorphism theorem [4, Corollary 8.2], the epimorphism $p : T(X) \to X$ that does not change the cardinality can only be an isomorphism.

□

As an example, we will now see how this formula applies to the known nontrivial tangent structure on $\mathbb{G}\mathrm{rp}$.

Example 3.4.4 ([54], Example 3.8)**.** *There is a tangent structure on* $\mathbb{G}\mathrm{rp}$ *whose tangent bundle endofunctor is given by*

$$T(G) = G \times \mathrm{Ab}(G)$$

where $\mathrm{Ab}(G)$ *denotes the abelianization. This implies that*

$$T^2(G) \cong G \times \mathrm{Ab}(G) \times \mathrm{Ab}(G) \times \mathrm{Ab}(G)$$

because $\mathrm{Ab}(\mathrm{Ab}(G)) \cong \mathrm{Ab}(G)$*. Define* $n_G := \#G$ *and* $k_G := \#\mathrm{Ab}(G)$*. Since* $\mathrm{Ab}(G)$ *is a quotient of* G*,* $k_G \leq n_G$ *and* k_G *divides* n_G*. Then* $\#T(G) = n_G \cdot k_G$ *and* $\#T^2(G) = n_G \cdot (k_G)^3$*. Thus the equation*

$$\#T^2(G) \cdot (\#G)^2 = (\#T(G))^3$$

becomes

$$n_G \cdot (k_G)^3 \cdot n_G^2 = (n_G \cdot k_G)^3.$$

Thus, in this example, the equation from Corollary 3.4.3(a) holds.

In fact, Lemay and Ikonicoff have also shown a result on differential bundles in [54, Example 4.16]: Every differential bundle is of the form $M \times A \xrightarrow{\pi_0} M$ for some abelian group A. This implies that

$$\frac{\#\mathrm{Ab}(M \times A)}{\#(M \times A)} = \#A = \frac{\#\mathrm{Ab}(M)}{\#M}.$$

We will now showcase the possibilities of a dimension by providing a different proof for this equation using the cardinality as a dimension.

Proposition 3.4.5. *Let* $(\mathbb{G}\mathrm{rp}, T, p, 0, +, \ell)$ *be the tangent structure from* [54] *described in Example 3.4.4. Let* $E \to M$ *be a differential bundle in* $(\mathbb{G}\mathrm{rp}, T, p, 0, +, \ell)$*. Then*

$$\frac{\#\mathrm{Ab}(E)}{\#E} = \frac{\#\mathrm{Ab}(M)}{\#M}$$

Proof. Due to Theorem 3.2.13, the equation

$$\#T(E) \cdot (\#M)^2 = \#T(M) \cdot (\#E)^2$$

holds. Since $\#T(E) = \#E \cdot \#\mathrm{Ab}(E)$ and $\#T(M) = \#M \cdot \#\mathrm{Ab}(M)$, this can be expanded into

$$\#\mathrm{Ab}(E) \cdot \#E \cdot (\#M)^2 = \#\mathrm{Ab}(E) \cdot \#M \cdot (\#E)^2.$$

Dividing both sides by $\#E^2 \cdot \#M^2$ leads to the equation

$$\frac{\#\mathrm{Ab}(E)}{\#E} = \frac{\#\mathrm{Ab}(M)}{\#M}.$$

□

In Example 3.6.8 we will further see that there is another nontrivial categorical dimension on the subcategory abelian groups, given by the rank over the integers. The reason for this is that every abelian group is a module over the integers. Since the cardinality is an $(\mathbb{N}_\infty, +)$-valued dimension on $\mathbb{S}\mathrm{et}^{\mathrm{op}}$ might suspect that the cardinality is also an $(\mathbb{N}_\infty, +)$-valued dimension on $\mathbb{G}\mathrm{rp}^{\mathrm{op}}$, the opposite category of groups, since it was already a dimension on $\mathbb{S}\mathrm{et}^{\mathrm{op}}$. However, Corollary 3.2.4 does not apply here, since the forgetful functor $\mathbb{G}\mathrm{rp}^{\mathrm{op}} \to \mathbb{S}\mathrm{et}^{\mathrm{op}}$ does only have a right adjoint, not a left-adjoint.

In fact the cardinality is not a dimension on $\mathbb{G}\mathrm{rp}^{\mathrm{op}}$. The reason is that the product in $\mathbb{G}\mathrm{rp}^{\mathrm{op}}$ is the coproduct in $\mathbb{G}\mathrm{rp}$, the free product of groups. The free product of groups does not fulfill desired Property (b') (additive w.r.t. products) of Remark 3.2.1. For finite groups A and B, the cardinality of $\#(A \otimes B)$ can even be infinite.

3.5 Characteristic, Krull dimension and the lowest common multiple

In this section we will show that $\mathbb{R}\mathrm{ing}_n$, the category of (in general nonunital) rings, has a categorical dimension given by the characteristic. A tangent structure on $\mathbb{C}\mathrm{Ring}_u$ (the subcategory of commutative unital rings with unital ring-homomorphisms) can be found in [81, Example 2] and Example 2.2.13(i). Since every categorical dimension on a tangent category is a tangent dimension, the characteristic will be a tangent dimension.

In this section we investigate different approaches to dimensions of rings. First we define several different categories of rings. The different choices we consider depend on whether or not the ring is commutative, whether or not the ring is unital, and whether or not the homomorphisms must preserve the unit.

Then we will show that the characteristic of a ring is a categorical dimension. This will imply that any tangent structure on the category of rings must satisfy certain constraints. This is part way of showing that any tangent structure on rings has to preserve the characteristic.

Then we will show that the Krull dimension is neither a categorical dimension on the category of rings, nor on its opposite. This is surprising since the opposite category of commutative unital rings is equivalent to the category of affine schemes and the dimension of the scheme corresponds to the Krull dimension of the ring. However, the dimension of an affine scheme is only compatible with the tangent structure for smooth schemes. We suspect the Krull dimension will be a categorical dimension on the opposite of the subcategory of commutative unital rings which corresponds to smooth schemes.

Let $\mathbb{R}\mathrm{ing}_n$ be the category of all rings (possibly without unit, the n stands for non-unital) and all ring homomorphisms (functions that preserve zero, addition and multiplication). Let $\mathbb{R}\mathrm{ing}_1$ denote the full subcategory of $\mathbb{R}\mathrm{ing}_n$ consisting of all unital rings, i.e. rings with a unit element. The morphisms may not preserve the unit. Let $\mathbb{R}\mathrm{ing}_u$ denote the subcategory of $\mathbb{R}\mathrm{ing}_1$ that only allows unit-preserving homomorphisms. We define $\mathbb{C}\mathrm{Ring}_n$, $\mathbb{C}\mathrm{Ring}_1$ and $\mathbb{C}\mathrm{Ring}_u$ as the full subcategories of $\mathbb{R}\mathrm{ing}_n$, $\mathbb{R}\mathrm{ing}_1$ and $\mathbb{R}\mathrm{ing}_u$ where the objects are commutative rings. For overview, we summarize this in Table 3.1.

Name	Objects	Morphisms
$\mathbb{R}\mathrm{ing}_n$	any rings	any ring homomorphisms
$\mathbb{R}\mathrm{ing}_1$	unital rings	any ring homomorphisms
$\mathbb{R}\mathrm{ing}_u$	unital rings	unit preserving ring homomorphisms
$\mathbb{C}\mathrm{Ring}_n$	any commutative rings	any ring homomorphisms
$\mathbb{C}\mathrm{Ring}_1$	unital commutative rings	any ring homomorphisms
$\mathbb{C}\mathrm{Ring}_u$	unital commutative rings	unit preserving ring homomorphisms

Table 3.1: The conventions we use for the different categories of rings

In other literature, $\mathbb{R}\mathrm{ing}$ sometimes denotes $\mathbb{R}\mathrm{ing}_n$ and sometimes $\mathbb{R}\mathrm{ing}_u$ while $\mathbb{C}\mathrm{Ring}$ most often denotes $\mathbb{C}\mathrm{Ring}_u$. For clarity we do not use undecorated $\mathbb{R}\mathrm{ing}$ and $\mathbb{C}\mathrm{Ring}$.

Before we define dimensions on the categories of rings, we will first describe pullbacks of rings.

Lemma 3.5.1. *Any pullback*

$$\begin{tikzcd} A \times_B C \arrow[r] \arrow[d] & A \arrow[d, "f"] \\ C \arrow[r, "g"'] & B \end{tikzcd}$$

in $\mathbb{R}\mathrm{ing}_n$, $\mathbb{R}\mathrm{ing}_1$, $\mathbb{R}\mathrm{ing}_u$, $\mathbb{C}\mathrm{Ring}_n$, $\mathbb{C}\mathrm{Ring}_1$ *or* $\mathbb{C}\mathrm{Ring}_u$, *is of the form*

$$A \times_B C = \{(a, b) \in A \times B \mid f(a) = g(b)\},$$

if it exists. The ring operations are defined componentwise.

Proof. The formula for $\mathbb{R}\mathrm{ing}_n$ is an easy application of the definition of a limit. The formula for $\mathbb{R}\mathrm{ing}_1$, $\mathbb{C}\mathrm{Ring}_n$ and $\mathbb{C}\mathrm{Ring}_u$ follows from the fact that the embedding of full subcategories preserves limits. The formula for $\mathbb{R}\mathrm{ing}_u$ is a direct application of the definition of a limit and the formula for $\mathbb{C}\mathrm{Ring}_u$ follows from the fact that $\mathbb{C}\mathrm{Ring}_u$ is a a full subcategory of $\mathbb{R}\mathrm{ing}_u$. $\square$

Pullbacks may not always exist in $\mathbb{R}\mathrm{ing}_1$ and $\mathbb{C}\mathrm{Ring}_1$. An example for this is the pullback

$$\begin{tikzcd}
2\mathbb{Z} \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{Z} \arrow[d, "x \mapsto (x,0)"] \\
\mathbb{Z} \arrow[r, "x \mapsto (x,[x])"'] & \mathbb{Z} \times \mathbb{Z}/2
\end{tikzcd}$$

in $\mathbb{R}\mathrm{ing}_n$ which does not exist in $\mathbb{R}\mathrm{ing}_1$ since $2\mathbb{Z}$, the set of even integers, has no multiplicative unit.

3.5.1 Characteristic as a dimension valued in the least common multiple monoid

In this section we will use a monoid structure on the natural numbers which is neither of the operations on the rig (semiring) $\mathbb{N}$. At first one might think that the characteristic of a ring is a strange choice for a dimension, since in general $\mathrm{char}(R \times R') \neq \mathrm{char}(R) + \mathrm{char}(R')$ and therefore desirable Property (b') (additive w.r.t. products) does not hold. However, the least common multiple is also a monoid structure on $\mathbb{N}^*_\infty$.

Definition 3.5.2. *Let $\mathbb{N}^*_\infty$ denote the set of nonzero natural numbers and infinity*

$$\mathbb{N}^*_\infty := \mathbb{N} \setminus \{0\} \sqcup \{\infty\} = \{1, 2, 3, ..., \infty\}.$$

Given $a, b \in \mathbb{N}^*_\infty$, we define their ***least common multiple*** *as*

$$\mathrm{lcm}(a, b) = \min(\{x \in \mathbb{N}^*_\infty \mid \exists s, t \in \mathbb{N}^*_\infty, x = s \cdot a = t \cdot b\}).$$

We define $\mathrm{lcm}(a, \infty) = \mathrm{lcm}(\infty, b) = \infty$ for all $a, b \in \mathbb{N}_\infty \setminus \{0\}$.

The least common multiple is a commutative monoid with unit 1. In particular lcm is associative, since the least common multiple of three numbers a, b, c is both equal to $\mathrm{lcm}(a, \mathrm{lcm}(b, c))$ and to $\mathrm{lcm}(\mathrm{lcm}(a, b), c)$.

The characteristic defined in Definition 3.5.3 is additive with respect to this monoid structure, i.e.

$$\mathrm{char}(R \times R') = \mathrm{lcm}(\mathrm{char}(R), \mathrm{char}(R')),$$

and thus Property (b') (additive w.r.t. products) holds with respect to this monoid structure.

In this section we will show that the characteristic of a unital ring is a dimension. The characteristic is defined as follows.

Definition 3.5.3. *Let R be a ring. The* ***characteristic*** $\mathrm{char}(R)$ *is defined as*

$$\mathrm{char}(R) = \min\{n \in \mathbb{N} \setminus \{0\} \mid n \cdot x = 0 \ , \forall x \in R\}.$$

If there is no number $n \in \mathbb{N}$ such that for all $x \in R$, $n \cdot x = 0$, we say the characteristic is ∞.

If the characteristic is finite, this is the classical definition in [43]. *Classically, the characteristic is defined to be zero if there is no number $n \in \mathbb{N}$ such that $n \cdot x = 0$ for all x.*

In the literature it is more common to call the characteristic zero when we call it infinity. We use this unusual convention since it will make it easier to define a monoid structure such that the characteristic is a categorical dimension.

Lemma 3.5.4. *Let $(R, +, \cdot, 0)$ be a ring of characteristic n. Then there exists an element $x_R \in R$ such that $k \cdot x_R \neq 0$ for all $k < n$. We call this element a* ***maximal order element****.*

Proof. The key insight for this proof is that the mulitplicative structure of R does not matter this is just a statement about the abelian group $(R, +, 0)$. Due to the characterization of finite exponent abelian groups in [56, Theorem 8.6], the abelian group $(R, +, 0)$ is isomorphic (as a group) to a product of cyclic groups.

$$R \overset{\varphi}{\cong} \prod_{i \in I} \mathbb{Z}/n_i\mathbb{Z}$$

where n is the least common multiple of the n_i. The element $x_R = \varphi^{-1}(1, 1, ...)$ fulfills the required property since $k \cdot (1, 1, ...) \neq 0$ for $k < n$. □

Section retraction pairs have a key role in the definition of categorical dimensions and in the definition of tangent categories. Thus we will frequently reason about characteristics using section retraction pairs. For this, it is good to keep in mind that sections can not decrease characteristic.

Lemma 3.5.5. *Given a section $A \xrightarrow{s} B$ with retraction $B \xrightarrow{r} A$ in Ring_n, the inequality*

$$\mathrm{char}(A) \leq \mathrm{char}(B)$$

holds. Additionally, $\mathrm{char}(B)$ *is a multiple of* $\mathrm{char}(A)$.

Proof. We distinguish the cases when A has finite characteristic and when A has infinite characteristic.

Case 1: A has infinite characteristic. Since A has infinite characteristic, for every $n \in \mathbb{N} \setminus \{0\}$, there is an $x_n \in A$ such that $n \cdot x_n \neq 0$. Then $s(x_n) \in B$ fulfills that

$$n \cdot x_n = n \cdot r(s(x_n)) = r(n \cdot s(x_n)) \neq 0.$$

Thus $n \cdot s(x_n) \neq 0$. Since n was arbitrary, this proves that B has infinite characteristic.

Case 2: A has finite characteristic. Let x_A be the element constructed in Lemma 3.5.4. Since $r \circ s = 1_A$, we know that for every $n < \mathrm{char}(A)$,

$$0 \neq n \cdot r(s(x_A)) = r(n \cdot s(x_A)).$$

Since r is a ring-homomorphism, this implies for $s(x_A) \in B$ that $n \cdot s(x_A) \neq 0$ implying that $\mathrm{char}(B) \geq \mathrm{char}(A)$.

On the other hand,

$$s(\mathrm{char(B)} \cdot x_A) = \mathrm{char}(B) \cdot s(x_A) = 0$$

and therefore

$$\mathrm{char(B)} \cdot x_A = r(s(\mathrm{char(B)} \cdot x_A)) = r(0) = 0$$

which implies that char(B) is a multiple of $\mathrm{char}(A)$. ◻

Proposition 3.5.6. *The characteristic* char *is an* $(\mathbb{N}^*_\infty, \mathrm{lcm})$*-valued dimension on* $\mathbb{R}\mathrm{ing}_n$*,* $\mathbb{R}\mathrm{ing}_1$ *and on* $\mathbb{R}\mathrm{ing}_u$*.*

Proof. We will first check the criteria of Definition 3.2.2 for $\mathbb{R}\mathrm{ing}_u$ and then for $\mathbb{R}\mathrm{ing}_n$. The result for $\mathbb{R}\mathrm{ing}_1$ follows from Corollary 3.2.4 since $\mathbb{R}\mathrm{ing}_1$ is a full subcategory of $\mathbb{R}\mathrm{ing}_n$.

In $\mathbb{R}\mathrm{ing}_u$, given any morphism $f : R \to R'$, the characteristic can not increase $\mathrm{char}(R') \leq \mathrm{char}(R)$ since $0 = f(0) = f(\mathrm{char}(R) \cdot 1)$. Thus, given a section retraction pair between R and R', $\mathrm{char}(R) = \mathrm{char}(R')$. Now let

$$\begin{tikzcd} A \times_B C \arrow[r] \arrow[d] & A \arrow[d, "r"] \\ C \arrow[r, "f"'] & B \end{tikzcd}$$

be any of the pullbacks from Definition 3.2.2 in $\mathbb{R}\mathrm{ing}_u$. Then f and r are both parts of section-retraction pairs and the characteristic of all 4 vertices is the same. Therefore

$$\mathrm{lcm}(\mathrm{char}(A), \mathrm{char}(C)) = \mathrm{lcm}(\mathrm{char}(A \times_B C), \mathrm{char}(B))$$

holds trivially.

Now let

$$\begin{tikzcd} A \times_B C \arrow[r] \arrow[d] & A \arrow[d, "r"] \\ C \arrow[r, "f"'] & B \end{tikzcd}$$

be any of the pullbacks from Definition 3.2.5 in $\mathbb{R}\mathrm{ing}_n$. Assuming that A and C have finite characteristic, let $x_A \in A$ and $x_C \in C$ be the maximal order elements constructed in Lemma 3.5.4. We will distinguish the cases when f is a section and when f is a retraction.

Case 1: Suppose f is a retraction, in particular it is surjective. Then there is a $y_C \in C$ such that $f(y_c) = r(x_A)$ and thus we have $(x_a, y_C) \in A \times_B C$ such that for $k \leq \mathrm{char}(A)$, $k \cdot (x_A, y_C) = (k \cdot x_A, k \cdot y_C) \neq 0$. Thus $\mathrm{char}(A \times_B C)$ is a multiple of $\mathrm{char}(A)$.

Analogously there is an $y_A \in A$ such that $(y_A, x_C) \in A \times_B C$ and thus $\mathrm{char}(A \times_B C)$ is a multiple of $\mathrm{char}(C)$. This shows that $\mathrm{char}(A \times_B C)$ is a multiple of $\mathrm{lcm}(\mathrm{char}(A), \mathrm{char}(C))$. Since $A \times_B C$ is a subset of $A \times C$, the equation

$$\mathrm{char}(A \times_B C) = \mathrm{lcm}(\mathrm{char}(A), \mathrm{char}(C))$$

holds. Due to Lemma 3.5.5, $\mathrm{char}(B)$ divides $\mathrm{char}(A)$, implying that

$$\mathrm{lcm}(\mathrm{char}(A \times_B C), \mathrm{char(B)}) = \mathrm{lcm}(\mathrm{lcm}(\mathrm{char}(A), \mathrm{char}(C)), \mathrm{char}(B)) = \mathrm{lcm}(\mathrm{char}(A), \mathrm{char}(C)).$$

This is exactly the equation we needed to show for Definition 3.2.2.

Case 2: Suppose f is a section. Then, due to Lemma 3.5.5, $\mathrm{char}(B)$ and $\mathrm{char}(C)$ both divide $\mathrm{char}(A)$.

Let $s : B \to A$ be a section for the retraction r. We can express the maximal order element $x_A \in A$ as $x_A = s(r(x_a)) + z_A$ with $r(z_A) = 0$, by choosing $z_A = x_A - s(r(x_A))$.

For $k = \mathrm{lcm}(\mathrm{char}(A \times_B C), \mathrm{char}(B))$, we observe that

$$k \cdot x_A = k \cdot s(r(x_A)) + k \cdot z_A.$$

Since k is a multiple of $\mathrm{char}(B)$, $k \cdot s(r(x_A)) = s(k \cdot r(x_A)) = 0$. Since $r(z_A) = 0$, we have the element $(z_A, 0) \in A \times_B C$ and since k is a mutiple of $\mathrm{char}(A \times_B C)$, $k \cdot (z_A, 0) = 0$, implying that $k \cdot z_A = 0$.

Thus $k \cdot x_A = 0$ and since x_A had maximal order this shows that $\mathrm{char}(A)$ divides $\mathrm{lcm}(\mathrm{char}(A \times_B C), \mathrm{char}(B))$. Since $\mathrm{char}(C)$ divides $\mathrm{char}(A)$, the equation $\mathrm{lcm}(\mathrm{char}(A), \mathrm{char}(C)) = \mathrm{char}(A)$ holds, implying that $\mathrm{lcm}(\mathrm{char}(A), \mathrm{char}(C)) = \mathrm{char}(A)$ divides $\mathrm{lcm}(\mathrm{char}(A \times_B C), \mathrm{char}(B))$. Since $A \times_B C$ is a subset of

$A \times C$, this shows that

$$\operatorname{lcm}(\operatorname{char}(A \times_B C), \operatorname{char}(B)) = \operatorname{lcm}(\operatorname{char}(A), \operatorname{char}(C)).$$

This is exactly the equation we needed to show for Definition 3.2.2.

Now it only remains to show that the equation

$$\operatorname{lcm}(\operatorname{char}(A \times_B C), \operatorname{char}(B)) = \operatorname{lcm}(\operatorname{char}(A), \operatorname{char}(C)).$$

also holds when $\operatorname{char}(A)$ or $\operatorname{char}(C)$ are infinite. If $\operatorname{char}(B) = \infty$, it holds trivially. Suppose $\operatorname{char}(B)$ is finite. Since in the case that f is a section $\operatorname{char}(C) \leq \operatorname{char}(B)$, and in the case that f is a retraction, C and A can be swapped, we can assume without loss of generality that $\operatorname{char}(A) = \infty$.

Let $n \in \mathbb{N} \setminus \{0\}$ be arbitrary. Let $x_n \in A$ be such that $n \cdot \operatorname{char}(B) \cdot x_n \neq 0$. Then $r(\operatorname{char}(B) \cdot x_n) = 0$ and thus $(\operatorname{char}(B) \cdot x_n, 0) \in A \times_B C$ fulfills that

$$n \cdot (\operatorname{char}(B) \cdot x_n, 0) \neq 0.$$

Since n was arbitrary, this shows that $\operatorname{char}(A \times_B C) = \infty$ and thus

$$\operatorname{lcm}(\operatorname{char}(A \times_B C), \operatorname{char}(B)) = \operatorname{lcm}(\operatorname{char}(A), \operatorname{char}(C)).$$

holds. This concludes the proof that char is a $(\mathbb{N}^*_\infty, \operatorname{lcm})$-valued dimension on $\mathbb{R}\mathrm{ing}_n$. Since $\mathbb{R}\mathrm{ing}_1$ is a full subcategory of $\mathbb{R}\mathrm{ing}_n$, Corollary 3.2.4 implies that it also is a $(\mathbb{N}^*_\infty, \operatorname{lcm})$-valued dimension on $\mathbb{R}\mathrm{ing}_1$. □

Corollary 3.5.7. *The characteristic* char *is an* $(\mathbb{N}^*_\infty, \operatorname{lcm})$*-valued dimension on* $\mathbb{CR}\mathrm{ing}_n$, $\mathbb{CR}\mathrm{ing}_1$ *and on* $\mathbb{CR}\mathrm{ing}_u$.

Proof. This follows directly from Proposition 3.5.6 and Corollary 3.2.4. □

As a consequence, the characteristic of the tangent bundle is now determined in $\mathbb{R}\mathrm{ing}_1$ and $\mathbb{R}\mathrm{ing}_u$. However, the characteristic of a tangent bundle in $\mathbb{R}\mathrm{ing}_u$ is determined anyways.

Proposition 3.5.8. *Let* $(T, p, 0, +, \ell, c)$ *be a tangent structure on* $\mathbb{R}\mathrm{ing}_u$. *Then for any unital ring* R,

$$\operatorname{char}(T(R)) = \operatorname{char}(R).$$

Proof. This simply follows from the fact that any unit-preserving ring homomorphism can only decrease characteristic. Since both $R \xrightarrow{0} T(R)$ and $T(R) \xrightarrow{p} R$ are unit-preserving ring homomorphisms, the characteristics have to equal. $\square$

For $\mathbb{R}\mathrm{ing}_1$, however the dimension brings an additional insight.

Corollary 3.5.9. *Let $(T, p, 0, +, \ell, c)$ be a tangent structure on $\mathbb{R}\mathrm{ing}_n$ or $\mathbb{R}\mathrm{ing}_1$. Then for every object R,*

$$\mathrm{lcm}(\mathrm{char}(T^2(R)), \mathrm{char}(R)) = \mathrm{char}(T(R))$$

Proof. This is Theorem 3.2.7 applied to the dimension defined in Proposition 3.5.6. $\square$

However, since we have a section retraction pair $R \xrightarrow{0} T(R), T(R) \xrightarrow{p} R$, we can even conclude more.

Corollary 3.5.10. *Let $(T, p, 0, +, \ell, c)$ be a tangent structure on $\mathbb{R}\mathrm{ing}_n$ or $\mathbb{R}\mathrm{ing}_1$. Then for every object R there is a number n_R such that*

$$\mathrm{char}(T(R)) = n_R \cdot \mathrm{char}(R)$$

and

$$\mathrm{char}(T^2(R)) = \mathrm{char}(T(R)).$$

Proof. Corollary 3.5.9 implies that the characteristic of $T(R)$ is a multiple of the characteristic of R.

Corollary 3.5.9 also implies that it is a multiple of the characteristic of $T^2(R)$. Due to the same argument applied to $T(R)$ instead of R, $\mathrm{char}(T^2(R))$ is a multiple of $\mathrm{char}(T(R))$. This implies that $\mathrm{char}(T(R)) = \mathrm{char}(T^2(R))$. $\square$

In Example 2.2.13(i) we constructed a tangent structure on $\mathbb{C}\mathrm{Ring}_u$, originally presented in [81, Example 2]. The tangent bundle endofunctor T sends a ring R to $R[x]/(x^2)$. Indeed, this tangent bundle endofunctor preserves the characteristic: $\mathrm{char}(T(R)) = \mathrm{char}(R)$.

All pullbacks of $\mathbb{C}\mathrm{Ring}_u$ are squares of ring homomorphisms that are pullbacks of underlying sets. Thus all pullbacks of $\mathbb{C}\mathrm{Ring}_u$ are also pullbacks in $\mathbb{C}\mathrm{Ring}_1$, implying that any tangent structure on $\mathbb{C}\mathrm{Ring}_u$ is also tangent structure on $\mathbb{C}\mathrm{Ring}_1$. Thus **??** is also an example of a tangent structure on $\mathbb{C}\mathrm{Ring}_1$ that preserves the characteristic.

It is unclear if an example of a tangent structure on $\mathbb{R}\mathrm{ing}_n$, $\mathbb{C}\mathrm{Ring}_n$, $\mathbb{R}\mathrm{ing}_1$ or $\mathbb{C}\mathrm{Ring}_1$ exists where the tangent bundle endofunctor does not preserve the characteristic. If it exists, it needs to fulfill the requirement of Corollary 3.5.10.

3.5.2 Krull dimension satisfying products but not pullbacks

The term "dimension of a ring" is commonly used for the Krull dimension of a commutative ring, which is defined through ascending chains of prime ideals and has extensive applications in algebraic geometry. Since

$$\dim(R \times R') = \max(\dim(R), \dim(R')),$$

desirable property (b') (additive w.r.t. products) of Remark 3.2.1 holds when choosing max as the monoid structure.

However, we will show that the pullback condition of Definition 3.2.2 does not hold and it thus is not a categorical dimension on $\mathbb{C}\mathrm{Ring}_u$.

Definition 3.5.11. [41, Page 217] *Given a commutative ring R with unit, its Krull dimension is the length n of the longest chain of prime ideals*

$$0 \subsetneq p_1 \subsetneq ... \subsetneq p_n \subsetneq R.$$

The Krull dimension fulfills some useful properties.

(a) Let F be a field. Then $\dim(F[x_1, ..., x_n]) = n$.

(b) Let R, R' be commutative rings then $\dim(R \times R') = \max(\dim(R), \dim(R'))$.

We show here that this is not an $\mathbb{N}_\infty$ valued dimension on $\mathbb{C}\mathrm{Ring}_u$ in the sense of Definition 3.2.2.

The following example will now show that it can not be an $(N_\infty, \max)$-valued dimension.

Example 3.5.12. *The diagram*

$$\begin{tikzcd} \mathbb{Q}[y] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{Q}[x,y]/(xy) \arrow[d, two heads] \\ \mathbb{Q} \arrow[r, hook] & \mathbb{Q}[x] \end{tikzcd} \tag{3.2}$$

is a pullback in $\mathbb{C}\mathrm{Ring}_u$, where the map $\mathbb{Q} \hookrightarrow \mathbb{Q}[x]$ includes a number as the constant term and the map $\mathbb{Q}[x,y]/(xy) \to \mathbb{Q}[x]$ evaluates at $y = 0$. It is a pullback because giving a pair (r,p) of $r \in \mathbb{Q}$ and $p \in \mathbb{Q}[x,y]/(xy)$ such that $p(x,0) = r$ is the same as giving a polynomial $p \in \mathbb{C}[x,y]/(xy)$ with no x-terms, i.e. a polynomial $p \in \mathbb{C}[y]$.

The very same example shows that it is not a categorical dimension on $\mathbb{C}\mathrm{Ring}_1$.

For a field F, the coproduct $F[x_1, ..., x_n] \otimes F[x_1, ..., x_k] = F[x_1, ..., x_{n+k}]$. Since $\dim(F[x_1, ..., x_n]) = n$, this suggests that for the opposite category $\mathbb{C}\mathrm{Ring}_u^{\mathrm{op}}$ desirable property (b') (additive w.r.t. products) from Remark 3.2.1 holds for $(\mathbb{N}_\infty, +)$.

The context of algebraic geometry would also suggest that the Krull dimension is a categorical dimension on $\mathbb{CRing}_u^{\mathrm{op}}$. The category $\mathbb{CRing}_u^{\mathrm{op}}$ is equivalent to the category of affine schemes and for smooth affine schemes the scheme-theoretical dimension equals the Krull dimension. However, using the counterexample in Example 3.5.13 that corresponds to a pullback of non-smooth schemes, we will show that it still is not a dimension in the sense of Definition 3.2.2.

Example 3.5.13. *Let $\mathbb{Q}$ denote the field of rational numbers. Let i denote the embedding $\mathbb{Q}[x] \hookrightarrow \frac{\mathbb{Q}[x,y]}{\langle x \cdot y \rangle}$. The diagram*

$$\begin{tikzcd}
\mathbb{Q}[x] \arrow[r, hook, "i"] \arrow[d, "\mathrm{ev}_1"'] & \frac{\mathbb{Q}[x,y]}{\langle xy \rangle} \arrow[d, "\mathrm{ev}_{(1,0)}"] \\
\mathbb{Q} \arrow[r, "1_{\mathbb{Q}}"'] & \mathbb{Q}
\end{tikzcd}$$

is a pushout square in $\mathbb{CRing}_u$ and thus corresponds to a pullback square in $\mathbb{CRing}_u^{\mathrm{op}}$. However, the Krull dimensions of the rings involved are

$$\dim(\mathbb{Q}[x]) = 1 \ , \ \dim(\mathbb{Q}) = 0 \ , \ \dim\left(\frac{\mathbb{Q}[x,y]}{\langle x \cdot y \rangle}\right) = 2$$

and thus

$$\dim(\mathbb{Q}) + \dim(\mathbb{Q}[x]) = 0 + 1 \neq 0 + 2 = \dim(0) + \dim\left(\frac{\mathbb{Q}[x,y]}{\langle x \cdot y \rangle}\right).$$

Remark 3.5.14. *The category of affine schemes is equivalent to the opposite category of $\mathbb{CRing}_u$ and the Krull dimension corresponds to the dimension of a scheme. However, the dimension of a scheme is only well-behaved with respect to the tangent structure for smooth schemes over a field, as one can read in* [88, Chapter 13]*. Thus the geometric intuition suggests that the Krull dimension is an $(\mathbb{N}, +)$-valued dimension in the sense of Definition 3.2.2 on the smooth objects of $\mathbb{CRing}_u^{\mathrm{op}}$. A formal proof would require a careful examination that the Krull dimension is additive under pushouts in the subcategory of $\mathbb{CRing}_u$ corresponding to smooth affine schemes. The examination of this is beyond the scope of this thesis.*

3.6 Transporting a dimension between modules and algebras with an adjunction

In this section we will show that $\mathbb{Mod}_R$, the category of modules over a PID R, and $\mathbb{Alg}_R$, the category of algebras over a PID R, have a categorical dimension given by the rank. In Example 3.6.1 we define a novel, somewhat degenerate tangent structure on $\mathbb{Mod}_R$ where the tangent bundle endofunctor doubles the dimension. Since every categorical dimension on a tangent category is a tangent dimension, the rank will be a tangent dimension.

First, we will show that the rank of an R-modules is a categorical dimension on the category $\mathbb{M}\mathrm{od}_R$ of R-modules and R-modules homomorphisms for any PID, R. In particular, this shows that the dimension of a vector space is a categorical dimension on the category $\mathbb{V}\mathrm{ect}_F$ for a field, F. As a consequence, a tangent structure on $\mathbb{M}\mathrm{od}_R$ must fulfill the equation in Corollary 3.6.7.

Next, we show that using the free-forgetful adjunction between R-algebras and R-modules, we obtain a categorical dimension on R-algebras. Finally, as an additional observation, we use the classical dimension on $\mathbb{V}\mathrm{ect}$ as a categorical dimension and show that there can only be two Cartesian tangent structures on $\mathbb{V}\mathrm{ect}$, and we identify them.

In [68][Section IV.2] Mac Lane shows that there is a free forgetful adjunction between $\mathbb{M}\mathrm{od}_R$ and $\mathbb{A}\mathrm{bGrp}$. Thus, by Corollary 3.2.4, the cardinality of the underlying abelian group is already an $(\mathbb{N}_\infty, \cdot)$-valued dimension on $\mathbb{M}\mathrm{od}_R$. In Section 3.6.1 we will show that the rank is a second categorical dimension on $\mathbb{M}\mathrm{od}_R$.

There is a somewhat degenerate tangent structure on the category $\mathbb{M}\mathrm{od}_R$ of modules over a ring R. In [81, Example 1] this is given as a tangent structure on abelian groups, but it works equally well for modules over a ring.

Example 3.6.1. *Let R be a commutative unital ring. The category $\mathbb{M}\mathrm{od}_R$ has a somewhat degenerate tangent structure given for an R-module V by*

- *the tangent bundle endofunctor $T(V) = V \times V$ with $T(f) = f \times f$,*
- *the projection $p_V = \pi_0 : V \times V \to V$,*
- *the zero section $0_V = \langle 1_V, \mathrm{const}_0 \rangle : V \to V \times V$,*
- *the addition $+_V = \langle \pi_0, \mathrm{add}(\pi_1, \pi_2) \rangle$, where $\mathrm{add}(v, w) = v + w$,*
- *the vertical lift $\ell_V = \langle \pi_0, \mathrm{const}_0, \mathrm{const}_0, \pi_1 \rangle : V \times V \to V \times V \times V \times V$*
- *the canonical flip $c_V = \langle \pi_0, \pi_2, \pi_1, \pi_3 \rangle : V \times V \times V \times V \to V \times V \times V \times V$.*

These maps fulfill all required equations and are natural transformations since all morphisms are R-linear. The universality of the vertical lift holds since the diagram

$$\begin{tikzcd}
V \times V \times V \arrow[r, "\langle \pi_0, \pi_2, \mathrm{const}_0, \pi_1 \rangle"] \arrow[d, "\pi_0"'] & V \times V \times V \times V \arrow[d, "\langle \pi_0, \pi_2 \rangle"] \\
V \arrow[r, "\langle 1_V, \mathrm{const}_0 \rangle"'] & V \times V
\end{tikzcd}$$

is a pullback square.

3.6.1 Rank of a module over a ring

The rank of is the most common generalization of the dimension of vector spaces to modules. A module's rank is a natural number or infinity (like in desired Property (a) (valued in $\mathbb{N}$) of Remark 3.2.1), invariant under isomorphism (like in desired Property (c) (isomorphism invariant) of Remark 3.2.1) and it is additive under cartesian products (like in desired Property (b') (additive w.r.t. products) of Remark 3.2.1). Desired Property (d) (embeddings to inequalities) holds since the rank of a submodule is less than or equal to the rank of the module itself. Desired Property (d')(the same as locally) does not make sense to ask, because an open subset of a module is in general not a module itself.

We will show below that it is indeed an $(\mathbb{N}_\infty, +)$-valued dimension in the sense of Definition 3.2.2, so the generalized version of desired property (b) (additive w.r.t. pullbacks) holds.

Recall that a **principal ideal domain** (PID) is a unital commutative ring without zero divisors where every ideal is generated by a single element. Throughout this section we fix a principal ideal domain R.

Definition 3.6.2. *Given a R-module A. Then **the rank** $\mathrm{rk}(A) \in \mathbb{N}_\infty$ is the cardinality of any maximal linear independent subset of A. If there is no finite maximal linearly independent subset of A, the rank is* $\mathrm{rk}(A) = \infty$.

It is a standard result, for example in [4], that any maximal linear independent subset of A has the same cardinality. A key property of the rank is its behaviour under short exact sequences.

Lemma 3.6.3. *Let R be a PID and let*

$$0 \to A \to B \to C \to 0$$

be a short exact sequence of R-modules with finite rank. Then $\mathrm{rk}(A) + \mathrm{rk}(C) = \mathrm{rk}(B)$

This is a standard result, for example Exercise 3.14 in chapter VI of [4], thus we omit the proof here.

Actually, as we show below, the requirement that A, B and C had finite rank was unnecessary.

Proposition 3.6.4. *Let R be a PID and let*

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

be a short exact sequence of R-modules. Then $\mathrm{rk}(A) + \mathrm{rk}(C) = \mathrm{rk}(B)$ *in* $\mathbb{N}_\infty$.

Proof. The case when A, B and C have finite rank is already covered in Lemma 3.6.3. Below we prove the formula for the cases when A, B or C have infinite rank.

(a) Let A have infinite rank. Then, for every $n \in \mathbb{N}$, there is an linearly independent subset $U \subseteq A$ with cardinality n. Since f is injective, $\{f(u)|u \in U\}$ is still linearly independent. Since n was arbitrary, this proves that B has also infinite rank. Thus the equation

$$\mathrm{rk}(A) + \mathrm{rk}(C) = \infty + \mathrm{rk}(C) = \infty = \mathrm{rk}(B)$$

holds.

(b) Let C have infinite rank. Then, for every $n \in \mathbb{N}$, there is an linearly independent subset $U \subseteq C$ with cardinality n. Since g is surjective, there are preimages $(b_u)_{u \in U}$ with $g(b_u) = u$. They form a linearly independent subset of B that has cardinality n. Since n was arbitrary, $\mathrm{rk}(B) = \infty$, proving that

$$\mathrm{rk}(A) + \mathrm{rk}(C) = \mathrm{rk}(A) + \infty = \infty = \mathrm{rk}(B).$$

(c) Let B have infinite rank and A have finite rank. Then $\mathrm{Im}(f) \subset B$ has finite rank. Thus, by extending a maximal linearly independent subset of $\mathrm{Im}(f)$ to an arbitrary sized linearly independent subset of B, we obtain an arbitrarily sized linearly independent subset of $B/\mathrm{Im}(f)$. Due to the uniqueness of cokernels, $B/\mathrm{Im}(f) \cong C$ and thus $\mathrm{rk}(C) = \infty$. Now the equation

$$\mathrm{rk}(A) + \mathrm{rk}(C) = \mathrm{rk}(A) + \infty = \infty = \mathrm{rk}(B)$$

holds.

□

For any R-module A, the addition $+ : A \times A \to A, (a, a') \mapsto a + a'$ and the negation $- : A \to A, a \mapsto -a$ are module-homomorphisms. Out of this we built the difference $f - g$ of module-homomorphisms. For $f : A \to B$ and $g : C \to B$ is defined as

$$f - g : A \times C \to B \ , \ (a, c) \mapsto f(a) - g(c).$$

We will use this difference in Lemma 3.6.5 and the proof of Proposition 3.6.6. Observe that $f - g$ is surjective if f or g is surjective.

Lemma 3.6.5. *Let A, B, C be R-modules. Let $A \times_B C$ denote the pullback arising as the limit of the span*

$A \xrightarrow{f} B \xleftarrow{g} C$. *Then the following sequence is exact*

$$0 \to A \times_B C \xrightarrow{\langle \pi_0, \pi_1 \rangle} A \times C \xrightarrow{f-g} \mathrm{Im}(f-g) \to 0$$

Proof. The sequence is exact at Im(f − g) since any map is surjective onto its image. The sequence is exact at $A \times_B C$ since $\langle \pi_0, \pi_1 \rangle$ is injective. The kernel of $f - g$ is exactly the elements (a, c) fulfilling $f(a) = g(c)$, thus it is equal to the image of $A \times_B C$ of $A \times C$. $\square$

Proposition 3.6.6. *Let R be a PID. Then assigning the rank to* $\mathrm{rk}(A)$ *to an R-module A is an* $(\mathbb{N}_\infty, +)$*-valued dimension* rk *on* Mod_R.

Proof. Let

$$\begin{tikzcd} A \times_B C \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "r"] \\ C \arrow[r, "f"'] & B \end{tikzcd}$$

be a pullback of R-modules of the form as the pullbacks in Definition 3.2.2. Then Lemma 3.6.5 shows that

$$0 \to A \times_B C \xrightarrow{\langle \pi_0, \pi_1 \rangle} A \times C \xrightarrow{f-r} \mathrm{Im}(f-r) \to 0$$

is a short exact sequence. Since r is surjective, $f - r$ is surjective i.e. $\mathrm{Im}(f - r) = B$, thus we obtain the short exact sequence

$$0 \to A \times_B C \xrightarrow{\langle \pi_0, \pi_1 \rangle} A \times C \xrightarrow{f-r} B \to 0.$$

Therefore the rank formula in Proposition 3.6.4 shows that $\mathrm{rk}(A \times C) = \mathrm{rk}(A \times_B C) + \mathrm{rk}(B)$. Due to the additivity of ranks under products this can be expanded into

$$\mathrm{rk}(A) + \mathrm{rk}(C) = \mathrm{rk}(A \times_B C) + \mathrm{rk}(B)$$

$\square$

Now we can apply Theorem 3.2.7 and obtain a clear dimension formula for any tangent structure.

Corollary 3.6.7. *Let R be a PID. Let $(T, p, 0, +, \ell, c)$ be a tangent structure on* Mod_R. *Then for any R-module A, the equation*

$$\mathrm{rk}(T^2(A)) + 2 \cdot \mathrm{rk}(A) = 3 \cdot \mathrm{rk}(A)$$

holds.

There are several special cases that follow from this

Examples 3.6.8. *1. Let F be a field and let $\mathbb{V}\mathrm{ect}_F$ denote the category of vector spaces over F with F-linear maps as morphisms. Then the dimension is an $(\mathbb{N}_\infty, +)$ valued dimension and, for any tangent structure on $\mathbb{V}\mathrm{ect}_F$,*

$$\dim(T^2(V)) + 2 \cdot \dim(V) = 3 \cdot \dim(V)$$

holds for any vector space V.

2. Let $\mathbb{A}\mathrm{bGrp}$ be the category of abelian groups (i.e. modules over the integers). Then the rank of the abelian group is an $(\mathbb{N}_\infty, +)$ valued dimension and, for any tangent structure on $\mathbb{A}\mathrm{bGrp}$,

$$\mathrm{rk}(T^2(G)) + 2 \cdot \mathrm{rk}(G) = 3 \cdot \mathrm{rk}(G)$$

holds for any abelian group G. Together with the cardinality from Theorem 3.4.2, this means that the category of abelian groups has two distinct nontrivial $\mathbb{N}_\infty$-valued dimensions.

In the following two sections we will explain the consequences for finite dimensional vector spaces and Algebras over a PID.

3.6.2 Module rank of an Algebra

In this section we utilize the free forgetful adjunction between modules and algebras to obtain a dimension of Algebras over a commutative ring R with unit. According to [68, Section IV.2], the forgetful functor $U : \mathbb{A}\mathrm{lg}_R \to \mathbb{M}\mathrm{od}_R$ that sends a R-Algebra to its underlying R-module has a left-adjoint given by the tensor algebra. Thus we obtain a dimension on the category of R-Algebras.

Proposition 3.6.9. *Let R be a PID and $\mathbb{A}\mathrm{lg}_R$ be the category of R-algebras and Algebra-homomorphisms.*

(a) The assignment rk *that sends an algebra A to the rank $\mathrm{rk}(U(A))$ of its underlying R-module is an $(\mathbb{N}_\infty, +)$-valued dimension on $\mathbb{A}\mathrm{lg}_R$.*

(b) For any tangent structure $(\mathbb{A}\mathrm{lg}_R, T, p, 0, +, \ell, c)$ on the category $\mathbb{A}\mathrm{lg}_R$ and for any object A,

$$\mathrm{rk}(U(T^2(A))) + 2 \cdot \mathrm{rk}(U(A)) = 3 \cdot \mathrm{rk}(U(T(A))).$$

Proof. Part (a) follows from Corollary 3.2.4. Part (b) follows from Theorem 3.2.7. □

In particular algebras over $\mathbb{Z}$ are the same as rings. Thus we obtain an additional dimension on the category of rings.

Corollary 3.6.10. *Let* $\mathrm{rk}_{\mathbb{Z}}$ *denote the rank of an abelian group over the integers.*

(a) The rank $\mathrm{rk}_{\mathbb{Z}}$ *is an* $\mathbb{N}_\infty$ *valued dimension on* $\mathbb{R}\mathrm{ing}_n$, $\mathbb{R}\mathrm{ing}_1$, $\mathbb{R}\mathrm{ing}_u$, $\mathbb{C}\mathrm{Ring}_n$, $\mathbb{C}\mathrm{Ring}_1$ *and* $\mathbb{C}\mathrm{Ring}_u$.

(b) For any ring R *and any tangent structure* $(T, p, 0, +, \ell, c)$ *on* $\mathbb{R}\mathrm{ing}_n$, $\mathbb{R}\mathrm{ing}_1$, $\mathbb{R}\mathrm{ing}_u$, $\mathbb{C}\mathrm{Ring}_n$, $\mathbb{C}\mathrm{Ring}_1$ *or* $\mathbb{C}\mathrm{Ring}_u$,

$$\mathrm{rk}_{\mathbb{Z}}(T^2(A)) + 2 \cdot \mathrm{rk}_{\mathbb{Z}}(A) = 3 \cdot \mathrm{rk}_{\mathbb{Z}}(T(A)).$$

Proof. (a) Since $\mathbb{R}\mathrm{ing}_n$ is the full subcategory of associative algebras in $\mathbb{A}\mathrm{lg}_{\mathbb{Z}}$, it follows from Proposition 3.6.9 and Corollary 3.2.4 that $\mathrm{rk}_{\mathbb{Z}}$ is a dimension on $\mathbb{R}\mathrm{ing}_n$. The categories $\mathbb{R}\mathrm{ing}_1, \mathbb{C}\mathrm{Ring}_u$ and $\mathbb{C}\mathrm{Ring}_1$ are full subcategories of $\mathbb{R}\mathrm{ing}_n$ and therefore Corollary 3.2.4 shows that $\mathrm{rk}_{\mathbb{Z}}$ is a dimension on them.

Pullbacks in $\mathbb{R}\mathrm{ing}_u$ are also pullbacks in $\mathbb{R}\mathrm{ing}_1$, because in either case commutative squares are pullbacks if and only if they are pullbacks of underlying sets. Thus any dimension on $\mathbb{R}\mathrm{ing}_1$ is also a dimension on $\mathbb{R}\mathrm{ing}_u$, in particular $\mathrm{rk}_{\mathbb{Z}}$. Since $\mathbb{C}\mathrm{Ring}_u$ is a full subcategory of $\mathbb{R}\mathrm{ing}_u$, Corollary 3.2.4 shows that $\mathrm{rk}_{\mathbb{Z}}$ is also a dimension on $\mathbb{C}\mathrm{Ring}_u$

(b) This now follows from 3.2.7.

□

Thereby we now have two distinct non-trivial dimensions on $\mathbb{R}\mathrm{ing}_1$ and $\mathbb{R}\mathrm{ing}_u$, the characteristic (as shown in Section 3.5) and the rank.

If we consider the tangent structure of Example **??**, we see that for any commmutative ring R with unit,

$$\mathrm{rk}_{\mathbb{Z}}(T(R)) = \mathrm{rk}_{\mathbb{Z}}(R[x]/(x^2)) = 2 \cdot \mathrm{rk}_{\mathbb{Z}}(R).$$

Thus the tangent bundle functor doubles this dimension in this case like in the case for classical manifolds.

3.6.3 Classification of Cartesian tangent structures on $\mathbb{F}\mathrm{inVect}$

The category of vector spaces over a field F is exactly the category $\mathbb{M}\mathrm{od}_F$ of modules over F. The classical dimension $\mathrm{Dim}(V)$ of a vector space V is exactly the rank $\mathrm{rk}(V)$. Due to Proposition 3.6.6, it is a $(\mathbb{N}_\infty, +)$-valued dimension.

All finite-dimensional vector spaces over a field F are isomorphic to powers F^n of the field F. Therefore the interaction between the tangent structure and the Cartesian structure dictates the entire tangent structure.

Theorem 3.6.11. *Let F be a field and let $\mathbb{F}\mathrm{inVect}_F$ be the full subcategory of $\mathbb{V}\mathrm{ect}_F$ consisting of finite-dimensional vector spaces. Then every Cartesian tangent structure $(T, p, 0, +, \ell, c)$ on $\mathbb{F}\mathrm{inVect}_F$ fulfills either*

$$T(V) \cong V \quad \forall V \in \mathrm{Obj}(\mathbb{F}\mathrm{inVect})$$

or

$$T(V) \cong V \times V \quad \forall V \in \mathrm{Obj}(\mathbb{F}\mathrm{inVect}).$$

Proof. Every finite-dimensional vector space is of the form $V = F^n$ where n is the dimension of V. Thus every Cartesian tangent structure will fulfill $T(V) \cong T(F^n) \cong T(F)^n$. Thus $\mathrm{Dim}(T(V)) = \mathrm{Dim}(T(F)) \cdot \mathrm{Dim}(V)$ which means the vector space dimension is a strong tangent dimension valued in $\mathbb{Z}$. Now Theorem 3.2.8 proves that either $\mathrm{Dim}(T(V)) = 2 \cdot \mathrm{Dim}(V)\ \forall V$ or $\mathrm{Dim}(T(V)) = \mathrm{Dim}(V)\ \forall V$. □

In fact, both situations are known. The trivial tangent structure with $T(V) \cong V$ is explained in Example 2.2.6. The tangent structure that doubles the dimension is the tangent structure of Example 3.6.1 for the case when R is a field.

3.7 Homotopy, excision and the opposite category of CW complexes

In this section we will show that $\mathbb{CW}^{\mathrm{op}}$, the opposite category of CW complexes and cellular maps, has a categorical dimension given by the Betti numbers. We are using Definition 3.2.2, which does not require a tangent structure. No tangent structure on $\mathbb{CW}^{\mathrm{op}}$ is known so far.

CW complexes are extensively used in topology to describe well-behaved topological spaces. Traditionally the dimension of a CW complex is the highest dimensional cell in the complex. However, this notion is not well-behaved under homotopy. We will show in this section that it is not a categorical dimension in the sense of Definition 3.2.2.

Rational homology is a tool that describes a topological space X in a homotopy-invariant way with a sequence of rational vector spaces $H_n(X, \mathbb{Q})_{n\in\mathbb{N}}$. The dimension of these vector spaces are known as the Betti numbers of the space.

The n-th Betti number B_n is often interpreted as “counting n-dimensional holes” in a space. Since the Betti numbers are about the dimension of the complement of our space, it may seem intuitive to have them be a categorical dimension on the opposite category of CW complexes.

A **CW complex** is a topological space X, constructed by an inductive construction out of standard cells

$D_k = \{\vec{x} \in \mathbb{R}^n | x^2\}$ for every $k \in \mathbb{N}$. The inductive construction can be found in Chapter 0 of [48]. The space obtained at the k-th stage of the inductive construciton is called X^k the **k-skeleton** of X.

A continuous map $f : X \to Y$ between CW complexes $X = \cup_{k\in\mathbb{N}} X^k$ and $Y = \cup_{k\in\mathbb{N}} Y^k$ is called **cellular** if $f(X^k) \subseteq Y^k$ for all $k \in \mathbb{N}$.

Definition 3.7.1. *We define the category* $\mathbb{C}\mathrm{W}$ *as the subcategory of* $\mathbb{T}\mathrm{op}$ *consisting of*

- *CW complexes of objects and*
- *cellular maps as morphisms.*

3.7.1 Not the classical dimension of CW complexes

The classical dimension of a CW complex is considered to be the the largest n of all n-cells used to construct it.

Definition 3.7.2. *Let* $X = \cup_{n\in\mathbb{N}} X^n$ *be a CW complex.*

(a) The smallest number $n \in \mathbb{N}$ *such that* $X = X^n$ *is the* ***classical dimension*** *of* X*.*

(b) If there is no such n*, the classical dimension is* ∞*.*

In particular for CW complexes X and X', the largest dimension of cells of $X \times X'$ is the sum of the largest dimensions of cells in X and X', so desirable property (b') (additive w.r.t. products) of Remark 3.2.1 holds with respect to the addition.

While the classical dimension is the most common notion of dimension for CW complexes, we show it is not a dimension in the sense of Definition 3.2.2. We will construct a pullback of retractions in CW that serves as a counterexample for the dimension formula in Definition 3.2.2.

The idea behind this counterexample is to glue together a space Bl that has 1-dimensional and 3-dimensional components. In the pullback, the 3-dimensional components are prevented from combining with each other by having them sent to the opposite end of the interval D^1.

Example 3.7.3. *We define the "balloon shaped" CW complex* Bl *obtained by gluing the unit interval* $D^1 = [0, 1]$ *with a 3-cell* $D^3 = \{(x, y, z) \in \mathbb{R}^3 | (x + 1)^2 + y^2 + z^2 \le 1\}$ *along the origin, obtaining a space that can explicitly be described as*

$$\{(x, 0, 0) \in \mathbb{R}^3 | 0 \le x \le 1\} \cup \{(x, y, z) \in \mathbb{R}^3 | (x + 1)^2 + y^2 + z^2 \le 1\}$$

We define two maps $l, r : \mathrm{Bl} \to D^1$*, both contracting the 3-cell away. However,* l *sends the end of the interval where the 3-cell was to the left and* r *sends the end of the interval where the 3-cell was to the right. In explicit*

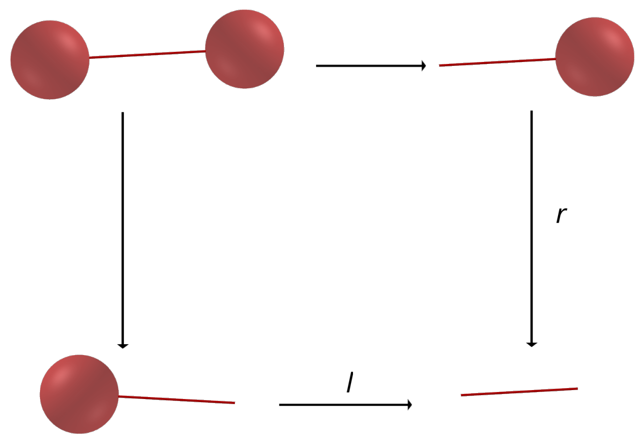

Figure 3.2: The pullback of the interval with a 3-cell at the left and the interval with a 3-cell at the right is an interval with a 3-cell on either side.

coordinates that means that

$$l(x,y,z) = \begin{cases} x & \text{if} \quad x \geq 0 \\ 0 & \text{if} \quad x < 0 \end{cases}$$

$$r(x,y,z) = \begin{cases} 1-x & \text{if} \quad x \geq 0 \\ 1 & \text{if} \quad x < 0. \end{cases}$$

It is easy to see that both l and r are retractions. As one can see directly from the Figure 3.2 the pullback

$$\begin{tikzcd} \mathrm{Bl} \times_{D^1} \mathrm{Bl} \arrow[r] \arrow[d] & \mathrm{Bl} \arrow[d, "r"] \\ \mathrm{Bl} \arrow[r, "l"'] & D^1 \end{tikzcd}$$

will be an interval with a 3-cell glued to both ends.

Since

$$\dim(\mathrm{Bl} \times_{D^1} \mathrm{Bl}) + \dim(D^1) = 3 + 1 \neq 3 + 3 = \dim(\mathrm{Bl}) + \dim(\mathrm{Bl}),$$

the condition in Definition 3.2.2 does not hold. Thus the classical dimension of CW complexes is not $(\mathbb{N}, +)$-*valued dimension on* $\mathbb{C}\mathrm{W}$ *in the sense of Definition 3.2.2.*

In general, combinatrics from the gluing (i.e. colimits) produces counterexamples proving that the dimension of CW complexes is not a dimension in the sense of Definition 3.2.2.

3.7.2 Betti numbers and excisive functors

In this section, we will show that the Betti numbers are a categorical dimension on the opposite category of CW complexes.

Intuitively, the nth Betti number counts the number of n-dimensional holes in a space X. Following this intuition, we check Property (b') (additive w.r.t. products) of Remark 3.2.1. The product in $\mathbb{CW}^{\mathrm{op}}$ is the disjoint union. Since the disjoint union of two complexes with n and k holes has $n+k$ holes in total, desired Property (b') (additive w.r.t. products) holds on $\mathbb{CW}^{\mathrm{op}}$ for the monoid $(\mathbb{N}_\infty, +)$.

We will show that the Betti numbers are indeed dimensions in the sense of Definition 3.2.2. Thus the sequence of all Betti numbers will be a categorical dimension valued in infinite sequences.

Eilenberg and Steenrod [40] define a generalized homology theory as an assignment sending a CW complex X to a sequence, $H_n(X)$, of abelian groups that fulfills a list of 7 axioms (including functoriality). Singular homology is the motivating example of a generalized homology theory. It is a standard result, for example in [48, Section 2.3], that the Mayer-Vietoris sequence still exists for any generalized homology theory. The Mayer-Vietoris sequence will be the main tool that we need in order to verify that the Betti number is a categorical dimension. Thus our argument will hold for any generalized homology theory.

Let H_n be denote a homology theory fulfilling the Eilenberg-Steenrod axioms. Let $\mathbb{CW}$ be the category of CW complexes and cellular maps. Then the n-th Betti number $B_n(X) = \mathrm{rk}(H_n(X))$ is the rank of the n-th homology of a space X. It is easy to see that this equals the dimension $\dim(H_n(X;\mathbb{Q}))$ of the n-th rational homology of X.

Proposition 3.7.4. *The n-th Betti number is an $\mathbb{N}_\infty$-valued dimension on* $\mathrm{CW}^{\mathrm{op}}$.

Proof. A pullback along a retraction in $\mathrm{CW}^{\mathrm{op}}$ corresponds to a pushout along a section s in CW.

$$\begin{tikzcd}
C \sqcup_A B & B \arrow[l] \\
C \arrow[u] & A \arrow[l, "f"] \arrow[u, hook, "s"']
\end{tikzcd}$$

Sections are exactly embeddings of retracts which are cellular embeddings of closed subspaces and therefore subcomplexes. Thus by Proposition 0.16 of [48] it has the homotopy extension property and is therefore a cofibration.

Section 10.7 of [69] shows that the pushout along any cofibration is homotopy-equivalent to the double mapping cylinder.

$$C \sqcup_A B \simeq {}_C M_B = (C \sqcup (A \times I) \sqcup B)/\sim,$$

where $(a, 0) \sim f(a)$ and $(a, 1) \sim s(a)$. This double mapping cylinder is the union of the two open sets $X = C \sqcup (A \times [0, 1))/\sim$ and $Y = (A \times (0, 1]) \sqcup B/\sim$. Their intersection $X \cap Y = A \times (0, 1)$ is homotopy-equivalent to A. Thus we obtain the Mayer Vietoris sequence

$$... \to H_{n+1}({}_C M_B) \xrightarrow{0} H_n(A) \xrightarrow{\langle H_n(s), H_n(f) \rangle} H_n(B) \oplus H_n(C) \longrightarrow H_n({}_C M_B) \xrightarrow{0} H_{n-1}(A) \to ...$$

where the maps $H_{n+1}({}_C M_B) \to H_n(A)$ are zero since $H_n(s)$ is a section and therefore the kernel of $(\langle H_n(s), H_n(f) \rangle)$ is zero. Therefore

$$B_n(C \sqcup_A B) + B_n(A) = B_n(C) + B_n(B).$$

and thus B_n is an $\mathbb{N}_\infty$-valued dimension on $\mathrm{CW}^{\mathrm{op}}$. □

Instead of considering the Betti numbers as an infinite sequence of separate dimensions, we can also consider them as a dimension valued in infinite sequences of numbers: Let $\mathbb{N}_\infty^\infty$ be the set of infinite sequences (or infinite vectors) in $\mathbb{N}_\infty$. It has a commutative monoid structure $+$, given by component-wise addition.

Corollary 3.7.5. *The sequence of Betti numbers*

$$(B_n)_{n \in \mathbb{N}} : X \mapsto (B_n(X))_{n \in \mathbb{N}}$$

is an $(\mathbb{N}_\infty^\infty, +)$*-valued dimension on* CW^{op}.

Corollary 3.7.6. *For every tangent structure* T *on* CW^{op}*, the Betti numbers* $(B_n)_{n \in \mathbb{N}}$ *satisfy*

$$B_n(T^2(X)) + 2 \cdot B_n(X) = 3 \cdot B_n(T(X)).$$

We will now explore a counterexample of a pushout of Hausdorff spaces along a section which is not homotopy-equivalent to the double mapping cylinder. This serves to show that our proof of Proposition 3.7.4 does not work in the more general setting of Hausdorff spaces and continuous maps.

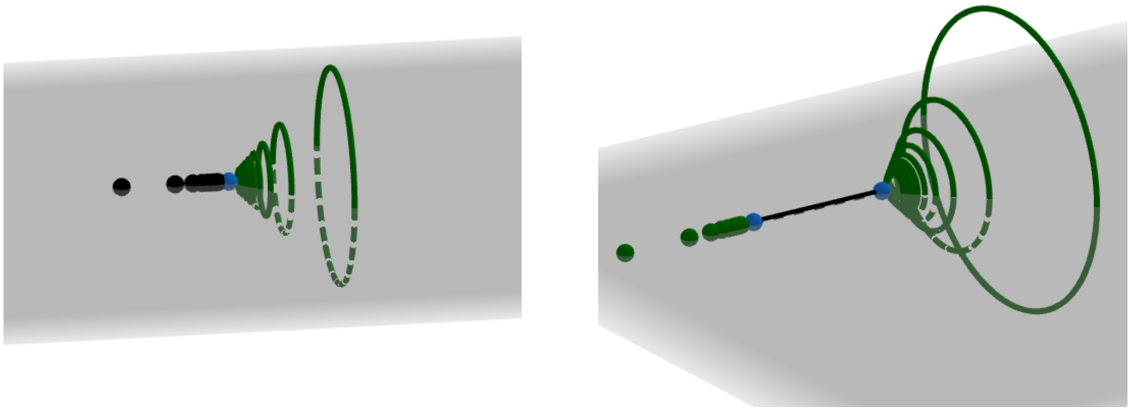

Figure 3.3: On the left is the pushout of A and B from Example 3.7.7, on the right the double mapping cylinder ${}_AM_B$. They are not homotopy-equivalent. The intuitive reason is that every open neighbourhood around the blue point on the left contains infinitely many dots and circles, whereas on the right there is no such point.

Example 3.7.7. *Consider the Hausdorff spaces*

$$
\begin{aligned}
A' &= \left\{ x \in \mathbb{R} | x = \frac{1}{n} \text{ for some } n \in \mathbb{N}_{>0} \right\}, \\
B' &= \left\{ (x,y,z) \in \mathbb{R}^3 | (x,y,z) = \left(\frac{1}{n}, \frac{1}{n}\sin(\varphi), \frac{1}{n}\cos(\varphi) \right) \text{ for some } n \in \mathbb{N}_{>0} \ , \ \varphi \in [0, 2\pi] \right\}, \\
A &= \{0\} \cup A', \text{ and} \\
B &= \{(0,0,0)\} \cup B'
\end{aligned}
$$

and the pushout diagram

$$
\begin{tikzcd}
* \arrow[r, "i_0"] \arrow[d, "i_{(0,0,0)}"'] & A \arrow[d] \\
B \arrow[r] \arrow[ur, phantom, "\ulcorner"] & P
\end{tikzcd}
$$

where the strict pushout is

$$P = \{(-x, 0, 0) | x \in A\} \cup B \subset \mathbb{R}^3.$$

As shown in Appendix A and visualized in Figure 3.3, the double mapping cylinder is not homotopy-equivalent to this strict pushout.

The most important property we used in the proof of Proposition 3.7.4 was that a generalized homology theory sends the pushout square

$$
\begin{tikzcd}
C \sqcup_A B & B \arrow[l] \\
C \arrow[u] & A \arrow[l, "f"] \arrow[u, hook, "s"']
\end{tikzcd}
$$

to a pullback square of modules and that the rank of a module is a dimension. In homotopy theory, homotopy functors that send homotopy pushout squares to homotopy pullback squares are generally known as **1-excisive functors**. Below we generalize Proposition 3.7.4 for 1-excisive functors.

In Proposition 3.7.8, we assume that homotopy limits/colimits are the same as strict limits/colimits. This allows us to compare the hypotheses in Definition 3.2.2 (which uses strict limits/colimits) to the properties of excisive functors (which are defined by their effect on homotopy limits/colimits).

Proposition 3.7.8. *Let $\mathbb{X}$ be a model category in which (strict) pushouts of sections along sections and (strict) pushouts of sections along retractions are homotopy pushouts. Let $\mathbb{Y}$ be a complete and cocomplete category. We will consider it as a model category with trivial model structure. Let $F : \mathbb{X} \to \mathbb{Y}$ be an 1-excisive functor. If* dim *is a categorical dimension on $\mathbb{Y}$, then the assignment* $\dim \circ F$ *that sends and object X of $\mathbb{X}$ to* $\dim(F(X))$ *is a categorical dimension on $\mathbb{X}^{\mathrm{op}}$.*

Proof. The pullbacks of Definition 3.2.2 in $\mathbb{X}^{\mathrm{op}}$ are pushouts in $\mathbb{X}$ and by assumption they are also homotopy pushouts. Since F is excisive, it sends them to pullbacks in $\mathbb{Y}$. The dimension formula for dim now implies the dimension formula for $\dim \circ F$. □

Remark 3.7.9. *It may be possible to define a* ***homotopy categorical dimension*** *using homotopy limits/colimits in place of the strict limits/colimits of Definition 3.2.2. This homotopy dimension would be preserved (analogously to Proposition 3.7.8) by any excisive functors between any model categories, without the requirement that homotopy limits/colimits and strict limits/colimits coincide.*

Given some higher tangent structure, like in [6]*, where the pullbacks of the tangent structure are replaced by homotopy pullbacks, this homotopy dimension should give rise to results analogous to Theorems 3.2.7, 3.2.8, 3.2.7 and 3.2.8.*

Remark 3.7.10. *Desired Property (d) (embeddings to inequalities) of Remark 3.2.1 does not hold for the Betti numbers as dimensions on $\mathbb{C}\mathrm{W}^{\mathrm{op}}$. There are monomorphisms $A \hookrightarrow A'$ in $\mathbb{C}\mathrm{W}^{\mathrm{op}}$ (i.e. epimorphisms $A' \to A$ in $\mathbb{C}\mathrm{W}$) such that $B_n(A) > B_n(A')$. One example of this is the map*

$$[0,1] = I \to S^1 = \{e^{ix}\} \subset \mathbb{C} \ , \ x \mapsto e^{3\pi x}.$$

Since it winds more than once around the circle it is surjective and thus an epimorphism and $B_1(I) = 0$ and $B_1(S^1) = 1$.

However, if we only consider sections $A \hookrightarrow A'$ in $\mathbb{C}\mathrm{W}^{\mathrm{op}}$ (i.e. retractions in $\mathbb{C}\mathrm{W}$), Property (d) (embeddings to inequalities) follows from functoriality of homology. In the future, we hope this insight can help us refine the definition of dimension into a functorial and universal construction.

3.8 Concluding remarks

The goal of this notion of dimension is not to generalize many classical notions of dimensions (though it generalizes at least some). The goal is to give an extra tool when searching for tangent structures on a given category. In particular, if one has a categorical dimension in mind and wishes for it to be compatible with a tangent structure (so that the categorical dimension becomes a strong dimension with respect to the tangent structure), then one only needs to consider tangent bundle endofunctors which leave the dimension invariant or for which the tangent bundle endofunctor doubles the dimensions. This can exclude many cases, thereby offering another tool in constructing tangent structures.

Ideas for further examples

Across the literature there are many more notions of "sizes" of objects. For example the magnitude for finite metric spaces and graphs, first introduced by Andrew Solow and Stephen Polasky in [84] and further elaborated by Tom Leinster in [62] has some promising properties that suggests it may be a dimension. In particular it is additive under coproducts which are a special case of pushouts, suggesting magnitude as a dimension on the opposite category of finite metric spaces and graphs. Here would we use Definition 3.2.2 which would help to put constraints on possible tangent structures on finite metric spaces and graphs.

Additionally, in Section 3.7 we saw that from any generalized homology theory one obtains an infinite family of dimensions. We suspect that similar results might holds for homologies of graphs and links. Finding dimensions in such traditionally non-geometric areas will make it easier to find tangent structures in them and obtain a geometric perspective to areas that are not traditionally considered as geometry.

Universal dimension

The notion of dimension in Definitions 3.2.2 and 3.2.5 is the notion of dimension tailored for the purpose of understanding the universality of the vertical lift.

In Remark 3.2.1 we listed some desirable properties of dimension, some of which our notion of dimension fulfills and some of which it does not fulfill. We hope that eventually we will discover a closely related notion of dimension that is universal with respect to some list of desired properties similar to Remark 3.2.1, but this list has not been determined here. Definition 3.2.2 does not fully characterize a dimension, as showcased by the fact that there are multiple nontrivial dimensions on given categories. For example the cardinality and the rank are both nontrivial dimensions on the category of abelian groups.

Desired Property (d) (embeddings to inequalities) of Remark 3.2.1 states that embeddings of submanifolds are sent to inequalities of dimensions: If $M' \hookrightarrow M$ is a submanifold of M, then $\dim(M') \leq \dim(M)$.

While Remark 3.7.10 shows that for the Betti numbers not all monomorphisms give rise to inequalities, all sections are sent to inequalities by all examples we gave.

We hope that this observation can motivate a future generalization of dimension which is not just an assignment of objects, but actually a functor.

Relations to the tangent category literature

In [64], Leung classifies tangent structures on a category $\mathbb{X}$ in terms of functors from the category $\mathbb{W}$eil of Weil-Algebras into the endofunctors of $\mathbb{X}$. The rank of the underlying $\mathbb{N}$-module is not a categorical dimension on $\mathbb{W}$eil, even though it is very close to being a dimension. Since it is a module over a rig (semiring) without negatives, Proposition 3.6.4 does not apply and not give a dimension. In the future we will see if the rank can still be used to construct a nontrivial dimension on a category similar to $\mathbb{W}$eil, where we also include negatives. We will continue to investigate if there is a way to construct a nontrivial categorical dimension for every Rosický tangent category (tangent category with negatives).

If we prove such a result for every Rosický tangent category, this will provide a tool to prove that a certain category has only the trivial Rosický tangent structure by proving that there is no nontrivial categorical dimension on it.

While we defined dimensions using retractions, in differential geometry any pullback along a submersion fulfills desired Property (b) (additive w.r.t. pullbacks). In [32], Cruttwell and Lanfranchi prove that display maps are a generalization of submersions in the setting of tangent categories. Thus, in order to gain a better understanding of dimensions, it seems worthwhile to investigate the behaviour of categorical dimensions with respect to pullbacks along display maps. In particular we would hope for a dimension formula for a pullback along display maps, generalizing the result from classical differential geometry.

Chapter 4

Differential bundles as functors from free modules

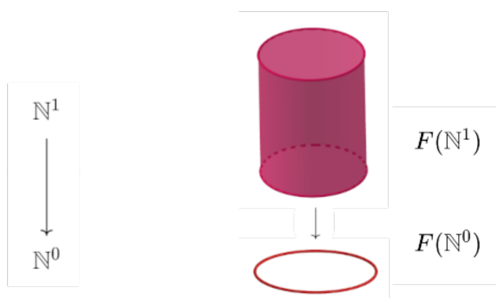

Figure 4.1: Any differential bundle can be encoded as the image of the differential bundle $\mathbb{N}^1 \to \mathbb{N}^0$ in the prototypically linear category $\mathbb{N}^\bullet$ of $\mathbb{N}$-modules. In this image, the cylinder $S^1 \times \mathbb{R}$ is an example of a vector bundle, the classical embodiment of differential bundles. [2]

In this chapter we study differential bundles, the categorification of vector bundles. We will show that differential bundles are closely related to modules over semirings by showing that all differential bundles are induced by a functor from the category $\mathbb{N}^\bullet$ of modules over $\mathbb{N}$ preserving certain pullbacks. We will achieve an equivalence between a category of differential bundles and a category of functors, thereby unifying the axiomatic and the functor approach to tangent categories and enabling easier generalizations in tangent infinity categories and homotopy theory.

This chapter is based on the preprint [82], with small corrections and without the background section of tangent categories (since this background was already given in Chapter 2). Therefore, like in [82], we use applicative composition order, i,e, we denote the composition of $f : A \to B$ and $g : B \to C$ as $g \circ f : A \to C$.

[2]Figure created using GeoGebra 3D

4.1 Introduction

In this chapter, we consider one particular geometric construction in tangent categories: differential bundles. Differential bundles in tangent categories were first discovered by Cockett and Cruttwell in 2018 [29]. They are a categorical generalization of vector bundles. Since many constructions in differential geometry are given by vector bundles (e.g. tangent and cotangent vectors form a vector bundle, de Rham cohomology is built from a vector bundle, and a Riemannian metric is a section of a certain vector bundle), these bundles are a key component of many constructions that we want to generalize in tangent categories. As defined in [29], a differential bundle is a pair of objects E, M in a tangent category together with a projection map $q : E \to M$ satisfying certain axioms. In particular the fiber, expressed as a certain pullback, has an additive structure that satisfies the conditions listed in Definition 2.3.1. A motivating example is the tangent bundle $T(M) \to M$ given as the endofunctor part of any tangent structure.

However, the definition makes it difficult to characterize or classify differential bundles. Unlike other algebraic theories such as internal monoids, the definition of a differential bundle is not given by functors from a certain structure category. The goal of this chapter is to provide exactly this kind of characterization.

To search for a structure category which characterizes differential bundles, we considered the aforementioned characterization of tangent categories through the category of Weil algebras discovered by Leung [64] and Garner [44]. This characterization utilizes actegories, a generalization of group-actions defined by Benabou in 1967 in [8]. Instead of a group acting on a set, an actegory completely describes the action of a monoidal category on another category. Leung and Garner showed that there is an equivalence between the 2-category of tangent categories and the 2-category of certain Weil algebra actegories. Later, Bauer, Burke and Ching used this characterization of tangent categories via Weil algebra actegories to generalize tangent categories into tangent infinity categories [6]. Importantly, they presented the structure of the subcategory of differential objects in a tangent category (the objects with trivial tangent structure) using the category $\mathbb{N}^\bullet$ of (free, finite-dimensional) $\mathbb{N}$ modules. This category, with a slight change, can also be used to characterize differential bundles in tangent categories.

The category $\mathbb{N}^\bullet$ of $\mathbb{N}$ modules is a tangent category in which the tangent bundle of a $\mathbb{N}$-module $\mathbb{N}^k$ is just the product $\mathbb{N}^k \times \mathbb{N}^k$ and the tangent bundle projection is just the projection on the first component $\pi_0 : \mathbb{N}^k \times \mathbb{N}^k \to \mathbb{N}^k$. The key property of the tangent category $\mathbb{N}^\bullet$ is that $\mathbb{N}^1 \to \mathbb{N}^0$ is a differential bundle. Since every object is a product of copies of $\mathbb{N}$, this implies that for every object $\mathbb{N}^k$, the map $\mathbb{N}^k \to \mathbb{N}^0$ is part of a differential bundle. We will continue to see that $\mathbb{N}^\bullet$ encodes the structure of differential bundles exactly.

In order to characterize differential bundles through functors, we define a special kind of functors preserving differential structures, called differential functors. They are defined to be tangent functors that preserve certain pullbacks with certain naturality conditions. We will have two flavors of them, lax and strong differential functors, where the difference is that strong differential functors are also required to preserve the terminal object.

Proposition 4.3.6 will then show that differential functors send differential objects (i.e. differential bundles over a terminal object) to differential bundles. This will immediately give us that every differential functor with domain $\mathbb{N}^\bullet$ induces a differential bundle in the target category.

Conversely, Proposition 4.4.9 constructs a differential functor from a differential bundle showing that every differential bundle comes from a differential functor. Proposition 4.4.9 is the technical crux of this chapter. This leads to Theorem 4.5.3, an equivalence of categories between differential bundles and differential functors.

This new characterization of differential bundles provides 1-categorical context for Michael Ching's and Kaya Arro's forthcoming work on differential bundles in tangent infinity categories and in particular on the differential bundles arising in Goodwillie functor calculus. They use E_∞, an infinity category analogue of $\mathbb{N}^\bullet$, to define a differential bundle in a tangent infinity category.

We also hope that this characterization of differential bundles will be useful in further developing the connections between classical differential geometry and the theory of tangent categories. Since differential bundles are the basic setup for connections [22], we hope we can use the alternate formulation of differential bundles in this chapter to reformulate connections and gain insights into their structure. In particular, we would like to examine the equivalence between horizontal and vertical connection in tangent categories in [22], with an eye towards extending this result. Similarly, we hope to consider dynamical systems in tangent categories using our characterization of differential bundles. Categorical dynamical systems were defined in [74] and dynamical systems in tangent categories were defined in [23]. A key part of the structure in [23] uses differential objects. We hope that our characterization of differential objects can be used to make the relationship between [74] and [23] concrete. See Section 4.6 for further details.

Finally, we hope that this perspective on differential bundles can help to find a correspondence between differential objects in a tangent category and its opposite.

Acknowledgements: I would like to thank Michael Ching for suggesting the main idea of this chapter. I would also like to thank Robin Cockett for going through the proofs in detail, finding the gaps and helping to fill them. Special thanks go to my supervisor Kristine Bauer for her support, discussing all the key proofs on the board in her office as well as for her support in structuring and communicating the result.

4.2 Characterizing Tangent categories as $\mathbb{W}\mathrm{eil}$-actegories

As presented in Definition 2.2.4, tangent categories are an axiomatization of manifolds, tangent bundles, and smooth maps. However, there is an alternative approach to tangent categories using actegories which was first developed by Garner and Leung in [44] and [64]. Garner and Leung noticed that the morphisms used in the axioms from Definition 2.2.4 correspond to morphisms in a certain category $\mathbb{W}\mathrm{eil}$ which are essential to the defining characteristics of $\mathbb{W}\mathrm{eil}$. Tangent categories can be equivalently defined using the action of $\mathbb{W}\mathrm{eil}$ on the desired tangent category.

In Section 4.2.1, we will begin by recalling what it means for a monoidal category $\mathbb{M}$ to act on another category, making it an $\mathbb{M}$-actegory. Then in Section 4.2.2, we will define the category $\mathbb{W}\mathrm{eil}$ and will present Leung-Garner's alternative definition of a tangent category, in which tangent categories are characterized as $\mathbb{W}\mathrm{eil}$-actegories. In this new characterization, one can also interpret the tangent functors and tangent transformations of Definitions 2.2.14 and 2.2.17 in terms of $\mathbb{W}\mathrm{eil}$-linear functors and $\mathbb{W}\mathrm{eil}$-linear natural transformations, which are the functors and natural transformations which preserve the action of $\mathbb{W}\mathrm{eil}$. Garner shows that there is a 2-equivalence of the 2-category of tangent categories, strong tangent functors and linear tangent transformations and the 2-category of $\mathbb{W}\mathrm{eil}$-acteogires, (strong) $\mathbb{W}\mathrm{eil}$-linear fuctors and $\mathbb{W}\mathrm{eil}$-linear natural transformations. In the final result of this section, we adapt Garner's argument to obtain similar equivalences for the 2-categories with lax tangent functors and tangent transformations which are not necessarily linear.

4.2.1 Actegories

We use actegories as a tool to encode tangent categories, tangent functors and tangent transformations. Here in Section 4.2.1 we define actegories and establish some basic properties. In Section 4.2.2 we will use actegories to characterize tangent categories.

Here we use the definition of monoidal categories as in [79, Appendix E2]. A monoidal category $(\mathbb{A}, \otimes, I)$ has a monoidal product $\otimes$ and unit I. A monoidal category also comes equipped with an associator, a, and left and right unitors, l and r, respectively. These are natural transformations, and their components at objects x, y, z in $\mathbb{A}$ will be denoted $a_{x,y,z} : (x \otimes y) \otimes z \cong x \otimes (y \otimes z)$, $l_x : I \otimes x \cong x$ and $r_x : x \otimes I \cong x$. When we want to emphasize these natural transformations, we denote the monoidal category structure by $(\mathbb{A}, \otimes, I, a, l, r)$. The notation $\mathbb{A}^{\otimes}$ is shorthand for $(\mathbb{A}, \otimes, I, a, l, r)$.

The following is an example of a monoidal category which is important for characterizing tangent categories.

Example 4.2.1. *Given a category $\mathbb{X}$, the category $\mathbb{E}\mathrm{nd}(\mathbb{X})$ of endofunctors of $\mathbb{X}$ has a monoidal structure* *given by the composition of endofunctors as monoidal product and the identity functor as unit, denoted as* $\mathbb{E}\mathrm{nd}(\mathbb{X})^\circ$.

Monoidal functors from any monoidal category $\mathbb{M}^\otimes$ into $\mathbb{E}\mathrm{nd}(\mathbb{X})^\circ$ describe $\mathbb{M}^\otimes$ acting on $\mathbb{X}$. These actions are made precise below in the notion of an actegory. Actegories were first defined more than 50 years ago in [8]. We present an equivalent definition from [16], a more modern reference that provides more details. While it is more common to denote the action by $\cdot$ or $\otimes$, we use T for it to resemble the tangent bundle endofunctor T.

Definition 4.2.2. [16, Definition 3.1.1] *Let* $\mathbb{M}^\otimes = (\mathbb{M}, \otimes, I, a, l, r)$ *be a monoidal category with symmetric monoidal product* $\otimes$ *and unit* I*. A (left)* $\mathbb{M}^\otimes$***-actegory*** $\mathbb{X}$ *(or left* $\mathbb{M}$*-action) is a category* $\mathbb{X}$ *equipped with a functor*

$$T_\bullet : \mathbb{M} \times \mathbb{X} \to \mathbb{X}$$

and two natural isomorphisms with components at $X \in \mathrm{Obj}(\mathbb{X})$ *and* $m, n \in \mathrm{Obj}(\mathbb{M})$

$$\eta_X : X \cong T_\bullet(I, X), \qquad \mu_X : T_\bullet(X, T_\bullet(m, n)) \cong T_\bullet(X \otimes m, n),$$

respectively called unitor and multiplicator, satisfying coherence laws expressed as the following commutative diagrams:

$$\begin{tikzcd}
T_\bullet(m, T_\bullet(n, T_\bullet(p, X))) \arrow[rr, "\mu_{m,n,T_\bullet(p,X)}"] \arrow[d, "T_\bullet(1_m, \mu_{n,p,X})"'] & & T_\bullet(m \otimes n, T_\bullet(p, X)) \arrow[d, "\mu_{m\otimes n,p,X}"] \\
T_\bullet(m, T_\bullet(n \otimes p, X)) \arrow[dr, "\mu_{m,n\otimes p,X}"'] & & T_\bullet((m \otimes n) \otimes p, X) \arrow[dl, "T_\bullet(a_{m,n,p}, 1_X)"] \\
& T_\bullet(m \otimes (n \otimes p), X) &
\end{tikzcd}$$

$$\begin{tikzcd}
T_\bullet(I, T_\bullet(m, X)) \arrow[rr, "\mu_{I,m,X}"] & & T_\bullet(I \otimes m, X)) \\
& T_\bullet(m, X) \arrow[ul, "\eta_{T_\bullet(m,X)}"] \arrow[ur, "T_\bullet(l_m, 1_X)"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
T_\bullet(m, T_\bullet(I, X)) \arrow[rr, "\mu_{m,I,X}"] & & T_\bullet(m \otimes I, X) \\
& T_\bullet(m, X) \arrow[ul, "T_\bullet(1_m, \eta_X)"] \arrow[ur, "T_\bullet(r_m, 1_X)"'] &
\end{tikzcd}$$

where $a_{m,n,p}$*,* l_m *and* r_m *are the natural isomorphisms encoding associativity, left-unitality, and right-unitality of* $\mathbb{M}$*.*

Any monoidal category $\mathbb{M}^\otimes$ is an actegory over itself with the functor $T_\bullet : \mathbb{M} \times \mathbb{M} \to \mathbb{M}$ given by the monoidal product, $T(M, N) = M \otimes N$. In particular this makes $\mathbb{W}\mathrm{eil}$ into a $\mathbb{W}\mathrm{eil}^\otimes$-actegory.

Remark 4.2.3. *For a given object* $m \in \mathbb{M}_0$*, we sometimes denote* $T_\bullet(m, -) : \mathbb{X} \to \mathbb{X}$ *as* $T_m : \mathbb{X} \to \mathbb{X}$ *and thus* $T_\bullet(m, X)$ *as* $T_m(X)$*.*

Functors between M-actegories should preserve the actegory structure. There are two main ways of defining these functors, depending on whether the action must be preserved up to natural isomorphism or up to natural transformation. In the latter case, there is also a choice of which direction the natural transformation should go. These choices are all encoded in the following definition.

Definition 4.2.4. [16, Definition 3.3.2] *Let $(\mathbb{X}, T_\bullet, \eta, \mu)$ and $(\mathbb{X}', T'_\bullet, \eta', \mu')$ be left $\mathbb{M}$-actegories. A **lax** $\mathbb{M}$**-linear functor** (F, α) between them is a functor $F : \mathbb{X} \to \mathbb{X}'$ equipped with a natural transformation*

$$\alpha : T'_\bullet \circ (1_\mathbb{M} \times F) \Rightarrow F \circ T_\bullet,$$

*i.e. a natural collection of maps $\alpha_{m,x} : T'_\bullet(m, F(x)) \to F(T_\bullet(m, x))$ for $x \in \mathrm{Obj}(\mathbb{X}), m \in \mathrm{Obj}(\mathbb{M})$, called **lineator** satisfying the following coherence laws for all $m, n \in \mathrm{Obj}(M)$ and $x \in \mathrm{Obj}(\mathbb{X})$:*

$$\begin{tikzcd}
T'_\bullet(m, T'_\bullet(n, F(x))) \arrow[rr, "T'_\bullet(m,\alpha_{n,x})"] \arrow[d, "\mu'_{m,n,F(x)}"'] & & T'_\bullet(m, F(T_\bullet(n, x))) \arrow[d, "\alpha_{m,T_\bullet(n,x)}"] \\
T'_\bullet(m \otimes n, F(x)) \arrow[dr, "\alpha_{m\otimes n,x}"'] & & F(T_\bullet(m, T_\bullet(n, x))) \arrow[dl, "F(\mu_{m,n,x})"] \\
& F(T_\bullet(m \otimes n, x)) &
\end{tikzcd}$$

$$\begin{tikzcd}
T'_\bullet(I_\mathbb{M}, F(X)) \arrow[rr, "\alpha_{I_\mathbb{M},X}"] \arrow[dr, "\eta'_{F(X)}"'] & & F(T_\bullet(I_\mathbb{M}, X)) \arrow[dl, "F(\eta_X)"] \\
& F(X) &
\end{tikzcd}$$

*We call (F, α) a **strong** $\mathbb{M}$**-linear functor**, if α is a natural isomorphism. An **oplax** $\mathbb{M}$**-linear functor** (F, α) is a lax $\mathbb{M}$-linear functor $(F, \alpha) : \mathbb{X}^{\mathrm{op}} \to \mathbb{X}'^{\mathrm{op}}$, i.e. a functor $F : \mathbb{X} \to \mathbb{X}'$ with a lineator $\alpha : F \circ T_\bullet \Rightarrow T'_\bullet \circ (1_\mathbb{M} \times F)$.*

Having defined actegories and functors between actegories, we will continue defining natural transformations between actegories, that will have a special linearity condition. Unlike [16] we will not always require this linearity condition and call the natural transformation linear if it fulfills the condition.

Definition 4.2.5. [16, Definition 3.3.9] *Let (F, α) and (F', α') be $\mathbb{M}$-linear functors between the $\mathbb{M}$-actegories $(\mathbb{X}, T_\bullet, \eta, \mu)$ and $(\mathbb{X}', T'_\bullet, \eta', \mu')$.*

*(a) A $\mathbb{M}$**-natural transformation** between (F, α) and (F', α') is simply a natural transformation $\varphi : F \Rightarrow F'$ of underlying functors.*

*(b) It is called a **linear** $\mathbb{M}$-**natural transformation** if in addition the diagram*

$$\begin{tikzcd} T_\bullet(m, F(X)) \arrow[r, "T_\bullet(1_m,\varphi)"] \arrow[d, "\alpha_{m,X}"'] & T_\bullet(m, F'(X)) \arrow[d, "\alpha'_{m,X}"] \\ F(T_\bullet(m, X)) \arrow[r, "\varphi(T_\bullet(m,X))"'] & F'(T_\bullet(m, X)) \end{tikzcd} \tag{4.1}$$

commutes.

Example 4.2.6. *Let* $\mathbb{F}\mathrm{inSet}$ *be the category of finite sets and* $\mathbb{T}\mathrm{op}$ *be the category of topological spaces. With the Cartesian product as monoidal product and the one point space* $* = \{0\}$ *as the monoidal unit,* $\mathbb{F}\mathrm{inSet}$ *is made into a monoidal category* $\mathbb{F}\mathrm{inSet}^\times$*. We will now define two actions making* $\mathbb{T}\mathrm{op}$ *into a* $\mathbb{F}\mathrm{inSet}$*-actegory:*

1. *The functor*

$$T_0 : \mathbb{F}\mathrm{inSet} \times \mathbb{T}\mathrm{op} \to \mathbb{T}\mathrm{op}, (A, X) \mapsto \mathrm{Disc}(A) \times X$$

   where Disc *sends the set* A *to the topological space that is* A *with the discrete topology. The pair* $(\mathbb{T}\mathrm{op}, T_0)$ *is a* $\mathbb{S}\mathrm{et}$*-actegory because the Cartesian product of sets with the discrete topology is the product of discrete topological spaces, the Cartesian product is associative and the one point set with the discrete topology is the one point space* $*$ *fulfilling* $* \times X \cong X$*.*

2. *The projection functor*

$$T_1 : \mathbb{F}\mathrm{inSet} \times \mathbb{T}\mathrm{op} \to \mathbb{T}\mathrm{op}, (A, X) \mapsto X$$

   also is trivially an actegory. All the coherence-morphisms become identities.

Now the pair $(1_{\mathbb{T}\mathrm{op}}, \pi_1)$*, comprised of the identity functor* $1_{\mathbb{T}\mathrm{op}} : \mathbb{T}\mathrm{op} \to \mathbb{T}\mathrm{op}$ *of topological spaces together with the natural transformation* $\pi_1 : T_0(A, X) = \mathrm{Disc}(A) \times X \Rightarrow X = T_1(A, X)$ *as lineator, is an oplax* $\mathbb{S}\mathrm{et}$*-linear functor.*

4.2.2 Tangent structures and Weil algebras

In [64], Leung presented an alternative characterization of tangent categories which was inspired by synthetic differential geometry (SDG). The characterization uses a monoidal category $\mathbb{W}\mathrm{eil}$ whose objects are certain commutative algebras over the monoid $\mathbb{N}$ (see Definition 4.2.7). Importantly, the category $\mathbb{W}\mathrm{eil}$ contains the dual numbers, i.e. the algebra $\mathbb{N}[x]/\langle x^2\rangle$. In SDG, the dual numbers play the role of an infinitesimal which represents tangent lines. Leung uses this idea to recast tangent categories: from a strong monoidal functor from $\mathbb{W}\mathrm{eil}$ to $\mathbb{E}\mathrm{nd}^\circ(\mathbb{X})$, one obtains a functor $T : \mathbb{X} \to \mathbb{X}$ as the image of the dual numbers. The natural transformations $p, 0, +, \ell, c$ of Definition 2.2.4 the images of certain maps $p, 0, +, \ell, c$, which are intentionally given the same names. The functor T together with these natural transformations fulfill Definition 2.2.4. Leung's result is summarized as Theorem 4.2.11.

In [44], Garner extended this result to show that there is a 2-equivalence between a 2-category of Weil algebras and a 2-category of tangent categories. That is, the equivalence extends to include strong tangent functors between tangent categories and linear tangent transformations between tangent functors. This result is summarized in Theorem 4.2.17. In Garner's equivalence, strong tangent functors correspond to strong $\mathbb{W}$eil-linear functors, and linear tangent transformations correspond to linear transformations. The goal of this section is to generalize this result to 2-categories of tangent categories with lax tangent functors and general tangent transformations. Our strategy for proving this will be to modify Garner's proof of Theorem 4.2.17. For this reason, in this section we review Weil algebras, Leung's correspondence, Garner's 2-equivalence and the proof with modifications as needed.

We start by revisiting Example 2.2.12.

Definition 4.2.7. [64, Remark 3.18] *Weil algebras are commutative algebras that are of the form*

$$\mathbb{N}[x_1, ..., x_n]/ < x_i x_j | i \sim j >$$

for some equivalence relation $\sim$. A morphism of Weil algebras is an $\mathbb{N}$-linear semi-ring homomorphism that preserves the unit. The category $\mathbb{W}$eil *consists of Weil algebras and morphisms of Weil algebras.*

In [64], Leung only asks for the relation $\sim$ to be reflexive and symmetric. He denotes the subcategory where $\sim$ is transitive as $\mathbb{W}\text{eil}_1$. Thus our category $\mathbb{W}$eil is what Leung calls $\mathbb{W}\text{eil}_1$.

Remark 4.2.8. *Every object of* $\mathbb{W}$eil *is generated by the dual numbers, $W = \mathbb{N}[x]/\langle x^2 \rangle$. The category* $\mathbb{W}$eil *has all finite coproducts and certain products. The coproduct is the tensor product $W \otimes W$, given by*

$$W \otimes W = \mathbb{N}[x, y]/ < x^2, y^2 > .$$

The product, $W \times W$ is explicitly:

$$W \times W = \mathbb{N}[x, y]/ < x^2, y^2, xy > .$$

Note that the coproduct $W \otimes W$ contains an element xy. However, the product $W^2 = W \times W$ does not contain xy. It is possible to use the explicit formulae for $W \otimes W$ and $W \times W$ to express coproducts and products in $\mathbb{W}$eil.

Every Weil Algebra A can be written uniquely as

$$A = W^{n_1} \otimes ... \otimes W^{n_k}$$

In fact, in [64, Definition 4.2]*, Leung uses this characterization to define the category* $\mathbb{W}$eil *which he calls Weil$_1$. Later, after Proposition 6.6 he remarks that the objects of* $\mathbb{W}$eil *are described by a relation $\sim$ that corresponds to a disjoint union of full graphs, to an equivalence relation in our perspective.*

Remark 4.2.9. *There are certain maps in* $\mathbb{W}$eil *that behave like the natural transformations used in Definition 2.2.4. For this reason, we use the same notation to denote these maps in* $\mathbb{W}$eil *as the natural transformations in a tangent category. Theorem 4.2.11 makes it clear that this ambiguity is justified. The maps in question are:*

$$\begin{aligned}
&p : W \to \mathbb{N} && ax + b \mapsto b \\
&+ : W \times W \mapsto W && ax_1 + bx_2 + c \mapsto (a+b)x + c \\
&0 : \mathbb{N} \mapsto W && a \mapsto 0 \cdot x + a \\
&c : \mathrm{W} \otimes W \to W \otimes W && ax_1x_2 + bx_1 + cx_2 + d \mapsto ax_1x_2 + cx_1 + bx_2 + d \\
&\ell : W \to W \otimes W && ax + b \mapsto ax_1x_2 + b
\end{aligned}$$

An immediate result of the observation that these maps behave like the tangent transformations is that $\mathbb{W}$eil itself is a tangent category using this structure. This is the content of the next Theorem.

Theorem 4.2.10. [64, Proposition 4.1] *The (endo)functor*

$$W \otimes \_ : \mathbb{W}\mathrm{eil} \to \mathbb{W}\mathrm{eil}$$

is the tangent functor of a tangent structure on $\mathbb{W}$eil.

The category $\mathbb{E}\mathrm{nd}(\mathbb{X})$ of endofunctors of a category $\mathbb{X}$ has a monoidal structure, given by the composition of endofunctors. The category $\mathbb{W}$eil has a monoidal structure, given by the coproduct $\otimes$.

A strong monoidal functor is a functor $F : \mathbb{X}^{\otimes} \to \mathbb{X}'^{\otimes}$ between monoidal categories together with natural isomorphisms

$$\mu : F(- \otimes -) \cong F(-) \otimes F(-) \quad \text{and} \quad \eta : F(I_{\mathbb{X}}) \cong I'_{\mathbb{X}}$$

fulfilling some compatibility properties that are explained in [51, Definition 1.27]. Using these monoidal structures, we can now present the anticipated characterization of tangent structures by Leung.

Theorem 4.2.11. [64, Theorem 14.1] *Giving a tangent structure* $(T, p, 0, +, \ell, c)$ *on* $\mathbb{X}$ *is equivalent to giving a strong monoidal functor* $\tilde{T} : \mathrm{Weil}^{\otimes} \to \mathrm{End}^{\circ}(\mathbb{X})$ *satisfying the following two conditions:*

1. *The diagrams of the form*

$$\begin{tikzcd} A \otimes (B \times C) \arrow[r, "1_A \otimes \pi_B"] \arrow[d, "1_A \otimes \pi_C"'] \arrow[dr, phantom, "\lrcorner" very near start] & A \otimes B \arrow[d, "1_A \otimes \epsilon_B"] \\ A \otimes C \arrow[r, "1_A \otimes \epsilon_C"'] & A \end{tikzcd}.$$

   are pullbacks in Weil *and* $\tilde{T}$ *preserves them.*

2. *The diagrams of the form*

$$\begin{tikzcd} W \times W \arrow[r, "\nu"] \arrow[d, "\mathrm{ev}_0"'] \arrow[dr, phantom, "\lrcorner" very near start] & W \otimes W \arrow[d, "1_W \otimes p"] \\ \mathbb{N} \arrow[r, "0"'] & W \end{tikzcd}$$

   where ν *is a shorthand for* $(1_W \otimes +) \circ (\ell \times (0_W \otimes 1_W))$*, are pullbacks in* Weil *and* $\tilde{T}$ *preserves them.*

The tangent bundle endofunctor T *is obtained from* $\tilde{T}$ *by setting* $T = \tilde{T}(W)$*, the natural transformations* $p, 0, +, \ell, c$ *in* $\mathbb{X}$ *are the images* $\tilde{T}(p), \tilde{T}(0), \tilde{T}(+), \tilde{T}(\ell), \tilde{T}(c)$ *of the morphisms in* Weil *with the same name.*

We will call the pullbacks of Theorem 4.2.11 the **tangent pullbacks**.[3] That the functor is monoidal means in particular that it sends the coproduct of algebras in $\mathbb{W}\mathrm{eil}$ to the composition of functors.

Remark 4.2.12. *As a convention, we denote a tangent bundle endofunctor like in Definition 2.2.4 as* $T : \mathbb{X} \to \mathbb{X}$ *(without decoration) and the corresponding* Weil*-action as* $T_{\bullet} : \mathrm{Weil} \to \mathrm{End}(\mathbb{X})$ *(with an* $\bullet$ *in the index). We use the same notation for the corresponding functor* $T_{\bullet} : \mathrm{Weil} \times \mathbb{X} \to \mathbb{X}$*.*

Remark 4.2.13. *As a special case of the correspondence in Theorem 4.2.11, the monoidal category* Weil *acts on itself by the tensor product. This monoidal functor* $\mathrm{Weil} \to \mathrm{End}(\mathrm{Weil})$ *encodes the tangent structure of Theorem 4.2.10.*

In order to make the correspondence between $\mathbb{W}\mathrm{eil}$-actegories and tangent categories more precise we use the language of 2-categories. For convenience, we review the main structures of 2-categories here - these will be required when we review Garner's proof of Theorem 4.2.17. For a full definition which includes the precise formulation of the associativity and unitality properties, see [79, Definition 1.7.8], [68, Chapter VII], or [55, Proposition 2.3.4]. A **2-category** consists of objects and for each pair of objects A, B a hom-category $\underline{\mathrm{Hom}}(A, B)$ with a unit $1_A \in \underline{\mathrm{Hom}}(A, A)$ and a composition bifunctor (i.e. functor originating in a product of categories) $\circ : \underline{\mathrm{Hom}}(B, C) \times \underline{\mathrm{Hom}}(A, B) \to \underline{\mathrm{Hom}}(A, C)$. The objects of $\underline{\mathrm{Hom}}(A, B)$ are called morphisms[4],

[3] In [64], the diagrams in Theorem 4.2.11.1 are called **foundational** pullbacks. Here we have used the terminology **tangent** pullbacks in order to include the pullbacks in the second part of the theorem.

[4] In the literature they are often called 1-morphisms

the morphisms of $\underline{\mathrm{Hom}}(A, B)$ are called 2-morphisms. The 2-morhpisms can be composed in two ways. Given morphisms $f, g, h : A \to B$ and 2-morphisms $\alpha : f \Rightarrow g$ and $\beta : g \Rightarrow h$, their vertical composition $\beta * \alpha : f \Rightarrow h$ is the composition in the category $\underline{\mathrm{Hom}}(A, B)$. Given morphisms $f_1, f_2 : A \to B$ and $g_1, g_2 : B \to C$ and 2-morphisms $\gamma : f_1 \Rightarrow f_2, \delta : g_1 \Rightarrow g_2$, their horizontal composition $\delta \circ \gamma : g_1 \circ f_1 \Rightarrow g_2 \circ f_2$ is the application of the bifunctor $\circ$.

Both compositions are strictly associative and unital. We denote the names of 2-categories with all capitals in simplified font. For example the 2-category of categories, functors and natural transformations we denote as CAT.

A **2-functor** $F : \mathsf{X} \to \mathsf{X}'$ between 2-categories X and X' is an assignment that sends objects to objects together with functors that send the hom-categories of X to the hom-categories of X' in a way that preserves compositions and units strictly. Thus it assigns morphisms to morphisms and 2-morphisms to 2-morphisms. For composable morphisms f and g, a 2-functor F fulfills $F(f \circ g) = F(f) \circ F(g)$ and $F(1) = 1$ (strict preservation of composition and unit). Since F is a functor of hom-categories it also preserves vertical composition of 2-morphisms, i.e. for $\alpha : f \Rightarrow g$ and $\beta : g \Rightarrow h$ $F(\beta * \alpha) = F(\beta) * F(\alpha)$. The precise definition can be found in [55, Definition 4.1.2].

A **2-natural transformation** between 2-functors $F, G : \mathsf{X} \to \mathsf{X}'$ is a collection of morphisms $\varphi_X : F(X) \to G(X)$ in X' for all objects X of X such that for every morphism $f : X \to X'$ of X the diagram

$$\begin{tikzcd} F(X) \arrow[r, "G(f)"] \arrow[d, "\varphi_X"'] & F(X') \arrow[d, "\varphi_{X'}"] \\ G(X) \arrow[r, "G(f)"'] & G(X') \end{tikzcd}$$

strictly commutes. The notions of 2-functors and 2-natural transformations are completely analogous to the classical notions of functors and natural transformations in 1-categories.

Definition 4.2.14. [55, Definition 6.2.10] *A* ***2-equivalence*** *is a 2-functor* $F : \mathsf{X} \to \mathsf{X}'$*, together with a 2-functor* $G : \mathsf{X}' \to \mathsf{X}$ *and 2-natural isomorphisms*

$$1_{\mathsf{X}'} \cong F \circ G \qquad 1_{\mathsf{X}} \cong G \circ F.$$

The Whitehead Theorem for 2-categories [55, Theorem 7.5.8] states that a 2-functor $F : \mathsf{X} \to \mathsf{X}'$ is a 2-equivalence if and only if it is

- essentially surjective, i.e. surjective on isomorphism-classes of objects,
- fully faithful on morphisms, i.e. bijective on morphisms, and

- fully faithful on 2-morphisms, i.e. bijective on 2-morphisms.

We will use the language of 2-equivalences to describe the correspondence between various 2-categories of tangent categories and $\mathbb{W}$eil-actegories. These 2-categories are our most important examples.

Example 4.2.15 (The 2-category of tangent categories)**.**

1. *Let* $\mathsf{TANG}^{\mathrm{lin}}_{\mathrm{strong}}$ *denote the 2-category with*

   - *tangent categories as in Definition 2.2.4 as objects,*
   - *strong tangent functors as in Definition 2.2.14 as morphisms, and*
   - *linear tangent transformations as in Definition 2.2.17 as 2-morphisms.*

   The 2-category $\mathsf{TANG}^{\mathrm{lin}}_{\mathrm{strong}}$ *is the 2-category used in Theorem 4.2.17.*

2. *Let* $\mathsf{TANG}_{\mathrm{lax}}$ *denote the 2-category of tangent categories, lax tangent functors as in Definition 2.2.14 and tangent transformations. The other 2-categories in this example are sub 2-categories of this one.*

3. *Let* $\mathsf{TANG}_{\mathrm{strong}}$ *denote 2-category of tangent categories, strong tangent functors and tangent transformations as in Definition 2.2.17.*

4. *Let* $\mathsf{TANG}^{\mathrm{lin}}_{\mathrm{lax}}$ *denote 2-category of tangent categories, lax tangent functors and linear tangent transformations.*

These are the four choices that arise when considering all combinations of morphisms and transformations from Definitions 2.2.14 and 2.2.17. We need all four combinations because in the next sections, strong functors will lead to differential objects and lax functors will lead to differential bundles. Differential objects and bundles have respective notions of additive and linear morphisms which will correspond to tangent transformations and linear tangent transformations.

In order to state the correspondence between tangent categories and $\mathbb{W}$eil-actegories we also need to have a 2-categorical description of $\mathbb{W}$eil-actegories, which is explained in the following example.

Example 4.2.16. [44, Section 3]

1. *Let* $\mathbb{W}\mathrm{eil}{-}\mathsf{ACT}^{\mathrm{lin}}_{\mathrm{strong}}$ *denote the 2-category with*

   - $\mathbb{W}$eil-*actegories as in Definition 4.2.2 preserving the tangent pullbacks of Definition 2.2.4 (the foundational pullbacks and the universality of the vertical lift) as objects,*
   - *strong* $\mathbb{W}$eil-*linear functors as in Definition 4.2.4 as morphisms, and*
   - *linear* $\mathbb{W}$eil-*natural transformations as in Definition 4.2.5 as 2-morphisms.*

2. *Let* $\mathbb{W}\mathrm{eil}{-}\mathsf{ACT}_{\mathrm{oplax}}$ *denote the 2-category of* $\mathbb{W}\mathrm{eil}$*-actegories preserving foundational pullbacks and the universality of the vertical lift, oplax* $\mathbb{W}\mathrm{eil}$*-linear functors as in Definition 4.2.4 and* $\mathbb{W}\mathrm{eil}$*-natural transformations. The other 2-categories in this example are sub 2-categories of this one.*

3. *Let* $\mathbb{W}\mathrm{eil}{-}\mathsf{ACT}_{\mathrm{strong}}$ *denote the 2-category of* $\mathbb{W}\mathrm{eil}$*-actegories preserving foundational pullbacks and the universality of the vertical lift, strong* $\mathbb{W}\mathrm{eil}$*-linear functors and* $\mathbb{W}\mathrm{eil}$*-natural transformations as in Definition 4.2.5.*

4. *Let* $\mathbb{W}\mathrm{eil}{-}\mathsf{ACT}^{\mathrm{lin}}_{\mathrm{oplax}}$ *denote the 2-category of* $\mathbb{W}\mathrm{eil}$*-actegories preserving foundational pullbacks and the universality of the vertical lift, oplax* $\mathbb{W}\mathrm{eil}$*-linear functors and linear* $\mathbb{W}\mathrm{eil}$*-natural transformations.*

For the 2-category $\mathsf{TANG}^{\mathrm{lin}}_{\mathrm{strong}}$, the following theorem, due to Garner [44], is useful, because it gives an equivalence of categories. This extends the correspondence between objects given in Theorem 4.2.11.

Theorem 4.2.17. [44, Theorem 9] *The 2-category* $\mathsf{TANG}^{\mathrm{lin}}_{\mathrm{strong}}$ *is equivalent to* $\mathbb{W}\mathrm{eil}{-}\mathsf{ACT}^{\mathrm{lin}}_{\mathrm{strong},t}$.

In the following we want to consider lax functors and natural transformations because these will encode differential bundles and additive morphisms between them. Therefore we will now do a slight generalization of Theorem 4.2.17 with different choices for the morphisms and 2-morphisms.

Concretely, we will now extend Theorem 4.2.17 to $\mathsf{TANG}^{\mathrm{add}}_{\mathrm{strong}}$, $\mathsf{TANG}^{\mathrm{add}}_{\mathrm{lax}}$ and $\mathsf{TANG}^{\mathrm{lin}}_{\mathrm{lax}}$.

Theorem 4.2.18. *There are equivalences of 2-categories:*

$$\mathsf{TANG}_{\mathrm{lax}} \simeq \mathbb{W}\mathrm{eil}{-}\mathsf{ACT}_{\mathrm{oplax}}$$
$$\mathsf{TANG}^{\mathrm{lin}}_{\mathrm{lax}} \simeq \mathbb{W}\mathrm{eil}{-}\mathsf{ACT}^{\mathrm{lin}}_{\mathrm{oplax}}$$
$$\mathsf{TANG}_{\mathrm{strong}} \simeq \mathbb{W}\mathrm{eil}{-}\mathsf{ACT}_{\mathrm{strong}}$$

Garner's proof of Theorem 4.2.17 proceeds by showing that there is a 2-functor between $\mathsf{TANG}^{\mathrm{lin}}_{\mathrm{strong}}$ and $\mathbb{W}\mathrm{eil}{-}\mathsf{ACT}^{\mathrm{lin}}_{\mathrm{strong}}$ which provides an essentially surjective correspondence between the objects and an isomorphism between the hom categories. In the remaining cases (lax or non-linear), Garner's proof will still provide a good framework. The construction of the equivalence in this case works analogously with only minor differences. However, the proof that there is a correspondence between 2-morphisms is significantly simplified in the non-linear case: the correspondence becomes trivial because Diagram 4.1 in Definition 4.2.5 does not need to commute in the non-linear case.

We will show now that lax tangent functors and tangent transformations correspond exactly to lax $\mathbb{W}\mathrm{eil}$-linear functors and natural transformations under 4.2.17. To prove this we use Garner's correspondence from [44, Theorem 9]. The following summary of Garner's proof is included in order to provide enough detail so that we can change the functors and natural transformations in Theorem 4.2.18 and this change does not impact the proof. We start with a useful lemma.

Lemma 4.2.19. *Let $(\mathbb{X}, T_\bullet)$ and $(\mathbb{X}', T'_\bullet)$ be objects of $\mathbb{W}\mathrm{eil}{-}\mathsf{ACT}_{\mathrm{oplax}}$ and let $F : \mathbb{X} \to \mathbb{X}'$ be a functor of underlying categories. Let $\hat{\alpha}, \hat{\beta} : F \circ T_\bullet \to T'_\bullet \circ (1_{\mathbb{W}\mathrm{eil}} \times F)$ be natural transformation fulfilling the lineator conditions of Definition 4.2.4 that coincide on the dual numbers*

$$\hat{\alpha}_W = \hat{\beta}_W : F \circ T_\bullet(W, -) \Rightarrow T'_\bullet(W, -) \circ F.$$

Then $\hat{\alpha}$ and $\hat{\beta}$ coincide: $\hat{\alpha} = \hat{\beta}$.

Proof. Every object V of $\mathbb{W}\mathrm{eil}$ is a coproduct of powers of the dual numbers, $V = W^{n_1} \otimes ... \otimes W^{n_k}$. Since $T_\bullet$ and $T'_\bullet$ preserve foundational pullbacks and morphisms into pullbacks are uniquely determined by their components, $\hat{\alpha}_W = \hat{\beta}_W$ implies $\hat{\alpha}_{W^n} = \hat{\beta}_{W^n}$. For $n = 0$, $\alpha_{\mathbb{N}} = \beta_{\mathbb{N}}$ since the conditions of 4.2.4 fully determine the lineator on the monoidal unit. Thus it only remains to show that the equality is also preserved for tensor products of Weil algebras. For that let V and V' be objects of $\mathbb{W}\mathrm{eil}$ such that $\hat{\alpha}_V = \hat{\beta}_V$ and $\hat{\alpha}_{V'} = \hat{\beta}_{V'}$. Then the pentagon condition from Definition 4.2.4 means that

$$\begin{tikzcd}
F \circ T_\bullet(V \otimes V', -) \arrow[r, "\cong"'] \arrow[dd, "\hat{\alpha}_{V \otimes V'}"'] & F \circ T_\bullet(V, -) \circ T_\bullet(V', -) \arrow[d, "\hat{\alpha}_V \circ T'_\bullet(V', -)"] \\
& T'_\bullet(V, -) \circ F \circ T_\bullet(V', -) \arrow[d, "T_\bullet(V, -) \circ \alpha_{V'}"] \\
T'_\bullet(V \otimes V', -) \circ F & T'_\bullet(V, -) \circ T'_\bullet(V', -) \circ F \arrow[l, "\cong"']
\end{tikzcd}$$

commutes (and the same for $\hat{\beta}_{V \otimes V}$). Thus $\hat{\alpha}_{V \otimes V}$ and $\hat{\beta}_{V \otimes V}$ are completely determined by their V and V' components and therefore coincide. Since every object of $\mathbb{W}\mathrm{eil}$ is a tensor product of powers of W this implies that $\hat{\alpha}$ and $\hat{\beta}$ coincide. □

Proof (of Theorem 4.2.17). To prove Theorem 4.2.17, Garner explicitly constructs a 2-functor

$$E : \mathsf{TANG}^{\mathrm{lin}}_{\mathrm{strong}} \to \mathbb{W}\mathrm{eil}{-}\mathsf{ACT}^{\mathrm{lin}}_{\mathrm{strong},t}$$

and verifies that it is an equivalence. The components of E are defined as follows.

On objects, E must send a tangent category to a $\mathbb{W}$eil-actegory. By Theorem 4.2.11, a tangent structure on $\mathbb{X}$ is given by a functor

$$T : \mathbb{W}\mathrm{eil} \to \mathbb{E}\mathrm{nd}(\mathbb{X})$$

The transpose of this functor is the functor associated to a Weil-actegory structure

$$T_\bullet : \mathbb{W}\mathrm{eil} \times \mathbb{X} \to \mathbb{X}$$

where $T_\bullet(A, x) = T(A)(x)$. Since T was monoidal, this is an actegory, and this defines $E(\mathbb{X})$.

On morphisms, E must send a (strong) tangent functor $(F, \alpha) : (\mathbb{X}, T) \to (\mathbb{X}', T')$ to a (strong) functor of Weil-actegories $(\hat{F}, \hat{\alpha}) : (\mathbb{X}, T_\bullet) \to (\mathbb{X}', T'_\bullet)$. Define $\hat{F} = F : \mathbb{X} \to \mathbb{X}'$. The natural transformation $\hat{\alpha} : F \circ T_\bullet \Rightarrow \hat{T}' \circ (1 \times F)$ is defined more subtly using an intermediary category whose construction we now review.

Let $\mathbb{K}_F$ be the 1-category whose objects are triples (A, B, τ) with $A \in \mathrm{End}(\mathbb{X}), B \in \mathrm{End}(\mathbb{X}')$ and $\tau : F \circ A \Rightarrow B \circ F$ a natural isomorphism. The morphisms $(\varphi, \psi) : (A, B, \tau) \to (A', B', \tau')$ of $\mathbb{K}_F$ are pairs of natural transformations $\varphi : A \Rightarrow A'$ and $\psi : B \Rightarrow B'$ satisfying $\tau' * F(\varphi) = \psi_F * \tau$. There is a tangent structure on $\mathbb{K}_F$ given by a tangent functor $T_{\mathbb{K}_F} : \mathbb{K}_F \to \mathbb{K}_F$ defined as follows:

$$T_{\mathbb{K}_F}(A, B, \tau) := (T \circ A, T' \circ B, T'(\tau) * \hat{\alpha}_A)$$

and defined on morphisms using T and T' in a similar way. By Theorem 4.2.11 this induces a functor $H : \mathbb{W}\mathrm{eil} \to \mathbb{E}\mathrm{nd}(\mathbb{K}_F)$. The first two components of $T_{\mathbb{K}_F}$ are the precomposition with T or T' respectively, so the first two components of $H(V)$ for any Weil algebra, V, are given by the precomposition with $T_\bullet$ and $\hat{T}'$. In particular for the object $(1_{\mathbb{X}}, 1_{\mathbb{X}'}, 1_F) \in \mathbb{K}_F$ and a Weil algebra V, there exists a natural transformation $\hat{\alpha}_V : F \circ T_\bullet(V, -) \Rightarrow T'(V, -) \circ (F \times 1_{\mathbb{W}\mathrm{eil}})$ such that

$$H(V)(1_{\mathbb{X}}, 1_{\mathbb{X}'}, 1_F) = (T_\bullet(V, -), \hat{T}'(V, -), \hat{\alpha}_V).$$

For a morphism $\theta : V \to V'$ in $\mathbb{W}\mathrm{eil}$, we have $H(\theta) = (T_\bullet(\theta, -), T'_\bullet(\theta, -))$ such that $\hat{\alpha}_{V'} * T_\bullet(\theta, -) = T'_\bullet(\theta, -) \circ \hat{\alpha}_V$ Therefore, the natural transformation $\hat{\alpha}_V$ is natural in $\mathbb{W}\mathrm{eil}$. Thus, we have produced a natural transformation $\hat{\alpha} : F \circ T_\bullet \to T'_\bullet \circ (1 \times F)$. The functor E sends a tangent functor (F, α_F) to the $\mathbb{W}$eil-linear functor $(F, \hat{\alpha})$ so defined. This is an object of $\mathbb{K}_F$, thus $\hat{\alpha}$ is an isomorphism, which means that strong tangent functors are sent to strong $\mathbb{W}$eil-linear functors. One can prove that $\hat{\alpha}$ fulfills the lineator conditions using that H is monoidal in order to expand $H(V \otimes V')(1_{\mathbb{X}}, 1_{\mathbb{X}'}, 1_F)$. The formula for the tangent structure $T_{\mathbb{K}_F}$ then forces the third component to be $T'_\bullet(V, \hat{\alpha}_{V'}) * \hat{\alpha}_V$ proving that the lineator property in Definition 4.2.4 holds.

To complete the definition of E, we must define E on 2-morphisms. For a 2-morphism $\varphi : (F, \alpha) \Rightarrow (F', \alpha')$ in $\mathrm{TAN}^{\mathrm{lin}}_{\mathrm{strong}}$, we must construct a natural transformation $F \Rightarrow F'$ fulfilling the linearity condition in Diagram

4.1. We choose $E(\varphi) := \varphi$ as the natural transformation. Garner indicates that a construction similar to the morphism argument using the auxiliary category $\mathbb{K}_F$ can be used to show that the linearity of φ in the sense of Definition 2.2.17 implies linearity of $E(\varphi) = \varphi$ in the sense of Definition 4.2.5. In order to point out the modifications needed for the proof of Theorem 4.2.18, we will make this analogy explicit. The analogue to $\mathbb{K}_F$ is an intermediary category $\mathbb{L}_\varphi$, defined next.

Suppose $\varphi : (F, \alpha) \to (F', \alpha')$ is a 2-morphism of $\mathsf{TANG}^{\text{lin}}_{\text{lax}}$, i.e. a linear tangent transformation. Then we define the 1-category $\mathbb{L}_\varphi$ whose objects are triples (V, a, b), where $V \in \text{Obj}(\mathbb{W}\text{eil})$ is a Weil algebra, $a : F \circ T_\bullet(V, -) \Rightarrow \hat{T}'(V, -) \circ F$ and $b : F' \circ T_\bullet(V, -) \Rightarrow T'_\bullet(V, -) \circ F'$ are natural transformations such that

$$\begin{tikzcd} F \circ T_\bullet(V,-) \arrow[r, "a"] \arrow[d, "\varphi"'] & \hat{T}'(V,-) \circ F \arrow[d, "\hat{T}'(V,-)\circ\varphi"] \\ F' \circ T_\bullet(V,-) \arrow[r, "b"'] & \hat{T}'(V,-) \circ F' \end{tikzcd}$$

commmutes. A morphism $(V, a, b) \to (V', a', b')$ of L_φ is a $\mathbb{W}$eil-morphism $\gamma : V \to V'$ such that the diagrams

$$\begin{tikzcd} F \circ T_\bullet(V,-) \arrow[r, "a"] \arrow[d, "F\circ T_\bullet(\gamma,-)"'] & \hat{T}'(V,-) \circ F \arrow[d, "\hat{T}'(\gamma,-)\circ F"] \\ F \circ T_\bullet(V',-) \arrow[r, "a'"'] & \hat{T}'(V',-) \circ F \end{tikzcd} \qquad \begin{tikzcd} F' \circ T_\bullet(V,-) \arrow[r, "b"] \arrow[d, "\hat{T}'(\gamma,-)\circ F'"'] & \hat{T}'(V,-) \circ F' \arrow[d, "\hat{T}'(\gamma,-)\circ F'"] \\ F' \circ T_\bullet(V',-) \arrow[r, "b'"'] & \hat{T}'(V',-) \circ F' \end{tikzcd}$$

commute.

Given objects (V, a, b) and (V', a', b') of $\mathbb{L}_\varphi$ such that the product $V \times V'$ exists in $\mathbb{W}$eil, the diagram

$$\begin{tikzcd} (V \times V', a \times_{1_F} a', b \times_{1_{F'}} b') \arrow[r, "\pi_1"'] \arrow[d, "\pi_0"] \arrow[dr, phantom, "\lrcorner", very near start] & (V', a', b') \arrow[d, "!"] \\ (V, a, b) \arrow[r, "!"'] & (\mathbb{N}, 1_F, 1_{F'}) \end{tikzcd} \tag{4.2}$$

is a pullback in $\mathbb{L}_\varphi$. The tuple $(V \times V', a \times_{1_F} a', b \times_{1_{F'}} b')$ is a well-defined object of $\mathbb{L}_\varphi$ because $T_\bullet$ and $T'_\bullet$ preserve foundational pullbacks and therefore

$$F \circ T_\bullet(V \times V', -) = F \circ T_\bullet(V, -) \times F \circ T_\bullet(V', -) \text{ and } T'_\bullet(V \times V', -) \circ F = T'_\bullet(V, -) \times F \circ T'_\bullet(V', -) \circ F.$$

Diagram 4.2 is a diagram in $\mathbb{L}_\varphi$ because $a \times_{1_F} a'$ and $b \times_{1_{F'}} b'$ are compatible with the projection maps π_0 and π_1. The diagram fulfills the universal property because any induced map into $(V \times V')$ is component-wise compatible with a, a', b and b' and thus compatible with $a \times a'$ and $b \times b'$.

There is a tangent structure on $\mathbb{L}_\varphi$ whose tangent functor, $T : \mathbb{L}_\varphi \to \mathbb{L}_\varphi$, is defined by tensoring with the dual numbers W:

$$T(V,a,b) = (W \otimes V, T'(a) \circ \alpha, T'(b) \circ \alpha').$$

Here $T'(a) \circ \alpha$ is shorthand for the composite

$$F \circ T_\bullet(W \otimes V, -) \xrightarrow{\cong} F \circ T \circ T_\bullet(V,-) \xrightarrow{\alpha} T' \circ F \circ T_\bullet(V,-) \xrightarrow{T'(a)} T' \circ \hat{T}'(V,-) \circ F \xrightarrow{\cong} \hat{T}'(W \otimes V, -) \circ F,$$

which includes the identification $T_\bullet(W,-)$ with T as in Remark 4.2.12. We define $T'(b) \circ \alpha'$ in the same way. The square

$$\begin{tikzcd}
F \circ T \circ T_\bullet(V,-) \arrow[r, "\alpha"] \arrow[d, "\varphi"'] & T \circ F \circ T_\bullet(V,-) \arrow[r, "T(a)"] \arrow[d, "T(\varphi)"'] & T \circ T_\bullet(V,-) \circ F \arrow[d, "T(T_\bullet(1_V,\varphi))"] \\
F' \circ T \circ T_\bullet(V,-) \arrow[r, "\alpha'"'] & T \circ F' \circ T_\bullet(V,-) \arrow[r, "T(b)"'] & T \circ T_\bullet(V,-) \circ F'
\end{tikzcd}$$

commutes because $\varphi : (F,\alpha) \to (F',\alpha')$ is a linear natural transformation (left square commutes) and (V,a,b) is an object of $\mathbb{L}_\varphi$ (right square commutes).

In the proof of [44, Theorem 9], Garner states that the projection from $\mathbb{K}_F$ to $\mathbb{E}\mathrm{nd}(\mathbb{X})$ preserves the tangent structure. Analogously, here the projection $\mathbb{L}_\varphi \to \mathbb{W}\mathrm{eil}$ preserves the tangent structure which is given by the tensor product with Weil algebras. This means that all foundational pullbacks in $\mathbb{L}_\varphi$ are pullbacks in the $\mathbb{W}$eil-component. The reason for this is that, when combining the pullback diagram of 4.2 with the monoidal structure of $\mathbb{W}$eil, foundational pullbacks in $\mathbb{W}$eil remain pullbacks in $\mathbb{L}_\varphi$ when taking the tensor product with a constant object.

Since $\mathbb{L}_\varphi$ is a tangent category Theorem 4.2.11 gives us a functor $G_\varphi : \mathbb{W}\mathrm{eil} \to \mathbb{E}\mathrm{nd}(\mathbb{L}_\varphi)$ which we sometimes also denote as $G_\varphi : \mathbb{W}\mathrm{eil} \times \mathbb{L}_\varphi \to \mathbb{L}_\varphi$. Since G_φ preserves foundational pullbacks, the first component of $G_\varphi(W^n,(V,a,b))$ will be $W^n \otimes V$. Since $G_\varphi : \mathbb{W}\mathrm{eil} \to \mathbb{E}\mathrm{nd}(\mathbb{L}_\varphi)$ is monoidal and every object of $\mathbb{W}$eil is a tensor product of powers of the dual numbers W, for any Weil-algebras V, V', the first component of $G_\varphi(V,(V',a,b))$ will be $V \otimes V'$. Let

$$G_\varphi(V,(\mathbb{N},1_F,1_{F'})) = (V, \tilde{\alpha}_V, \tilde{\alpha}'_V)$$

Then $\tilde{\alpha}_V : F \circ T_\bullet(V,-) \Rightarrow T'_\bullet(V,-) \circ F$ and $\tilde{\alpha}'_V : F' \circ T_\bullet(V,-) \Rightarrow T'_\bullet(V,-) \circ F'$ are natural in V because G_φ is a functor. In order to show that $\tilde{\alpha}$ and $\tilde{\alpha}'$ fulfill the lineator condition of Definition 4.2.4 we can expand $G_\varphi(V \otimes V',(\mathbb{N},1_F,1_{F'}))$ using monoidality into $G_\varphi(V, G_\varphi(V',(\mathbb{N},1_F,1_{F'}))) = G_\varphi(V,(V',\tilde{\alpha}_{V'},\tilde{\alpha}'_{V'}))$ and the tangent structure T on $\mathbb{L}_\varphi$ together with the fact that every object of $\mathbb{W}$eil is built from W then shows that

$$(V \otimes V', \tilde{\alpha}_{V\otimes V'}, \tilde{\alpha}'_{V\otimes V'}) = G_\varphi(V \otimes V',(\mathbb{N},1_F,1_{F'})) = (V \otimes V', T'_\bullet(V,\tilde{\alpha}_{V'}) * \tilde{\alpha}_V, T'_\bullet(V,\tilde{\alpha}'_{V'}) * \tilde{\alpha}'_V)$$

proving the lineator property of Definition 4.2.4.

Since

$$G_\varphi(W,(\mathbb{N},1_F,1_{F'}))=(W\otimes\mathbb{N},T'(1_F)\circ\alpha,T'(1_{F'})\circ\alpha')=(W,\alpha,\alpha'),$$

the components $\hat{\alpha}_W=\alpha=\tilde{\alpha}_W$ and $\hat{\alpha}'_W=\alpha'=\tilde{\alpha}'_W$. Thus Lemma 4.2.19 shows that for any Weil-algebra V, $\hat{\alpha}_V=\tilde{\alpha}_V$ and $\hat{\alpha}'_V=\tilde{\alpha}'_V$.

Therefore

$$G_\varphi(V,(\mathbb{N},1_F,1_{F'}))=(V,\hat{\alpha}_V,\hat{\alpha}'_V),$$

is an object of $\mathbb{L}_\varphi$ which implies that the diagram

$$\begin{tikzcd}
F\circ T_\bullet(V,-) \arrow[r, "\hat{\alpha}"] \arrow[d, "\varphi"'] & \hat{T}'(V,-)\circ F \arrow[d, "\hat{T}'(V,-)\circ\varphi"] \\
F'\circ T_\bullet(V,-) \arrow[r, "\hat{\alpha}'"'] & \hat{T}'(V,-)\circ F'
\end{tikzcd}$$

commutes. Thus φ is a linear natural transformation of actegories, so we may define $E(\varphi)=\varphi$.

Garner's proof concludes by showing that E is essentially surjective on objects and fully faithful on morphisms and 2-morphisms.

It is essentially surjective on objects because of Theorem 4.2.11.

It is full on morphisms because for any strong $\mathbb{W}$eil-linear functor $(F,\hat{\alpha})$, the W-component $(F,\hat{\alpha}_W)$ is a strong tangent functor such that $E(F,\hat{\alpha}_W)=(F,\hat{\alpha})$.

It is faithful on morphisms because $E(F,\alpha)=(F,\hat{\alpha})$ where $\hat{\alpha}_W=\alpha$. Thus, two different strong tangent functors (F,α) and (F',α') will be sent to two $\mathbb{W}$eil-linear functors $(F,\hat{\alpha})$ and $(F',\hat{\alpha}')$ which are different because the W component of $\hat{\alpha}$ is α and analogous for $\hat{\alpha}'$.

It is full on 2-morphisms because the W component of the linearity condition of a $\mathbb{W}$eil-natural transformation implies the linearity condition of a tangent transformation.

It is faithful on 2-morphisms because the data of a 2-morphism in $\mathsf{TANG}^{\mathrm{lin}}_{\mathrm{strong}}$ and in $\mathbb{W}\mathrm{eil}{-}\mathsf{ACT}^{\mathrm{lin}}_{\mathrm{strong},t}$ coincide.

□

Proof (of Theorem 4.2.18). There are three cases to prove, and in each case we adapt Garner's proof as necessary to account for the differences between the 2-categories which are involved. We take each of them in turn.

Tangent Categories with lax functors and linear transformations In order to prove that $\mathsf{TANG}^{\text{lin}}_{\text{lax}} \cong \mathbb{W}\text{eil}{-}\mathsf{ACT}^{\text{lin}}_{\text{oplax}}$, we need to relax the condition of strong morphisms to oplax morphisms. We need to define a functor

$$E' : \mathsf{TANG}^{\text{lin}}_{\text{lax}} \to \mathbb{W}\text{eil}{-}\mathsf{ACT}^{\text{lin}}_{\text{oplax}}$$

and show that E' is essentially surjective on objects, fully faithful on morphisms and fully faithful on 2-morphisms. Let $(\mathbb{X}, T)$ be an object of $\mathsf{TANG}^{\text{lin}}_{\text{lax}}$. The functor E' is defined on objects by letting $E'(\mathbb{X})$ be the Weil-actegory whose Weil action is given by the adjoint $T_\bullet : \mathbb{W}\text{eil} \times \mathbb{X} \to \mathbb{X}$ of the tangent functor $T : \mathbb{W}\text{eil} \to \mathbb{E}\text{nd}(\mathbb{X})$, exactly as was the case for the functor E in the proof of Theorem 4.2.17. Like in that proof, given a lax tangent functor $(F, \alpha) : (\mathbb{X}, T) \to (\mathbb{X}', T')$ let $\mathbb{K}'_F$, be the category of triples $(A \in \mathbb{E}\text{nd}(\mathbb{X}), B \in \mathbb{E}\text{nd}(\mathbb{X}'), \tau : F \circ A \Rightarrow B \circ F)$ with morphisms $(\varphi, \psi) : (A, B, \tau) \to (A', B', \tau')$ of $\mathbb{K}'_F$ given by pairs of natural transformations $\varphi : A \Rightarrow A'$ and $\psi : B \Rightarrow B'$ satisfying $\tau' * F(\varphi) = \psi_F * \tau$. The only difference between $\mathbb{K}_F$ and $\mathbb{K}'_F$ is that τ is not necessarily an isomorphism. The functor $T_{\mathbb{K}'_F} : \mathbb{K}'_F \to \mathbb{K}'_F$ given by

$$T_{\mathbb{K}'_F}(A, B, \tau) := (T \circ A, T' \circ B, T'(\tau) \circ \alpha_A).$$

is a tangent bundle functor. This is because the only difference from the proof of Theorem 4.2.17 is that in that proof, τ and α needed to be isomorphisms. Since this did not contribute to the tangent structure, the same proof works in this case. This tangent structure gives rise to a monoidal functor $H : \mathbb{W}\text{eil} \to \mathbb{E}\text{nd}(\mathbb{K}_F)$, and evaluation of this functor at the object $(1_{\mathbb{X}}, 1_{\mathbb{X}'}, 1_F) \in \mathbb{K}_F$ gives us, for each Weil algebra V,

$$H(V)(1_{\mathbb{X}}, 1_{\mathbb{X}'}, 1_F) = (T_\bullet(V, -), \hat{T}'(V, -), \hat{\alpha}_V)$$

where $\hat{\alpha}_V : F \circ T_\bullet(V, -) \Rightarrow T'(V, -) \circ (F \times 1_{\mathbb{W}\text{eil}})$ is a natural transformation. The functor E' sends (F, α) to $(F, \hat{\alpha})$ so defined. Since $\hat{\alpha}$ is the third component of an object of $\mathbb{K}'_F$, it is not required to be an isomorphism. This means that lax tangent functors are sent to lax Weil-linear functors. The functor E' is defined on 2-morphisms exactly like in the original proof (of Theorem 4.2.17).

The difference to the proof of Theorem 4.2.17 is that we don't just allow strong Weil-linear functors but also lax Weil-linear functors as morphisms in the target. Therefore every object in $\mathbb{W}\text{eil}{-}\mathsf{ACT}^{\text{lin}}_{\text{oplax}}$ is still isomorphic to the image of an object of $\mathsf{TANG}^{\text{lin}}_{\text{lax}}$. (It even is isomorphic by a strong functor which in particular is a lax functor). Thereby E' is automatically essentially surjective on objects. It also is fully faithful on 2-morphisms for the very same reasons as in the proof of 4.2.17.

Next we need to show that E' is fully faithful on morphisms. So let $\mathbb{X}$ and $\mathbb{X}'$ be tangent categories and $(F, \hat{\alpha}) : E(\mathbb{X}) \to E(\mathbb{X}')$ a map of the corresponding Weil-actegories. The maps $\alpha(W, -)$ constitute a natural transformation $F(T_\bullet(W,-)) \Rightarrow \hat{T}'F(-)$ which, since $T_\bullet(W,-) \cong T$ in both domain and codomain, determines and is determined by one $\alpha : F \circ T \Rightarrow T' \circ F$.

Tangent categories with strong functors In order to prove that $\mathsf{TANG}_{\text{strong}} \cong \mathbb{W}\text{eil}{-}\mathsf{ACT}_{\text{strong},t}$, we need to relax the condition of linear natural transformations to any natural transformations. We need to define a functor

$$E'' : \mathsf{TANG}_{\text{strong}} \to \mathbb{W}\text{eil}{-}\mathsf{ACT}_{\text{strong},t}$$

and prove that it is essentially surjective on objects, fully faithful on morphisms and fully faithful on 2-morphisms. On objects it is defined through the adjunction and on morphisms through the category $\mathbb{K}_F$, like E from the proof of Theorem 4.2.17.

The only substantial difference to the proof of Theorem 4.2.17 is on 2-morphisms. Since the transformations are not linear, we choose $E''(\varphi) = \varphi$, but now we do not need to verify linearity. We defined E'' to be the identity on 2-morphisms, thus E'' is automatically fully faithful on 2-morphisms. It is essentially surjective on objects and fully faithful on morphisms because on objects and morphisms it is the same as E from the original proof (of Theorem 4.2.17).

Tangent categories with lax functors In order to prove that $\mathsf{TANG}_{\text{lax}} \cong \mathbb{W}\text{eil}{-}\mathsf{ACT}_{\text{oplax}}$ we need to relax both conditions, the strong functors to lax functors and the linear natural transformations to general natural transformations. We need to define a functor

$$E''' : \mathsf{TANG}_{\text{lax}} \to \mathbb{W}\text{eil}{-}\mathsf{ACT}_{\text{oplax}}$$

and prove that it is essentially surjective on objects, fully faithful on morphisms and fully faithful on 2-morphisms. We define E''' to be the same as E' on objects and morphisms and the same as E'' on 2-morphisms. From the analogous results about E' and E' it now follows that E''' is essentially surjective, fully faithful on morphisms and fully faithful on 2-morphisms.

$\square$

Remark 4.2.20. *Inspired by Theorem 4.2.18 we will from now on describe tangent categories through their* $\mathbb{W}$eil-*actegory structure* $(\mathbb{X}, T_\bullet : \mathbb{W}\text{eil} \times \mathbb{X} \to \mathbb{X})$, *lax/strong tangent functors through their corresponding colax/strong* $\mathbb{W}$eil-*linear functors and (linear) tangent transformations through their (linear)* $\mathbb{W}$eil-*natural transformations. When describing a tangent category through its* $\mathbb{W}$eil-*actegory structure, we call it a* ***tangent actegory****.*

Recall from Remark 4.2.3, that given a $\mathbb{W}$eil *algebra A we will denote the functor $T_\bullet(A,-) : \mathbb{X} \to \mathbb{X}$ by T_A. In addition, given a tangent actegory $(\mathbb{X}, T_\bullet)$ we say $(E, M, q, \zeta, \sigma, \lambda)$ is a differential object in the* $\mathbb{W}$eil-*actegory $(\mathbb{X}, T_\bullet)$ if it is a differential object in the corresponding tangent category $(\mathbb{X}, T_W)$.*

4.2.3 The tangent actegory $\mathbb{N}^\bullet$

In this section we will study the category $\mathbb{N}^\bullet$ from Definition 2.2.9 with the tangent structure $(\mathbb{N}^\bullet, D)$ defined in Example 2.2.11. The tangent category $\mathbb{N}^\bullet$ is the universal tangent category with a differential object, in the sense that it will classify differential bundles and differential objects in Theorem 4.5.3.

Due to Theorem 4.2.17 this tangent structure has a corresponding $\mathbb{W}$eil-actegory structure via the functor $D_\bullet$ satisfying

$$D_\bullet(A, \mathbb{N}^k) = \underline{A} \otimes \mathbb{N}^k,$$

where $\underline{A}$ is the underlying additive monoid of A. Under the correspondence of Theorem 4.2.17 this encodes the tangent structure of Example 2.2.11, because

$$D_\bullet(W, \mathbb{N}^k) = W \otimes \mathbb{N}^k \cong \mathbb{N}^k \times \mathbb{N}^k.$$

One can check that the images of the maps in $\mathbb{W}$eil also provide the $p, 0, +, \ell$ and c from Example 2.2.11. as well. Equivalently this can be seen as an example of Leung's Theorem 4.2.11. The functor $\mathbb{W}\text{eil} \to \text{Fun}(\mathbb{N}^\bullet, \mathbb{N}^\bullet)$ is the adjoint of the tensor product sending the Weil algebra A to the endofunctor $\underline{A} \otimes -$.

However, the equivalences in Theorems 4.2.17 and 4.2.18 do not only provide a correspondence on objects. They also give a correspondence on morphisms and 2-morphisms. We will showcase this correspondence for the functor $F : \mathbb{N}^\bullet \to \mathbb{S}\text{mMan}$ from Example 2.2.15.

Example 4.2.21. *The functor $F : \mathbb{N}^\bullet \to \mathbb{S}\text{mooth}$ from Example 2.2.15 that sends $\mathbb{N}^k$ to $\mathbb{R}^k$ can be seen as a strong* $\mathbb{W}$eil-*linear functor of actegories in the following way: The underlying functor is the functor from Example 2.2.15. For the lineator we choose the identity:*

$$\hat{\alpha}_{W^{n_1} \otimes ... \otimes W^{n_k}, \mathbb{N}^m} = 1 : F(D_\bullet(W^{n_1} \otimes ... \otimes W^{n_k}, \mathbb{N}^m)) \to T_\bullet(W^{n_1} \otimes ... \otimes W^{n_k}, F(\mathbb{N}^m)).$$

We can do this because the domain can be simplified into

$$F(D_\bullet(W^{n_1} \otimes ... \otimes W^{n_k}, \mathbb{N}^m)) = F(\mathbb{N}^{m\cdot(n_1+1)\cdot...\cdot(n_k+1)}) = \mathbb{R}^{m\cdot(n_1+1)\cdot...\cdot(n_k+1)}.$$

The target can be simplified into

$$T_\bullet(W^{n_1} \otimes ... \otimes W^{n_k}, F(\mathbb{N}^m)) = T_\bullet(W^{n_1} \otimes ... \otimes W^{n_k}, \mathbb{R}^m) = \mathbb{R}^{m\cdot(n_1+1)\cdot...\cdot(n_k+1)}.$$

Since the domain and the target coincide, we can choose the lineator $\hat{\alpha}$ to be the identity.

This example is important since the main result of this chapter, the classification in Theorem 4.5.3, will state that such functors exactly correspond to differential bundles.

Recall from Definition 2.2.9 that morphisms in $\mathbb{N}^\bullet$ from $\mathbb{N}^m$ to $\mathbb{N}^n$ are given by matrices with components $f = (f_{ij})_{1\leq i\leq n, 1\leq j\leq m}$ where $f_{ij} \in \mathbb{N}$ for all i, j. Lemma 4.2.22 will now re-write the application of a matrix as an iterated sum.

For this we use the additive bundle structure $\zeta : \mathbb{N}^0 \to \mathbb{N}^1$ and $\sigma : \mathbb{N}^2 \to \mathbb{N}^1$ on $\mathbb{N}^1$ from Example 2.3.4. Let $\sigma_k : \mathbb{N}^k \to \mathbb{N}^1$ be defined recursively by $\sigma_0 = \zeta : \mathbb{N}^0 \to \mathbb{N}^1$ and $\sigma_k = \sigma \circ (1_\mathbb{N} \times \sigma_{k-1}) : \mathbb{N}^k \to \mathbb{N}^1$. Recall that $\Delta_k : \mathbb{N} \to \mathbb{N}^k$ is the diagonal.

Lemma 4.2.22. *Let*

$$f = (f_{ij})_{1\leq i\leq n, 1\leq j\leq m} : \mathbb{N}^m \to \mathbb{N}^n$$

be an arbitrary morphism in $\mathbb{N}^\bullet$. Then f can be decomposed into components

$$f = \langle f_1, ..., f_n \rangle$$

with components $f_i : \mathbb{N}^m \to \mathbb{N}$ that are given by composition

$$f_i = \sigma_m \circ \langle \sigma_{f_{i1}} \circ \Delta_{f_{i1}} \circ \pi_1, ..., \sigma_{f_{im}} \circ \Delta_{f_{im}} \circ \pi_m \rangle$$

Proof. This is just a way to formulate the application of a matrix $f = (f_{ij})_{1\leq i\leq n, 1\leq j\leq m}$ to a vector v

$$f(v)_i = \sum_{j=1}^{m} f_{ij} \cdot v_j$$

using the addition from Example 2.3.4. We first replace the multiplication with scalars by a sum of constant terms

$$\sum_{j=1}^{m} f_{ij} \cdot v_j = \sum_{j=1}^{n} \sum_{r=1}^{f_{ij}} v_j$$

and then replace the sums with σ to obtain

$$\sum_{j=1}^{n}\sum_{r=1}^{f_{ij}} v_j = \sigma_m \circ \langle \sigma_{f_{i1}}(v_1, ..., v_1), ..., \sigma_{f_{im}}(v_m, ..., v_m)\rangle$$

$$= \sigma_m \circ \langle \sigma_{f_{i1}} \circ \Delta_{f_{i1}}(v_1), ..., \sigma_{f_{im}} \circ \Delta_{f_{im}}(v_n)\rangle$$

$$= \sigma_m \circ \langle \sigma_{f_{i1}} \circ \Delta_{f_{i1}} \circ \pi_1, ..., \sigma_{f_{im}} \circ \Delta_{f_{im}} \circ \pi_n\rangle(v).$$

□

4.3 Differential functors

This section introduces differential functors from a tangent actegory, $(\mathbb{X}, T_\bullet)$, to another, $(\mathbb{X}', T'_\bullet)$. Like a tangent functor, a differential functor will consist of a functor $F : \mathbb{X} \to \mathbb{X}'$ together with a natural transformation $\hat{\alpha} : F \circ T \Rightarrow T' \circ (1 \times F)$, but F and $\hat{\alpha}$ will be required to preserve additional structure (as in Definition 4.3.3). The key difference between differential functors and the tangent functors of Definition 2.2.14 or the $\mathbb{W}$eil-linear functors of Definition 4.2.4 is that differential functors preserve differential bundles (Definition 2.3.1) and bundle morphisms (Definition 2.3.5).

We construct a category of differential functors and differential natural transformations. In light of the results of Section 4.2.1, we construct differential functors from $\mathbb{W}$eil-linear functors and we use Weil-actegories in place of the tangent categories of Definition 2.2.4. In light of the Theorems 4.2.17 and 4.2.18 , we use $\mathbb{W}$eil-actegories as our model of tangent categories. Differential functors will be $\mathbb{W}$eil-linear functors with additional structure. We begin this section by providing a definition of what it means to be a differential bundle in a Weil-actegory by defining it to be what corresponds to Definition 2.3.1 under the correspondence of Theorem 4.2.18. In Section 3, we considered strong and lax $\mathbb{W}$eil-linear morphisms as well as tangent transformations and linear tangent transformations. This begs the question of what choices we should make in defining both objects and morphisms of “the” category of differential functors. Instead of making a single choice, we will define all four possible categories as sub-categories of the category of morphisms in the 2-category $\mathsf{TANG}_{\text{lax}}$ and denote them $\mathbb{D}\text{iffFun}, \mathbb{D}\text{iffFun}^{\text{strong}}, \mathbb{D}\text{iffFun}_{\text{lin}}$ and $\mathbb{D}\text{iffFun}_{\text{lin}}^{\text{strong}}$.

Finally, we will establish the fact that both lax and strong differential functors send differential objects to differential bundles in Proposition 4.3.5. Given a lax differential functor $(F, \alpha) : (\mathbb{X}, T) \to (\mathbb{X}', T')$, we define an evaluation functor $\texttt{Eval} : \mathbb{D}\text{Obj}(\mathbb{X}) \times \mathbb{D}\text{iffFun}(\mathbb{X}, \mathbb{X}') \to \mathbb{D}\text{Bun}(\mathbb{X}')$ which sends a functor F and a differential object A to the bundle obtained by evaluating F at the object A over $F(*)$, where $*$ is the terminal object of $\mathbb{X}$. Analogous statements hold for $\mathbb{D}\text{iffFun}^{\text{strong}}, \mathbb{D}\text{iffFun}_{\text{lin}}$ and $\mathbb{D}\text{iffFun}_{\text{lin}}^{\text{strong}}$.

The fact that this assignment is functorial is established in Propopositions 4.3.5, 4.3.6, 4.3.7 and Theorem 4.3.10. Each of the four propositions is for one of the four different choices of categories of differential functors. Its correspondence to the classical evaluation functor $\mathtt{eval} : \mathbb{X} \times \mathrm{Fun}(\mathbb{X}, \mathbb{X}') \to \mathbb{X}'$, which sends A, F to $F(A)$, makes the diagram

$$\begin{tikzcd} \mathbb{D}\mathrm{Obj}(\mathbb{X}) \times \mathbb{D}\mathrm{iffFun}(\mathbb{X}, \mathbb{X}') \arrow[r, "\mathtt{Eval}"] \arrow[d, hook] & \mathbb{D}\mathrm{Bun}(\mathbb{X}') \arrow[d, "\pi_E"] \\ \mathbb{X} \times \mathrm{Fun}(\mathbb{X}, \mathbb{X}') \arrow[r, "\mathtt{eval}"] & \mathbb{X}' \end{tikzcd}$$

commute. Here the functor π_E sends a differential bundle $(E, M, q, \zeta, \sigma, \lambda)$ to the total space of the bundle, E.

There are variations of the functor $\mathtt{Eval}$ whose target is either $\mathbb{D}\mathrm{Bun}(\mathbb{X}')$ or $\mathbb{D}\mathrm{Obj}(\mathbb{X}')$, depending on whether (F, α) is lax or strong. All the variations of $\mathtt{Eval}$ are lined up in Corollary 4.3.11.

Theorem 4.3.10 and Corollary 4.3.11 give us a way to produce new differential bundles in $\mathbb{X}'$ out of differential objects in $\mathbb{X}$. We conclude this section with the Example 4.3.12 using the tangent category $(\mathbb{N}^\bullet, T)$ of Example 2.2.11. In subsequent sections, we will show that in fact every differential bundle in $\mathbb{X}$ is obtained from $\mathbb{N}^\bullet$.

We begin by identifying objects of a Weil-actegory which are differential objects in the corresponding tangent category. The following definition makes this precise. Recall Definitions 2.3.1 and 2.3.5 for the definition of a differential bunle in a tangent category.

Recall from Remark 4.2.3 that we denote the functor $T_\bullet(A, -) : \mathbb{X} \to \mathbb{X}$ by T_A.

Definition 4.3.1. *Let* $(\mathbb{X}, T_\bullet : \mathbb{W}\mathrm{eil} \times \mathbb{X} \to \mathbb{X})$ *be a tangent actegory, and let* A *be a Weil algebra.*

1. *A tuple* $(E, M, q : E \to M, \zeta : M \to E, \sigma : E_2 \to E, \lambda : E \to T_W(E))$ *is a* $\mathbb{W}\mathrm{eil}$***-differential bundle*** *in* $(\mathbb{X}, T_\bullet)$ *if it is a differential bundle in the sense of Definition 2.3.1 in the corresponding tangent category* $(\mathbb{X}, T_W)$*. We will abbreviate* $(E, M, q, \zeta, \sigma, \lambda)$ *to* (E, M) *when the name of the structure maps are not needed.*

2. *Given* $\mathbb{W}\mathrm{eil}$*-differential bundles* (E, M) *and* (E', M')*, a pair of morphisms* $f : E \to E', g : M \to M'$ *is a* $\mathbb{W}\mathrm{eil}$***-differential bundle morphism*** *if* (f, g) *is a differential bundle morphism in the corresponding tangent category* $(\mathbb{X}, T_W)$*.*

3. *A* $\mathbb{W}\mathrm{eil}$*-differential bundle morphism is linear/additive if it is linear/additive in the corresponding tangent category* $(\mathbb{X}, T_W)$*.*

Let $\mathbb{W}\mathrm{eil}{-}\mathbb{D}\mathrm{Bun}(\mathbb{X})$*,* $\mathbb{W}\mathrm{eil}{-}\mathbb{D}\mathrm{Bun}_{\mathrm{add}}(\mathbb{X})$ *and* $\mathbb{W}\mathrm{eil}{-}\mathbb{D}\mathrm{Bun}_{\mathrm{lin}}(\mathbb{X})$ *be the categories of* $\mathbb{W}\mathrm{eil}$*-differential bundles* *with any, linear or additive morphisms. Denote* $\mathbb{W}\mathrm{eil}{-}\mathbb{D}\mathrm{Obj}(\mathbb{X})$*,* $\mathbb{W}\mathrm{eil}{-}\mathbb{D}\mathrm{Obj}_{\mathrm{add}}(\mathbb{X})$ *and* $\mathbb{W}\mathrm{eil}{-}\mathbb{D}\mathrm{Obj}_{\mathrm{lin}}(\mathbb{X})$ *the full subcategories of* $\mathbb{W}\mathrm{eil}$*-differential objects with any, linear or additive morphisms.*

Lemma 4.3.2. *Let* $(\mathbb{X}, T_\bullet)$ *be a tangent actegory with corresponding tangent category* $(\mathbb{X}, T_W)$*. There are isomorphisms of categories*

$$\mathbb{D}\mathrm{Bun}(\mathbb{X}, T_W) \cong \mathbb{W}\mathrm{eil}{-}\mathbb{D}\mathrm{Bun}(\mathbb{X}, T_\bullet) \quad , \quad \mathbb{D}\mathrm{Obj}(\mathbb{X}, T_W) \cong \mathbb{W}\mathrm{eil}{-}\mathbb{D}\mathrm{Obj}(\mathbb{X}, T_\bullet) \ ,$$

$$\mathbb{D}\mathrm{Bun}_{\mathrm{add}}(\mathbb{X}, T_W) \cong \mathbb{W}\mathrm{eil}{-}\mathbb{D}\mathrm{Bun}_{\mathrm{add}}(\mathbb{X}, T) \ , \ \mathbb{D}\mathrm{Obj}_{\mathrm{add}}(\mathbb{X}, T_W) \cong \mathbb{W}\mathrm{eil}{-}\mathbb{D}\mathrm{Obj}_{\mathrm{add}}(\mathbb{X}, T_\bullet) \ ,$$

$$\mathbb{D}\mathrm{Bun}_{\mathrm{lin}}(\mathbb{X}, T_W) \cong \mathbb{W}\mathrm{eil}{-}\mathbb{D}\mathrm{Bun}_{\mathrm{lin}}(\mathbb{X}, T_\bullet) \ , \ \mathbb{D}\mathrm{Obj}_{\mathrm{lin}}(\mathbb{X}, T_W) \cong \mathbb{W}\mathrm{eil}{-}\mathbb{D}\mathrm{Obj}_{\mathrm{lin}}(\mathbb{X}, T_\bullet) \ ,$$

.

Proof. By Definition 4.3.1 there are functors in both directions that do not change the data of an object or a morphism. The fact that these functors are inverse to one another is immediate. □

Lemma 4.3.2 allows us to slightly abuse terminology by using the word "differential bundle" for objects on both sides of the equivalence (using the conventions from Remarks 4.2.12 and 4.2.20). In particular we will also use $\mathbb{D}\mathrm{Bun}(\mathbb{X}, T_\bullet)$ to denote $\mathbb{W}\mathrm{eil}{-}\mathbb{D}\mathrm{Bun}_{\mathrm{add}}(\mathbb{X}, T)$.

Definition 4.3.3 (Strong and lax differential functors)**.** *Suppose* $(\mathbb{X}, T_\bullet)$ *and* $(\mathbb{X}', T'_\bullet)$ *are* $\mathbb{W}\mathrm{eil}$*-actegories and suppose that that* $\mathbb{X}$ *has a terminal object* $*$*.*

(a) A ***lax differential functor*** *from* $(\mathbb{X}, T_\bullet)$ *to* $(\mathbb{X}', T'_\bullet)$ *is an oplax* $\mathbb{W}\mathrm{eil}$*-linear functor,* $(F : \mathbb{X} \to \mathbb{X}', \hat{\alpha} : F \circ T_\bullet \Rightarrow T'_\bullet \circ (1 \times F))$*, such that for all objects* A *of* $\mathbb{W}\mathrm{eil}$*,*

  (a) $T'_A \circ F$ *preserves pullbacks over the terminal object* $*$*, and*

  (b) $\hat{\alpha}_{A,-}$ *is Cartesian, i.e. for each* $f \in \mathrm{Hom}_{\mathbb{X}}(X, Y)$*, the naturality diagram*

$$\begin{tikzcd} F \circ T_A(X) \arrow[r, "\hat{\alpha}_{A,X}"] \arrow[d, "F \circ T_A(f)"'] \arrow[dr, phantom, "\lrcorner", very near start] & T'_A \circ F(X) \arrow[d, "T'_A \circ F(f)"] \\ F \circ T_A(Y) \arrow[r, "\hat{\alpha}_{A,Y}"] & T'_A \circ F(Y) \end{tikzcd}$$

  for the natural transformation $\hat{\alpha}_{A,-} : F \circ T_A \Rightarrow T'_A \circ F$ *is a pullback.*

(b) A ***strong differential functor*** *is a lax differential functor such that* $F(*)$ *is a terminal object in* $\mathbb{X}'$*.*

If the tangent bundle of the terminal object is terminal, $T(*) \cong *$ in both $\mathbb{X}$ and $\mathbb{X}'$, a consequence of this definition is that a strong differential functor is a strong $\mathbb{W}\mathrm{eil}$-linear functor, i.e. the natural transformation

α is actually a natural isomorphism. Let $1_{\mathbb{X}}$ (resp. $1_{\mathbb{X}'}$) denote the terminal object in $\mathbb{X}$ (resp. $\mathbb{X}'$). The reason a strong differential functor is a strong $\mathbb{W}$eil-linear functor is that the naturality diagram of α with respect to $! : X \to 1_{\mathbb{X}}$

$$\begin{tikzcd} F(T_A(X)) \arrow[r, "\hat{\alpha}_{(A,X)}"] \arrow[d, "!"'] & T'_A(F(X)) \arrow[d, "!"] \\ 1_{\mathbb{X}'} = F(T_A(1_{\mathbb{X}})) \arrow[r, "1_*"'] & T'_A(F(1_{\mathbb{X}})) = 1_{\mathbb{X}'} \end{tikzcd}$$

is a pullback and thus $\hat{\alpha}_{(A,X)}$ is an isomorphism.

Example 4.3.4. *Let* $(\mathbb{S}\mathrm{mMan}, T)$ *be the tangent category of smooth manifolds with smooth maps. It was defined in Example 2.2.7 using the axiomatic formulation of tangent categories in Definition 2.2.4. It can be turned into a* $\mathbb{W}$eil-*actegory using Theorem 4.2.18).*

(a) Let $(-)^2 : \mathbb{S}\mathrm{mMan} \to \mathbb{S}\mathrm{mMan}$ *denote the squaring functor. For smooth manifolds* M *and* N *and a smooth map* $f : M \to N$*, this functor is defined as* $(-)^2(M) = M^2 = M \times M$ *and* $(-)^2(f) = f \times f$*. This functor, together with the canonical isomorphism* $\alpha : T(M \times M) \to T(M) \times T(M)$*, is a tangent functor functor as in Definition 2.2.14. Under Theorem 4.2.18 it corresponds to a* $\mathbb{W}$eil-*linear functor* $(F, \hat{\alpha})$ *which is a strong differential functor.*

(b) Let $S^1 \times -$ *be the product with the circle functor. For smooth manifolds* M *and* N *and a smooth map* $F : M \to N$*, this functor is defined as* $(S^1 \times -)(M) = S^1 \times M$ *and* $(S^1 \times -)(f) = 1_{S^1} \times f$*. This functor, together with the map* $\beta := 0_{S^1} \times 1_{T(M)} : S^1 \times T(M) \to T(S^1) \times T(M) \cong T(S^1 \times M)$*, is a tangent functor and thus corresponds to a* $\mathbb{W}$eil-*linear functor* $(F, \hat{\beta})$*. It is a lax differential functor but not a strong differential functor because it sends the terminal object (the one point space* $*$*) to* $S^1 \times * \cong S^1$ *which is not terminal.*

An important difference between these examples is that $S^1 \times * \ncong *$ *whereas* $* \times * \cong *$*. This showcases the difference between a lax differential functor such as* $S^1 \times -$ *and a strong differential functor such as* $(-)^2$*.*

The main goal of this section is to construct an evaluation functor

$$\mathtt{Eval} : \mathbb{D}\mathrm{Obj}_{\mathrm{add}}(\mathbb{X}) \times \mathbb{D}\mathrm{iffFun}(\mathbb{X}, \mathbb{X}') \to \mathbb{D}\mathrm{Bun}_{\mathrm{add}}(\mathbb{X}')$$

in Theorem 4.3.10, that sends a differential object and a differential functor to a differential bundle. The explicit definition of the category $\mathbb{D}$iffFun is in Definition 4.3.8.

The next three propositions show (in order) that this assignment is well-defined on objects, that it is well-defined on morphisms of $\mathbb{D}$Obj and that it is well-defined on morphisms of $\mathbb{D}$Bun.

First we describe the assignment on objects. Recall notation convention for differential objects from Remark 2.3.2.

Proposition 4.3.5. *Let* $(\mathbb{X}, T_\bullet)$ *and* $(\mathbb{X}', T'_\bullet)$ *be tangent actegories and let* $(F, \hat{\alpha}) : (\mathbb{X}, T_\bullet) \to (\mathbb{X}', T'_\bullet)$ *be an oplax* Weil-*linear functor. Let* $(X, \sigma, \zeta, \lambda)$ *be a differential object in* $\mathbb{X}$.

(a) If $(F, \hat{\alpha})$ *is a lax differential functor, then then* $(F(X), F(1), F(!), F(\sigma), F(\zeta), \hat{\alpha}_X \circ F(\lambda))$ *is a differential bundle.*

(b) If $(F, \hat{\alpha})$ *is a strong differential functor, then then* $(F(X), F(1), F(!), F(\sigma), F(\zeta), \hat{\alpha}_X \circ F(\lambda))$ *is a differential object.*

Proof. (a) Let $(\mathbb{X}, T, p, 0, +, \ell)$ and $(\mathbb{X}', T', p', 0', +', \ell')$ be the tangent structures corresponding to the tangent actegories $(\mathbb{X}, T_\bullet)$ and $(\mathbb{X}', T'_\bullet)$. Denote $\hat{\alpha}_W$ by $\alpha : F \circ T \Rightarrow T' \circ F$.

To prove the statement, we simply need to check that $(F(X), F(1), F(!), F(\sigma), F(0), \hat{\alpha}_X \circ F(\lambda))$ satisfies Definition 2.3.1 in the case where the tangent functor is given by T'_W. Finite pullback powers of $F(!) : F(X) \to F(*)$ exist and the k-th pullback power equals $F(X^k)$ and $F = T'_{\mathbb{N}} \circ F$ preserves pullbacks over $*$ by Definition 4.3.3 . In order to show that these pullbacks are preserved by the n-th power of the tangent functor $T' = T'_W$ we need to show that

$$\begin{tikzcd} T'^n(F(X^2)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & T'^n(F(X)) \arrow[d] \\ T'^n(F(X)) \arrow[r] & T'^n(F(*)) \end{tikzcd}$$

is a pullback. This is the case as $T'^n \circ F = T'_{W^{\otimes n}} \circ F$ preserves pullbacks over the terminal object.

The remainder of Properties 1-4 in Definition 2.3.1 follows from functoriality. In order to show Property 5, we need to show that $(\hat{\alpha}_X \circ F(\lambda), 0')$ is an additive bundle morphism of type

$$(F(X), F(*), F(!), F(\sigma), F(\zeta)) \to (T'(F(X)), T'(F(*)), T'(F(!)), T'(F(\sigma)), T'(F(\zeta)))$$

and that $(\hat{\alpha}_X \circ F(\lambda), F(\zeta))$ is an additive bundle morphism of type

$$(F(X), F(*), F(!), F(\sigma), F(\zeta)) \to (T'(F(X)), F(X), p'_{F(X)}, +'_{F(X)}, 0'_{F(X)}).$$

This means we need to show that the three diagrams of Definition 2.2.3 commute for each of these morphisms. Each of them will follow from functoriality of F, naturality of α, the commuting diagrams in Definition 2.2.14 and Definition 2.3.1.

To show that $(\alpha \circ F(\lambda), 0')$ preserves the projection, we must show that $T'F(!) \circ (\alpha_X \circ F(\lambda) = 0_{\mathbb{X}'} \circ F(!)$. We proceed as follows:

$$\begin{aligned} T'(F(!)) \circ \alpha \circ F(\lambda) &= \alpha \circ F(T(!)) \circ F(\lambda) \\ &= \alpha \circ F(0) \circ F(!) \\ &= 0' \circ F(!) \end{aligned}$$

To show that $(\alpha \circ F(\lambda), 0')$ preserves the addition, we must show that $T'(F(\sigma)) \circ (\alpha \circ F(\lambda))_2 = \alpha \circ F(\lambda) \circ T'(F(\sigma))$. Remembering that for the differential object X, the projection $q : X \to 1$ is the unique morphism $! : X \to 1$ to the terminal object 1 , we proceed as follows:

$$\begin{aligned} T'(F(\sigma)) \circ \langle \alpha \circ F(\lambda) \circ \pi_0, \alpha \circ F(\lambda) \circ \pi_1 \rangle &= \alpha \circ F(T(\sigma)) \circ \langle F(\lambda) \circ \pi_0, F(\lambda) \circ \pi_1 \rangle \\ &= \alpha \circ F(T(\sigma) \circ \langle \lambda \circ \pi_0, \lambda \circ \pi_1 \rangle) \\ &= \alpha \circ F(\lambda) \circ F(\sigma) \end{aligned}$$

To show that $(\alpha \circ F(\lambda), 0')$ preserves the zero section, we must show that $T'(F(\zeta)) \circ 0' = \alpha \circ F(\lambda) \circ F(\zeta)$. We proceed as follows:

$$\begin{aligned} T'(F(\zeta_X)) \circ 0' &= T'(F(\zeta_X)) \circ \alpha \circ F(0) \\ &= \alpha \circ F(T(\zeta_X)) \circ F(0) \\ &= \alpha \circ F(\lambda_X) \circ F(\zeta_X) \end{aligned}$$

To show that the additive bundle morphism $(\alpha \circ F(\lambda), F(\zeta))$ preserves the projection, we must show that $p \circ \alpha \circ F(\lambda) = F(\zeta) \circ F(!)$. We proceed as follows:

$$\begin{aligned} p' \circ \alpha \circ F(\lambda) &= F(p) \circ F(\lambda) \\ &= F(\zeta) \circ F(!) \end{aligned}$$

To show that the additive bundle morphism $(\alpha \circ F(\lambda), F(\zeta))$ preserves the addition, we must show that $+' \circ (\alpha \circ F(\lambda))_2 = \alpha \circ F(\lambda) \circ F(\sigma)$. We proceed as follows:

$$
\begin{aligned}
+ \circ \langle \alpha \circ F(\lambda) \circ \pi_0, \alpha \circ F(\lambda) \circ \pi_1 \rangle &= \alpha \circ F(+) \circ \langle F(\lambda) \circ \pi_0, F(\lambda) \circ \pi_1 \rangle \\
&= \alpha \circ F(+ \circ \langle \lambda \circ \pi_0, \lambda \circ \pi_1 \rangle) \\
&= \alpha \circ F(\lambda) \circ F(\sigma)
\end{aligned}
$$

To show that the additive bundle morphism $(\alpha \circ F(\lambda), F(\zeta))$ preserves the zero section, we must show that $0' \circ F(\zeta) = \alpha \circ F(\lambda) \circ F(\sigma)$. We proceed as follows: The following equation shows that the additive bundle morphism $(\lambda, 0)$ preserves the zero section.

$$
\begin{aligned}
\alpha \circ F(\lambda) \circ F(\zeta) &= \alpha \circ F(0) \circ F(\zeta) \\
&= 0'_{F(X)} \circ F(\zeta)
\end{aligned}
$$

In order to show that the vertical lift $\alpha \circ F(\lambda)$ is compatible with ℓ', we must show that $T'(\alpha \circ F(\lambda)) \circ \alpha \circ F(\lambda) = \ell' \circ \alpha \circ F(\lambda)$. We proceed as follows:

$$
\begin{aligned}
T'(\alpha) \circ T'(F(\lambda)) \circ \alpha \circ F(\lambda) &= T'(\alpha) \circ \alpha \circ F(T(\lambda)) \circ F(\lambda) \\
&= T'(\alpha) \circ \alpha \circ F(\ell) \circ F(\lambda) \\
&= \ell' \circ \alpha \circ F(\lambda)
\end{aligned}
$$

Since $\alpha \circ F(0_X) = 0_{F(X)}$ (which is part of Definition 2.2.14 of a lax tangent functor) the map $\mu' : F(X^2) \to T'(F(X))$ used in the universality of the vertical lift is

$$\mu' = T'(F(\sigma)) \circ \langle \alpha \circ F(0_X) \circ \pi_0, \alpha \circ F(\lambda) \circ \pi_1 \rangle = \alpha \circ F(T(\sigma) \circ \langle 0_X, \lambda \rangle) = \alpha \circ F(\mu).$$

Thus, Diagram 2.2 of Definition 2.3.1, the universality diagram is

$$
\begin{tikzcd}
F(X^2) \arrow[r, "F(\mu_X)"] \arrow[d, "F(!)"'] \arrow[dr, phantom, "\lrcorner", very near start] & F(T(X)) \arrow[r, "\alpha"] \arrow[d, "F(T(!))"'] \arrow[dr, phantom, "\lrcorner", very near start] & T'(F(X)) \arrow[d, "T(F(!))"] \\
F(*) \arrow[r, "F(0)"'] & F(T(*)) \arrow[r, "\alpha"'] & T'(F(*)).
\end{tikzcd}
$$

The left square is a pullback since F preserves pullbacks over the terminal object. The right square is a pullback because it is the naturality square of α.

(b) By part (a) of this proof, $F(X) \to F(*)$ is a differential bundle. Since F is a strong morphism,

$F(*) = *'$ and $F(X)$ is a differential bundle over the terminal object which by Definition 2.3.1, is a differential object.

□

Now we show that the evaluation of a differential functor sends additive morphisms of differential objects to additive morphisms of differential bundles. This defines the evaluation functor of Theorem 4.3.10 on morphisms in the first component.

Proposition 4.3.6. *Let $(F, \alpha) : (\mathbb{X}, T) \to (\mathbb{X}', T')$ be a differential functor, $(A, \zeta, \sigma, \lambda)$ and $(B, \zeta', \sigma', \lambda')$ be differential objects in $\mathbb{X}$ and $f : A \to B$ be an additive morphism of differential objects. Then*

$$(F(f), 1_{F(*)}) : (F(A), F(*), F(!), F(\zeta), F(\sigma), \alpha \circ F(\lambda)) \to (F(B), F(*), F(!), F(\zeta'), F(\sigma'), \alpha \circ F(\lambda'))$$

is an additive morphism of differential bundles.

Proof. Due to Definition 2.3.5, showing that $(F(f), 1_{F(*)})$ is an additive morphism of differential bundles amounts to showing that

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "F(!)"'] & F(B) \arrow[d, "F(!)"] \\ F(*) \arrow[r, "1_{F(*)}"'] & F(*) \end{tikzcd} \qquad \begin{tikzcd} F(A_2) \arrow[r, "F(f_2)"] \arrow[d, "F(\sigma)"'] & F(B_2) \arrow[d, "F(\sigma')"] \\ F(A) \arrow[r, "F(f)"'] & F(B) \end{tikzcd} \qquad \begin{tikzcd} F(*) \arrow[r, "1_{F(*)}"] \arrow[d, "F(\zeta)"'] & F(*) \arrow[d, "F(\zeta)"] \\ F(A) \arrow[r, "F(f)"'] & F(B) \end{tikzcd}$$

commute. They all commute by functoriality because they are images of the diagrams expressing that f was a morphism of differential objects. □

The $\mathbb{W}$eil-natural transformations from Definition 4.2.5 provide a notion of morphisms between differential functors.

The following proposition shows that $\mathbb{W}$eil-natural transformations between differential functors induce additive morphisms of differential bundles and linear $\mathbb{W}$eil-natural transformations between differential functors induce linear morphisms of differential bundles. Thereby it shows that the evalutation functor of Theorem 4.3.10 is well-defined on morphisms in the second component.

Proposition 4.3.7. *Let $(\mathbb{X}, T), (\mathbb{X}', T')$ be tangent categories and let $(F, \alpha), (F', \alpha') : (\mathbb{X}, T) \to (\mathbb{X}', T')$ be differential functors.*

(a) If A is a differential object in $\mathbb{X}$ and $\varphi : F \Rightarrow F'$ is a natural transformation of differential functors, then the pair (φ_A, φ_*) consisting of $\varphi_A : F(A) \to F'(A)$ and $\varphi_* : F(*) \to F'(*)$ is an additive morphism of differential bundles.

(b) If $\varphi : F \Rightarrow F'$ is a linear natural transformation of differential functors, then the pair (φ_A, φ_*) consisting of φ_A and φ_* is a linear morphism of differential bundles.

Proof. (a) A morphism between differential functors is exactly a natural transformation $\varphi : F \Rightarrow F'$. The pair of maps (φ_A, φ_*) forms an additive morphism of differential bundles as in Definition 2.3.5 since they commute with the images of $\sigma_A, 0_A$ and q_A by naturality.

(b) In order to show that the morphism (φ_A, φ_*) of differential bundles is linear, we need to show that

$$\begin{tikzcd} TF(A) \arrow[r, "T(\varphi_A)"] & TF'(A) \\ F(A) \arrow[u, "\alpha_W \circ F(\lambda)"] \arrow[r, "\varphi_A"'] & F'(A) \arrow[u, "\alpha'_W \circ F'(\lambda)"'] \end{tikzcd}$$

commutes. This diagram factors as

$$\begin{tikzcd} T(F(A)) \arrow[r, "T(\varphi_{(A)})"] & T(F'(A)) \\ F(T(A)) \arrow[u, "\alpha_W"] \arrow[r, "\varphi_{T(A)}"] & F'(T(A)) \arrow[u, "\alpha'_W"'] \\ F(A) \arrow[u, "F(\lambda_A)"] \arrow[r, "\varphi_A"'] & F'(A) \arrow[u, "F'(\lambda_A)"'] \end{tikzcd}$$

where the lower part commutes by naturality of φ and the upper part commutes if φ as linear.

$\square$

The different flavors of functors and natural transformations in Propositions 4.3.5 and 4.3.7 define four full subcategories of the hom-categories $\underline{\mathrm{Hom}}(\mathbb{X}, \mathbb{X}')$ in the four variants of $\mathbb{W}\mathrm{eil}{-}\mathsf{ACT}^{\bullet}_{\bullet,t}$.

Definition 4.3.8. *Let $(\mathbb{X}, T_\bullet)$ and $(\mathbb{X}', T'_\bullet)$ be tangent actegories.*

(a) The category $\mathbb{D}\mathrm{iffFun}(\mathbb{X}, \mathbb{X}')$ has as objects lax differential functors from $\mathbb{X}$ to $\mathbb{X}'$ and the morphisms between them are $\mathbb{W}\mathrm{eil}$-natural transformations as in Definition 4.2.5.

(b) The category $\mathbb{D}\mathrm{iffFun}^{\mathrm{strong}}(\mathbb{X}, \mathbb{X}')$ has as objects strong differential functors from $\mathbb{X}$ to $\mathbb{X}'$ and the morphisms between them are $\mathbb{W}\mathrm{eil}$-natural transformations.

(c) The category $\mathbb{D}\mathrm{iffFun}_{\mathrm{lin}}(\mathbb{X}, \mathbb{X}')$ has as objects lax differential functors from $\mathbb{X}$ to $\mathbb{X}'$ and the morphisms between them are linear $\mathbb{W}\mathrm{eil}$-natural transformations as in Definition 4.2.5.

(d) The category $\mathbb{D}\mathrm{iffFun}^{\mathrm{strong}}_{\mathrm{lin}}(\mathbb{X}, \mathbb{X}')$ has as objects strong differential functors from $\mathbb{X}$ to $\mathbb{X}'$ and the morphisms between them are linear $\mathbb{W}\mathrm{eil}$-natural transformations.

Next we will define a functor `Eval` that evaluates differential functors on differential objects. The key insight here is that one does not just get any object, but an object together with the structure that makes it a differential bundle. For strong differential functors one even obtains differential objects.

This means strong differential functors preserve differential objects, lax differential functors almost preserve them by sending them to differential bundles. In addition they preserve additive and linear maps between differential objects (or send them to additive and linear maps of differential bundles). Thus strong differential functors preserve the category of differential objects, lax differential functors send it into the category of differential bundles.

In order to capture this more precisely we formulate this result as an evaluation functor.

Let `eval` $: \mathbb{X} \times \mathrm{Fun}(\mathbb{X}, \mathbb{X}') \to \mathbb{X}'$ be the usual evaluation functor. Let $u : \mathbb{D}\mathrm{Obj}(\mathbb{X}) \to \mathbb{X}$ be the functor that sends a differential object to its underlying object. Let $U : \mathbb{D}\mathrm{iffFun}(\mathbb{X}, \mathbb{X}') \to \mathrm{Fun}(\mathbb{X}, \mathbb{X}')$ be the functor that sends a differential functor to its underlying functor. Let $t : \mathbb{D}\mathrm{Bun}(\mathbb{X}') \to \mathbb{X}'$ be the functor that sends a differential bundle (E, M, q, ζ, σ) to its total bundle space E. We will construct a functor `Eval` so that `eval` factors through $\mathbb{D}\mathrm{Bun}(\mathbb{X}')$ in the sense that the following diagrams commute.

$$\begin{tikzcd}
\mathbb{D}\mathrm{Obj}_{\mathrm{add}}(\mathbb{X}) \times \mathbb{D}\mathrm{iffFun}(\mathbb{X}, \mathbb{X}') \arrow[r, "\mathtt{Eval}"] \arrow[d, "u \times U"'] & \mathbb{D}\mathrm{Bun}_{\mathrm{add}}(\mathbb{X}') \arrow[d, "t"] \\
\mathbb{X} \times \mathrm{Fun}(\mathbb{X}, \mathbb{X}') \arrow[r, "\mathtt{eval}"'] & \mathbb{X}'
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{D}\mathrm{Obj}_{\mathrm{add}}(\mathbb{X}) \times \mathbb{D}\mathrm{iffFun}^{\mathrm{strong}}(\mathbb{X}, \mathbb{X}') \arrow[r, "\mathtt{Eval}"] \arrow[d, "u \times U"'] & \mathbb{D}\mathrm{Obj}_{\mathrm{add}}(\mathbb{X}') \arrow[d, "u"] \\
\mathbb{X} \times \mathrm{Fun}(\mathbb{X}, \mathbb{X}') \arrow[r, "\mathtt{eval}"'] & \mathbb{X}'
\end{tikzcd}
\tag{4.3}$$

Definition 4.3.9. *Let $(\mathbb{X}, T, p, 0, +, \ell, c)$ and $(\mathbb{X}', T', p', 0', +', \ell', c')$ be tangent categories. Define the assignment of objects and morphisms* `Eval` $: \mathbb{D}\mathrm{Obj}_{\mathrm{add}}(\mathbb{X}) \times \mathbb{D}\mathrm{iffFun}_{\mathrm{add}}(\mathbb{X}, \mathbb{X}') \to \mathbb{D}\mathrm{Bun}_{\mathrm{add}}(\mathbb{X}')$ *by the following rule:*

- *On objects,* `Eval` *sends a differential object A and a differential functor $F : \mathbb{X} \to \mathbb{X}'$, to the differential bundle in $\mathbb{X}'$ that is given by $(F(A), F(*), F(q), F(\sigma), F(0), \alpha \circ F(\lambda))$.*

- *On morphisms* `Eval` *sends an additive differential object morphism $(f, 1_*) : A \to A'$ and a natural transformation $\varphi : F \Rightarrow F'$ to the differential bundle morphism $(\varphi_{A'} \circ F(f), \varphi_*)$.*

Proposition 4.3.5 shows that `Eval` is well-defined because $(F(A), F(*), F(q), F(\sigma), F(0), \alpha \circ F(\lambda))$ is indeed a differential bundle. The pair $(\varphi_{A'} \circ F(f), \varphi_*)$ is a differential bundle morphism because it is the composition of $(\varphi_{A'}, \varphi_*)$ with $(F(f), 1_{F(*)})$, both of which are differential bundle morphisms by Propositions 4.6 and 4.7.

Theorem 4.3.10. *Let $(\mathbb{X}, T, p, 0, +, \ell, c)$ and $(\mathbb{X}', T', p', 0', +', \ell', c')$ be tangent categories. Then the assign-*

ment

$$\texttt{Eval} : \mathbb{D}\mathrm{Obj}_{\mathrm{add}}(\mathbb{X}) \times \mathbb{D}\mathrm{iffFun}(\mathbb{X}, \mathbb{X}') \to \mathbb{D}\mathrm{Bun}_{\mathrm{add}}(\mathbb{X}')$$

as defined in Definition 4.3.9 is a functor and the diagrams (4.3) commute.

Proof. The assignments of objects and morphisms are given by Propositions 4.3.5, 4.3.6 and 4.3.7. Thus it only remains to show that `Eval` preserves identity and compositions. By its definition on morphisms from the usual evaluation, `Eval` sends the identity to the identity. For composition let $A \xrightarrow{(f,1_*)} B \xrightarrow{(g,1_*)} C$ be additive morphisms of differential objects and let $F \overset{\varphi}{\Rightarrow} G \overset{\psi}{\Rightarrow} H$ be natural transformations between differential functors.

Then

$$\texttt{Eval}(g, \psi) \circ \texttt{Eval}(f, \varphi) = (\psi_C, \psi_*) \circ (G(g), 1_{G(*)}) \circ (\varphi_B, \varphi_*) \circ (F(f), 1_{F(*)}).$$

Due to naturality of φ this is equal to

$$(\psi_C, \psi_*) \circ (\varphi_B, \varphi_*) \circ (F(g), 1_{F(*)}) \circ (F(f), 1_{F(*)}) = (\psi_C \circ \varphi_B, \psi_* \circ \varphi_*) \circ (F(g) \circ F(f), 1_{F(*)})$$

which equals $\texttt{Eval}((g, \psi) \circ (f, \varphi)$.

The diagrams (4.3) commute because the first component of `Eval` sends A, F to the evaluation $F(A)$. $\square$

The following Corollary summarizes the linear and strong versions of Theorem 4.3.10 succinctly. By a slight abuse of notation, we use $\texttt{Eval}_A$ to denote all four cases.

Corollary 4.3.11. *Suppose $\mathbb{X}$ and $\mathbb{X}'$ are tangent categories and A is a differential object in $\mathbb{X}$. Then there are four evaluation functors, namely*

(a) $\texttt{Eval}_A : \mathbb{D}\mathrm{iffFun}(\mathbb{X}, \mathbb{X}') \to \mathbb{D}\mathrm{Bun}_{\mathrm{add}}(\mathbb{X}')$*,*

(b) $\texttt{Eval}_A : \mathbb{D}\mathrm{iffFun}_{\mathrm{lin}}(\mathbb{X}, \mathbb{X}') \to \mathbb{D}\mathrm{Bun}_{\mathrm{lin}}(\mathbb{X}')$*,*

(c) $\texttt{Eval}_A : \mathbb{D}\mathrm{iffFun}^{\mathrm{strong}}(\mathbb{X}, \mathbb{X}') \to \mathbb{D}\mathrm{Obj}_{\mathrm{add}}(\mathbb{X}')$*, and*

(d) $\texttt{Eval}_A : \mathbb{D}\mathrm{iffFun}^{\mathrm{strong}}_{\mathrm{lin}}(\mathbb{X}, \mathbb{X}') \to \mathbb{D}\mathrm{Obj}_{\mathrm{lin}}(\mathbb{X}')$*.*

Proof.

Case (a) is just the functor `Eval` of theorem 4.3.10 evaluated on A in the first component. Cases (b), (c) and (d) follow from (a) in terms of functoriality as they are just a restriction to a subcategory. Proposition 4.3.6.b and Proposition 4.3.7.b ensure that the range is correct. $\square$

A typical example for a category with a differential object is $\mathbb{N}^\bullet$ from Definition 2.2.9.

Example 4.3.12. *Let $\mathbb{N}^\bullet$ be the category from Definition 2.2.9. Then any differential functor F from $\mathbb{N}^\bullet$ into a tangent category $\mathbb{X}$ induces a differential bundle in $\mathbb{X}$. Concretely the functor will produce a differential bundle with*

$$M = F(\mathbb{N}^0), \qquad E = F(\mathbb{N}^1), \qquad E_n = F(\mathbb{N}^n).$$

4.4 Inducing a differential functor from $\mathbb{N}^\bullet$

In this section we focus in particular on the differential object $\mathbb{N}$ in $\mathbb{N}^\bullet$. The main construction is the functor `Ind` that is an inverse for $\texttt{Eval}_{\mathbb{N}^\bullet}$ sending a differential bundle $(E, M, q, \zeta, \sigma, \lambda)$ in a tangent category $\mathbb{X}$ to a functor $\mathbb{N}^\bullet \to \mathbb{X}$ whose evaluation on $\mathbb{N}$ returns the differential bundle $(E, M, q, \zeta, \sigma, \lambda)$ as in Example 4.3.12. This means $\mathbb{N} \to *$ is an initial differential bundle in the sense that there is a differential functor from $\mathbb{N} \to *$ to every other differential bundle. In Section 4.5 we will see in which sense this functor is unique.

Recall from Example 2.2.11 that the category $\mathbb{N}^\bullet$ from Definition 2.2.9 forms a Cartesian tangent category (as defined in Example 2.3.3) and thus a tangent actegory $(\mathbb{N}^\bullet, D_\bullet)$. We will not explicitly use that the tangent structure is Cartesian. Objects of $(\mathbb{N}^\bullet, D_\bullet)$ are of the form $\mathbb{N}^k$ for $k \geq 0$, morphisms of $\mathbb{N}^\bullet$ are matrices, and the tangent functor is defined by $D_W(\mathbb{N}^k) = \mathbb{N}^k \times \mathbb{N}^k$. As we saw in Example 2.3.4. the natural numbers $\mathbb{N}^1$ are a differential object in $\mathbb{N}^\bullet$.

In Lemma 4.4.1 below, we show that the set of bundle morphisms from the pullback powers of the bundle E over M to E is a commutative monoid. This result is important because it allows us to use linear combinations, resembling the matrix addition, as a tool to build morphisms between pullback powers.

The set of natural numbers $\mathbb{N} = \{0, 1, 2, ...\}$ forms a rig (semiring), and every commutative monoid M can be seen as a module over the rig $\mathbb{N}$ by defining $n \cdot x$ for $n \in \mathbb{N}$ and $x \in X$ as the n-fold addition of x to itself, i.e. $n \cdot x = \sum_{i=1}^{n} x$.

The following Lemma holds for any additive bundle, though we are particularly interested in the case of differential bundles. Thus, unlike in Definition 2.2.2, we use the notation (E, M, q, ζ, σ) instead of $(X, A, +, 0, p)$, which resembles the notation of differential bundles. We define a commutative monoid structure on certain morphism sets of the slice category.

Lemma 4.4.1. *Let (E, M, q, ζ, σ) be an additive bundle in a category $\mathbb{X}$.*

1. *For every $n \in \mathbb{N}$, the set*

$$\mathrm{Hom}_{/M}(E_n, E) := \{f : E_n \to E | q \circ f = q \circ \pi_0\}$$

together with the addition

$$s : \mathrm{Hom}_{/M}(E_n, E) \times \mathrm{Hom}_{/M}(E_n, E) \to \mathrm{Hom}_{/M}(E_n, E) \quad , \quad (f, g) \mapsto \sigma \circ \langle f, g\rangle$$

and the zero element

$$z = \zeta \circ q \circ \pi_0 \in \mathrm{Hom}_{/M}(E_n, E)$$

is a commutative monoid and thereby an $\mathbb{N}$*-module*

2. *For* $g : E_n \to E_m$ *and* $f : E_m \to E$*, fulfilling* $q \circ \pi_0 = q \circ f$ *and* $q \circ \pi_0 = q \circ \pi_0 \circ g$*,*

$$(n \cdot f) \circ g = n \cdot (f \circ g)$$

and for morphisms $(f_i)_{1 \le i \le k} : E_m \to E$ *fulfilling* $q \circ \pi_0 = q \circ f$*,*

$$\left(\sum_{i=1}^{k} f_i\right) \circ g = \sum_{i=1}^{k} (f_i \circ g),$$

where $\sum$ *and* $\cdot$ *are from the* $\mathbb{N}$*-module structure defined in part 1.*

Proof. 1. The definition of s is well-defined because f and g fulfill $q \circ f = q \circ g = q \circ \pi_0$ and therefore $\langle f, g\rangle : E_n \to E_2$ is defined.

The addition s is commutative and associative, because σ is. The element z is the unit for s because ζ is the unit for σ.

2. Given a commutative monoid, the $\mathbb{N}$-module structure is defined as repeated addition, i.e. $n \cdot f := \sum_{i=1}^{n} f$. Therefore the second equation implies the first one. The second equation holds because

$$\left(\sum_{i=1}^{k} f_i\right) \circ g = \sigma \circ \langle f_1, ..., f_k\rangle \circ g = \sigma \circ \langle f_1 \circ g, ..., f_k \circ g\rangle = \sum_{i=1}^{k} (f_i \circ g).$$

The condition that $q \circ \pi_0 = q \circ f$ and $q \circ \pi_0 = q \circ \pi_0 \circ g$ is required because the commutative monoid structure from Part 1 is only defined for morphisms in the slice category $\mathrm{Hom}_{/M}(E_n, E) := \{f : E_n \to E | q \circ f = q \circ \pi_0\}$.

□

Using this module structure we now can define a functor F_E.

Definition 4.4.2. *Given a differential bundle* $(E, M, q, \zeta, \sigma, \lambda)$ *in a tangent category (formulated as a* Weil-*actegory)* $(\mathbb{X}, T_\bullet)$*, then there is an assignment of objects and morphisms,* $F_E : \mathbb{N}^\bullet \to \mathbb{X}$*, defined*

- *on objects as* $F_E(\mathbb{N}^k) = E_k$*, in particular* $F_E(\mathbb{N}^0) = M$*, and*

- *on morphisms (i.e. matrices)* $A = (a_{ij})_{1\le i\le n, 1\le j\le m} : \mathbb{N}^m \to \mathbb{N}^n$ *as*

$$F_E(A) = \left\langle \sum_{j=1}^{m} a_{1j} \cdot \pi_j, ..., \sum_{j=1}^{m} a_{nj} \cdot \pi_j \right\rangle$$

where $\sum$ *and* $\cdot$ *are from the module structure in Lemma 4.4.1.*

Proposition 4.4.3. *The assignment* F_E *of Definition 4.4.2 is a functor* $F_E : \mathbb{N}^\bullet \to \mathbb{X}$.

The proof is essentially the same as the proof from classical linear algebra that composition of linear maps corresponds to matrix multiplication, just in a more general setting.

Proof. In order to show that this assignment is a functor, we need to check that it preserves composition and identity.

It preserves the identity because, for the identity matrix $\mathbb{I}_n : \mathbb{N}^n \to \mathbb{N}^n$, the functor evaluates as

$$F_E(\mathbb{I}_n) = \left\langle \sum_{j=1}^{n} (\mathbb{I}_n)_{1j} \cdot \pi_j, ..., \sum_{j=1}^{n} (\mathbb{I}_n)_{nj} \cdot \pi_j \right\rangle = \langle \pi_1, ..., \pi_n \rangle = 1_{E_n}.$$

In order to show that F_E preserves composition, let $A = (a_{ij})_{1\le i\le m, 1\le j\le n} : \mathbb{N}^n \to \mathbb{N}^m$ and $B = (b_{ij})_{1\le i\le n, 1\le j\le k} : \mathbb{N}^k \to \mathbb{N}^n$ be composable morphisms of $\mathbb{N}^\bullet$. Then the composition of their images gives in the h-th component

$$\begin{aligned}
(F_E(A) \circ F_E(B))_h &= \left(\sum_{i=1}^{n} a_{hi} \cdot \pi_i \right) \circ \left\langle \sum_{j=1}^{k} b_{1j} \cdot \pi_j, ..., \sum_{j=1}^{k} b_{nj} \cdot \pi_j \right\rangle \\
&= \sum_{i=1}^{n} \left(a_{hi} \cdot \pi_i \circ \left\langle \sum_{j=1}^{k} b_{1j} \cdot \pi_j, ..., \sum_{j=1}^{k} b_{nj} \cdot \pi_j \right\rangle \right) \\
&= \sum_{i=1}^{n} \left(a_{hi} \cdot \sum_{j=1}^{k} b_{ij} \cdot \pi_j \right) \\
&= \sum_{j=1}^{k} \left(\sum_{i=1}^{n} a_{hi} b_{ij} \right) \pi_j = \sum_{j=1}^{k} (AB)_{hj} \pi_j = (F_E(AB))_h.
\end{aligned}$$

□

In the following propositions we will construct a natural transformation $\hat{\alpha} : F_E \circ D_\bullet \to T_\bullet \circ (1_{\mathbb{W}\mathrm{eil}} \times F_E)$ in order to make $(F_E, \hat{\alpha})$ a differential functor. We will define it first for the generators of $\mathbb{N}^\bullet$ and $\mathbb{W}$eil and then construct the other components by pullback powers. For a differential bundle $E \xrightarrow{q} M$, the tangent bundle $T(E)$ has a projection $T(q) : T(E) \to T(M)$ and a projection $p : T(E) \to E$, over both of which we take pullback powers. In order to keep track which base the pullback power is taken over, we will introduce a notation which resembles fractions. This notation $\left(\frac{A}{B}\right)^k$ evokes the idea of products in the slice category $\mathbb{X}/B$, and is more compact and more informative than the somewhat standard notation $A \times_B \cdots \times_B A$ (which makes it hard to keep track of the number k of factors).

Notation 4.4.4. *Let $\mathbb{X}$ be a category with pullbacks and let*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\beta"'] & B \arrow[d, "\alpha"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

be a commutative square in $\mathbb{X}$. The object $\left(\frac{A}{B}\right)^k := A \times_B ... \times_B A$ is the k-th pullback power of $A \xrightarrow{f} B$. The pullbacks $\left(\frac{A}{B}\right)^k$, $\left(\frac{A}{C}\right)^k$ and $\left(\frac{B}{D}\right)^k$ are defined analogously. Then, the morphisms $\beta : A \to C$ and $\alpha : B \to D$ induce a morphism

$$\left(\frac{\beta}{\alpha}\right)^k : \left(\frac{A}{B}\right)^k \to \left(\frac{C}{D}\right)^k$$

between the pullback powers. Analogously the morphisms $f : A \to B$ and $g : C \to D$ induce a morphism

$$\left(\frac{f}{g}\right)^k : \left(\frac{A}{C}\right)^k \to \left(\frac{B}{D}\right)^k.$$

Proposition 4.4.5. *In the circumstances of Notation 4.4.4, let $\left(\frac{\left(\frac{A}{B}\right)^k}{\left(\frac{C}{D}\right)^k}\right)^m$ be the m-th pullback power of $\left(\frac{A}{B}\right)^k \xrightarrow{\left(\frac{\beta}{\alpha}\right)^k} \left(\frac{C}{D}\right)^k$. Then*

$$\left(\frac{\left(\frac{A}{B}\right)^k}{\left(\frac{C}{D}\right)^k}\right)^l \cong \left(\frac{\left(\frac{A}{C}\right)^l}{\left(\frac{B}{D}\right)^l}\right)^k.$$

In particular when $C = D$,

$$\left(\frac{\left(\frac{A}{B}\right)^k}{C}\right)^l \cong \left(\frac{\left(\frac{A}{C}\right)^l}{\left(\frac{B}{C}\right)^l}\right)^k.$$

Proof. This is the fact that we can commute limits, stated in our notation. $\square$

Now we will combine this fraction calculus of pullbacks with the notation more common for differential bundles and tangent categories, where

$$E_k = \left(\frac{E}{M}\right)^k \quad \text{and} \quad T_k(A) = \left(\frac{T(A)}{A}\right)^k.$$

denote the k-fold pullback power of $q : E \to M$ and $p : T(A) \to A$ respectively. Recall that these pullbacks are preserved by the tangent structure due to Definitions 2.3.1 and 2.2.4.

Corollary 4.4.6.

(a) Let $(E, M, q, \zeta, \sigma, \lambda)$ be a differential bundle in a tangent actegory $(\mathbb{X}, T_\bullet)$. Then

$$T_\bullet(W^k, E_l) \cong \left(\frac{T_k E}{T_k M}\right)^l \cong \left(\frac{T(E_l)}{E_l}\right)^k.$$

(b) In particular for the differential bundle $\mathbb{N}^1 \to \mathbb{N}^0$ in $(\mathbb{N}^\bullet, D_\bullet)$

$$D_\bullet(W^k, \mathbb{N}^l) \cong \left(\frac{\left(\frac{\mathbb{N}^2}{\mathbb{N}}\right)^k}{\mathbb{N}^0}\right)^l \cong \left(\mathbb{N}^{2l}\right)^k.$$

Proof. Since $T_k E = \left(\frac{T(E)}{E}\right)^k$, part (a) is just Proposition 4.4.5 applied to the square

$$\begin{tikzcd}
T(E) \arrow[r, "p_E"] \arrow[d, "T(q)"'] & E \arrow[d, "q"] \\
T(M) \arrow[r, "p_M"'] & M
\end{tikzcd}$$

that commutes by naturality of p. Part (b) is part (a) for the differential bundle $\mathbb{N}^1 \to \mathbb{N}^0$ in $(\mathbb{N}^\bullet, D_\bullet)$. $\square$

Lemma 4.4.7. *Let $(E, M, q, \zeta, \sigma, \lambda)$ be a differential bundle in a tangent actegory $(\mathbb{X}, T_\bullet)$. Then the functor F_E from Proposition 4.4.3 evaluates as*

$$F_E\left(\left(\frac{\mathbb{N}^l}{\mathbb{N}}\right)^k\right) \cong \left(\frac{E_l}{E}\right)^k \qquad \text{and} \qquad F_E\left(\frac{\mathbb{N}^{2l}}{\mathbb{N}^l}\right)^k \cong \left(\frac{E_{2l}}{E_l}\right)^k.$$

Proof. Since $\left(\frac{\mathbb{N}^l}{\mathbb{N}}\right)^k = \mathbb{N}^{(l-1)\cdot k+1}$ and $\left(\frac{\mathbb{N}^{2l}}{\mathbb{N}^l}\right)^k = \mathbb{N}^{(k+1)l}$, the proof amounts to the following equations:

$$F_E\left(\left(\frac{\mathbb{N}^l}{\mathbb{N}}\right)^k\right) = F_E(\mathbb{N}^{(l-1)\cdot k+1}) = E_{(l-1)\cdot k+1} \cong \left(\frac{E_l}{E}\right)^k$$

$$F_E\left(\frac{\mathbb{N}^{2l}}{\mathbb{N}^l}\right)^k = F_E(\mathbb{N}^{(k+1)l}) = E_{(k+1)l} \cong \left(\frac{E_{2l}}{E_l}\right)^k.$$

$\square$

Given a differential bundle $(E, M, q, \zeta, \lambda)$ in a tangent actegory $(\mathbb{X}, T_\bullet)$, we are now finally able to construct the differential functor

$$\texttt{Ind}(E) = (F_E, \hat{\alpha}) : (\mathbb{N}^\bullet, D_\bullet) \to (\mathbb{X}, T_\bullet)$$

that sends $\mathbb{N}$ to E. The underlying functor F_E is the functor F_E from Proposition 4.4.3.

Definition 4.4.8. *Given a differential bundle $(E, M, q, \zeta, \sigma, \lambda)$ in a tangent actegory $(\mathbb{X}, T_\bullet)$, we define a set of morphisms*

$$\hat{\alpha} : F_E \circ D_A(\mathbb{N}^k) \to T_A \circ F_E(\mathbb{N}^k)$$

for every object A of Weil *and $\mathbb{N}^k$ of $\mathbb{N}^\bullet$. We define it inductively out of the differential bundle structure of $(E, M, q, \zeta, \sigma, \lambda)$ by*

$$\hat{\alpha}_{W,\mathbb{N}^0} := 0 : M \to T(M), \tag{4.4}$$

$$\hat{\alpha}_{\mathbb{N},\mathbb{N}^1} := 1_E : E \to E \tag{4.5}$$

$$\hat{\alpha}_{W,\mathbb{N}^1} := T(\sigma) \circ \langle 0 \circ \pi_0, \lambda \circ \pi_1 \rangle : E_2 \to T(E), \tag{4.6}$$

$$\hat{\alpha}_{W^k,\mathbb{N}^1} := \left(\frac{\hat{\alpha}_{W,\mathbb{N}^1}}{E}\right)^k : F_E \circ D_{W^k}(\mathbb{N}^1) = \left(\frac{E_2}{E}\right)^k \to \left(\frac{TE}{E}\right)^k = T_{W^k} \circ F_E(\mathbb{N}^1), \tag{4.7}$$

$$\hat{\alpha}_{W^k,\mathbb{N}^l} := \left(\frac{\left(\frac{\hat{\alpha}_{W,\mathbb{N}}}{E}\right)^k}{\left(\frac{\hat{\alpha}_{W,\mathbb{N}^0}}{E}\right)^k}\right)^l : F_E \circ D_{W^k}(\mathbb{N}^l) = \left(\frac{\left(\frac{E_2}{E}\right)^k}{M}\right)^l \to \left(\frac{\left(\frac{TE}{E}\right)^k}{\left(\frac{T(M)}{M}\right)^k}\right)^l = T_{W^k} \circ F_E(\mathbb{N}^l), \tag{4.8}$$

and

$$\hat{\alpha}_{A\otimes B,\mathbb{N}^l} := T_A(\hat{\alpha}_{B,\mathbb{N}^l}) \circ \hat{\alpha}_{A,D_B(\mathbb{N}^l)} : \quad F_E \circ D_A \circ D_B(\mathbb{N}^l) \to T_A \circ T_B \circ F_E(\mathbb{N}^l), \tag{4.9}$$

where $\left(\frac{A}{B}\right)^n$ denotes the n-th pullback power of A over B. The expression $E_n = \left(\frac{E}{M}\right)^n$ continues to denote the n-th pullback power of E over M.

In Proposition 4.4.9 we show that $\hat{\alpha}$ is a natural transformation $F_E \circ D_\bullet \to (T_\bullet \circ 1_{\mathbb{W}\mathrm{eil}} \times F_E)$ and that $(F_E, \hat{\alpha})$ is a differential functor. This is the key technical observation needed for Theorem 4.4.10. Before that, we now showcase how Equations 4.6-4.9 determine $\hat{\alpha}$ for a nontrivial Weil algebra. For this example we use the component of $\hat{\alpha}$ at $W \otimes W$. The component of $\hat{\alpha}$ is

$$\hat{\alpha}_{W\otimes W,\mathbb{N}} = T_W(\alpha_{W,\mathbb{N}}) \circ \hat{\alpha}_{W,D_W(\mathbb{N})}.$$

The parts unpack as

$$\hat{\alpha}_{W,D_W(\mathbb{N})} = \langle T(\sigma) \circ \langle 0 \circ \pi_0, \lambda \circ \pi_1 \rangle, T(\sigma) \circ \langle 0 \circ \pi_2, \lambda \circ \pi_3 \rangle \rangle : E_4 \to T(E_2) \cong \left(\frac{T(E)}{T(M)} \right)^2$$

$$T_W(\hat{\alpha}_{W,\mathbb{N}}) = T\left(T(\sigma) \circ \langle 0 \circ \pi_0, \lambda \circ \pi_1 \rangle\right) : T(E_2) \to T \circ T(E) = T^2(E)$$

and the composition is

$$\hat{\alpha}_{W \otimes W,\mathbb{N}} = T^2(\sigma \circ (\sigma \times_M \sigma)) \circ \langle T(0) \circ 0 \circ \pi_0, T(0) \circ \lambda \circ \pi_1, T(\lambda) \circ 0 \circ \pi_2, T(\lambda) \circ \lambda \circ \pi_3 \rangle.$$

Proposition 4.4.9. *Given a differential bundle $E \xrightarrow{q} M$ in $\mathbb{X}$, Proposition 4.4.3 and Equations 4.6-4.9 define a differential functor*

$$\mathtt{Ind}(E) = (F_E, \hat{\alpha}) : (\mathbb{N}^\bullet, D_\bullet) \to (\mathbb{X}, T_\bullet).$$

Proof. We have already shown in Proposition 4.4.3 that F_E is a functor. By Definition, F_E preserves pullbacks over the terminal object. Thus, in order to show that $(F_E, \hat{\alpha})$ is a differential functor, it remains to show that $\hat{\alpha}$ is natural and Cartesian.

To show that the transformation $\hat{\alpha}$ is natural, we show that the construction is natural with respect to the generating maps of $\mathbb{N}^\bullet$ as in Lemma 4.2.22. This proof essentially follows from the fact that $\hat{\alpha}$ is constructed out of these maps, and naturality with respect to them follows as a consequence. However, checking this carefully is technical and tedious. For this reason, we postpone full details of this proof until Appendix B.

Next we need to show that $\hat{\alpha}$ is Cartesian, i.e. that for any Weil algebra A, the naturality diagrams of $\hat{\alpha}_{A,-}$ with respect to all morphisms in $\mathbb{N}^\bullet$ are pullback diagrams. We start by checking the naturality diagrams

$$\begin{tikzcd} F_E \circ D_{W^n}(\mathbb{N}) \arrow[r, "\hat{\alpha}_{W^n,\mathbb{N}}"] \arrow[d, "F_E \circ D_{W^n}(\Delta_k)"'] & T_{W^n} \circ F_E(\mathbb{N}) \arrow[d, "T_{W^n} \circ F_E(\Delta_k)"] \\ F_E \circ D_{W^n}(\mathbb{N}^k) \arrow[r, "\hat{\alpha}_{W^n,\mathbb{N}^k}"'] & T_{W^n} \circ F_E(\mathbb{N}^k) \end{tikzcd} \qquad \begin{tikzcd} F_E \circ D_{W^n}(\mathbb{N}^k) \arrow[r, "\hat{\alpha}_{W^n,\mathbb{N}^k}"] \arrow[d, "F_E \circ D_{W^n}(\sigma_k)"'] & T_{W^n} \circ F_E(\mathbb{N}^k) \arrow[d, "T_{W^n} \circ F_E(\sigma_k)"] \\ F_E \circ D_{W^n}(\mathbb{N}) \arrow[r, "\hat{\alpha}_{W^n,\mathbb{N}}"'] & T_{W^n} \circ F_E(\mathbb{N}) \end{tikzcd}$$

of $\hat{\alpha}(W^n, -)$ with respect to the generating morphisms $\Delta_k : \mathbb{N} \to \mathbb{N}^k$ and $\sigma_k : \mathbb{N}^k \to \mathbb{N}$ of $\mathbb{N}^\bullet$. Since F_E preserves the pullbacks involved and thus preserves the diagonal, the naturality diagram of $\hat{\alpha}_{W^n,-}$ with respect to $\Delta_k : \mathbb{N} \to \mathbb{N}^k$ is

$$\begin{tikzcd} \left(\frac{E_2}{E}\right)^n \arrow[r, "\hat{\alpha}_{W^n,\mathbb{N}}"] \arrow[d, "\left(\frac{(\Delta_k)_2}{\Delta_k}\right)^n"'] & T_n(E) \arrow[d, "T_n(\Delta_k)"] \\ \left(\frac{E_{2k}}{E_k}\right)^n \arrow[r, "\hat{\alpha}_{W^n,\mathbb{N}^k}"'] & T_n(E_k) \end{tikzcd}.$$

It is a pullback because it is the n-th pullback power of

$$\begin{tikzcd} E_2 \arrow[r, "\hat{\alpha}_{W,\mathbb{N}}"] \arrow[d, "(\Delta_k)_2"'] & T(E) \arrow[d, "T(\Delta_k)"] \\ E_{2k} \arrow[r, "\hat{\alpha}_{W,\mathbb{N}^k}"'] & T(E_k) \end{tikzcd} \quad \text{over} \quad \begin{tikzcd} E \arrow[r, "\hat{\alpha}_{\mathbb{N},\mathbb{N}^1}=1_E"] \arrow[d, "\Delta_k"'] & E \arrow[d, "\Delta_k"] \\ E_k \arrow[r, "\hat{\alpha}_{\mathbb{N},\mathbb{N}^k}=1_{E_k}"'] & E_k. \end{tikzcd}$$

The second diagram is trivially a pullback. The first diagram is a pullback because in

$$\begin{tikzcd} E_2 \arrow[r, "\hat{\alpha}_{W,\mathbb{N}}"] \arrow[d, "(\Delta_k)_2"'] & T(E) \arrow[d, "T(\Delta_k)"] \\ E_{2k} \arrow[r, "\hat{\alpha}_{W,\mathbb{N}^k}"] \arrow[d, "q\circ\pi_0"'] & T(E_k) \arrow[d, "T(q\circ\pi_0)"] \\ M \arrow[r, "\hat{\alpha}_{W,\mathbb{N}^0}"'] & T(M) \end{tikzcd}$$

the bottom and the outer squares are pullbacks, because of the universality of the vertical lift in Definition 2.3.1. Thus the top square is a pullback due to the pasting law for pullbacks [68, Exercise III.4.8].

The functor F_E preserves the addition σ by its definition and the functor D_{W^n} sends the addition σ to two copies of the addition $\sigma \times \sigma$. Thus the fact that the naturality diagram of $\alpha_{W^k,-}$ for $\sigma_k : \mathbb{N}^k \to \mathbb{N}$ is a pullback diagram is completely analogous. Lemma 4.2.22 shows that all morphisms in $\mathbb{N}^\bullet$ are generated from σ_k and Δ_k. Thus the naturality diagrams of $\hat{\alpha}_{W^n,-}$ with respect to any morphisms in $\mathbb{N}^\bullet$ are pullbacks.

If the naturality diagrams of $\hat{\alpha}_A$ and $\hat{\alpha}_B$ are pullbacks, then the naturality diagrams of $\hat{\alpha}_{A\otimes B}$ are pullbacks because both subsquares are pullbacks:

$$\begin{tikzcd} F_E \circ D_A \circ D_B(X) \arrow[r, "\hat{\alpha}_{A,T_B(X)}"] \arrow[d] & T_A \circ F_E \circ D_B(X) \arrow[r, "T_A(\hat{\alpha}_{B,X})"] \arrow[d] & T_A \circ T_B \circ F_E(X) \arrow[d] \\ F_E \circ D_A \circ D_B(Y) \arrow[r, "\hat{\alpha}_{A,T_B(Y)}"'] & T_A \circ F_E \circ D_B(Y) \arrow[r, "T_A(\hat{\alpha}_{B,X})"'] & T_A \circ T_B \circ F(Y) \end{tikzcd}$$

The left square is a pullback by the assumption that the naturality diagrams of $\hat{\alpha}_A$ are pullbacks. The right square is a pullback since the naturality diagrams of $\hat{\alpha}_B$ are pullbacks built out of the universality diagram for the vertical lift. The universality diagram is a pullback preserved by the tangent functor, thus the right square is still a pullback.

Since every object of $\mathbb{W}$eil is a tensor product of powers of the dual numbers W, this shows that for any Weil algebra A, the naturality diagrams of $\hat{\alpha}_{A,-}$ are pullbacks, i.e. $\hat{\alpha}_{A,-}$ is Cartesian. This was all that remained to show that $(F_E, \hat{\alpha})$ is a differential functor. □

Theorem 4.4.10. *Let $\mathbb{X}$ be a tangent category and $(E, M, q, \zeta, \sigma, \lambda)$ and $(E', M', q', \zeta', \sigma', \lambda')$ be differential* *bundles in* $\mathbb{X}$*. Denote* $\texttt{Ind}(E) = (F_E, \hat{\alpha})$ *and* $\texttt{Ind}(E') = (F'_E, \hat{\alpha}')$*.*

(a) *An additive morphism of differential bundles induces a natural transformation* $\varphi : F_E \Rightarrow F_{E'}$ *between the induced functors.*

(b) *A linear morphism of differential bundles induces a linear natural transformation.*

(c) *This defines two functors*

$$\texttt{Ind} : \mathbb{D}\mathrm{Bun}_{\mathrm{add}}(\mathbb{X}) \to \mathbb{D}\mathrm{iffFun}(\mathbb{N}^\bullet, \mathbb{X})$$

and

$$\texttt{Ind} : \mathbb{D}\mathrm{Bun}_{\mathrm{lin}}(\mathbb{X}) \to \mathbb{D}\mathrm{iffFun}_{\mathrm{lin}}(\mathbb{N}^\bullet, \mathbb{X}).$$

Proof. (a) Suppose $(f, g) : (E, M, p, \zeta, \sigma, \lambda) \to (E', M', p', \zeta', \sigma', \lambda')$ is an additive morphism of differential bundles, i.e. a pair of maps $f : E \to E', g : M \to M'$ compatible with the projection, the addition and zero as in Definition 2.3.5. We define a natural transformation, $\varphi : F_E \Rightarrow F_{E'}$, by defining its $\mathbb{N}^k$ components for all $k \in \mathbb{N}$ as follows:

$$\varphi_{\mathbb{N}^k} = \begin{cases} g : M \to M' & \text{if } k = 0 \\ f_k : E_k \to E'_k & \text{if } k > 0. \end{cases}$$

The transformation φ is natural with respect to $+_k : \mathbb{N}^k \to \mathbb{N}$ (in particular for $k = 0$, $0 : \mathbb{N}^0 \to \mathbb{N}$) and $\Delta_k : \mathbb{N} \to \mathbb{N}^k$ (in particular for $k = 0$, $p : \mathbb{N} \to \mathbb{N}^0$). That is, the diagrams

$$\begin{tikzcd} F_E(\mathbb{N}^k) \arrow[r, "\varphi_{\mathbb{N}^k}"] \arrow[d, "F_E(\sigma_k)"'] & F_{E'}(\mathbb{N}^k) \arrow[d, "F_{E'}(\sigma_k)"] \\ F_E(\mathbb{N}) \arrow[r, "\varphi_{\mathbb{N}^1}"'] & F_{E'}(\mathbb{N}) \end{tikzcd} \qquad \begin{tikzcd} F_E(\mathbb{N}^0) \arrow[r, "g"] \arrow[d, "F_E(\zeta)"'] & f_{E'}(\mathbb{N}^0) \arrow[d, "F_{E'}(\zeta')"] \\ F_E(\mathbb{N}) \arrow[r, "f"'] & F_{E'}(\mathbb{N}) \end{tikzcd}$$

$$\begin{tikzcd} F_E(\mathbb{N}^1) \arrow[r, "f"] \arrow[d, "F_E(\Delta_k)"'] & F_{E'}(\mathbb{N}^1) \arrow[d, "F_{E'}(\Delta_k)"] \\ F_E(\mathbb{N}^k) \arrow[r, "f_k"'] & F_{E'}(\mathbb{N}^k) \end{tikzcd} \qquad \begin{tikzcd} F_E(\mathbb{N}^1) \arrow[r, "f"] \arrow[d, "F_E(!)"'] & F_{E'}(\mathbb{N}^1) \arrow[d, "F_{E'}(!)"] \\ F_E(\mathbb{N}^0) \arrow[r, "g"'] & F_{E'}(\mathbb{N}^0) \end{tikzcd}$$

commute. Evaluating the functor F_E, we see that they are in fact

$$\begin{tikzcd} E_k \arrow[r, "f_k"] \arrow[d, "\sigma_k"'] & E'_k \arrow[d, "\sigma'_k"] \\ E \arrow[r, "f"'] & E' \end{tikzcd} \quad \begin{tikzcd} M \arrow[r, "g"] \arrow[d, "\zeta"'] & M' \arrow[d, "\zeta'"] \\ E \arrow[r, "f"'] & E' \end{tikzcd} \quad \begin{tikzcd} E \arrow[r, "f"] \arrow[d, "\Delta_k"'] & E' \arrow[d, "\Delta_k"] \\ E_k \arrow[r, "f_k"'] & E'_k \end{tikzcd} \quad \begin{tikzcd} E \arrow[r, "f"] \arrow[d, "q"'] & E' \arrow[d, "q'"] \\ M \arrow[r, "g"'] & M' \end{tikzcd}.$$

The first diagram commutes because (f,g) is additive and thus σ and σ' are compatible with (f,g). The second diagram commutes because (f,g) is additive and thus ζ and ζ' are compatible with (f,g). The third diagram commutes by the definition of f_k. The fourth diagram commutes because q and q' are compatible with (f,g), see Definition 2.3.5.

The two classes of morphisms Δ_k and σ_k (including $\Delta_0 = q$ and $\sigma_0 = \zeta$) generate all morphisms in $\mathbb{N}^\bullet$ as their induced map into pullbacks and composition, according to Lemma 4.2.22.

All morphisms in $\mathbb{N}^\bullet$ can be decomposed into morphisms whose components are composites of σ_k's and Δ_k's using pullbacks and projections. The pullbacks and projections are preserved by F_E since F_E is a differential functor. Therefore, φ is natural with respect to all morphisms in $\mathbb{N}^\bullet$.

(b) If (f,g) is a linear morphism, the diagram

$$\begin{tikzcd}
T(E) \arrow[r, "T(\varphi_{\mathbb{N}^1})=T(f)"] & T(E') \\
E \arrow[u, "F_E(\lambda)=\lambda"] \arrow[r, "\varphi_{\mathbb{N}^1}=f"'] & E' \arrow[u, "\lambda'=F_{E'}(\lambda)"']
\end{tikzcd}$$

commutes by Definition 2.3.5. Thus $\varphi \circ \lambda = \lambda' \circ \varphi$. Because 0 is natural with respect to f, the diagram

$$\begin{tikzcd}
T(E) \arrow[r, "T(\varphi_{\mathbb{N}^1})=T(f)"] & T(E') \\
E \arrow[u, "0_E"] \arrow[r, "\varphi_{\mathbb{N}^1}=f"'] & E \arrow[u, "0_{E'}"']
\end{tikzcd}$$

commutes. Recall from Definition 4.4.8, that the natural transformation $\hat{\alpha} : F \circ T_\bullet \Rightarrow T_\bullet \circ (1_{\mathbb{W}\mathrm{eil}} \times F)$ is constructed using $\sigma, 0$ and λ in Definition 4.4.8. The natural transformation φ commutes with all parts of $\hat{\alpha}$ and therefore with $\hat{\alpha}$, e.g. for $W^2 \in \mathbb{W}\mathrm{eil}$

$$\begin{tikzcd}
E_3 \arrow[r, "\alpha_{W^2,\mathbb{N}}=(T(\sigma)\circ\langle 0\circ\pi_0,\lambda\circ\pi_1\rangle)_2"] \arrow[d, "\varphi_{\mathbb{N}^3}"'] & T_2(E) \arrow[d, "T_2(\varphi_{\mathbb{N}})"] \\
E'_3 \arrow[r, "\alpha'_{W^2,\mathbb{N}}=(T(\sigma')\circ\langle 0\circ\pi_0,\lambda'\circ\pi_1\rangle)_2"'] & T_2(E')
\end{tikzcd}$$

commutes. This is what is required for it to be linear.

(c) The assignment on objects and morphisms follows from (a), (b) and Proposition 4.4.9. Functoriality holds, since taking the pullback of morphisms preserves compositions and the identity.

$\square$

We will now see that φ is uniquely defined as soon as we specify $\varphi_{\mathbb{N}^1}$ and $\varphi_{\mathbb{N}^0}$ as we did in the proof of

Theorem 4.4.10. This is important because we will use it to prove that `Eval` from Theorem 4.3.10 is faithful.

Proposition 4.4.11. *Natural transformations between differential functors $(F,\alpha),(F',\alpha') : \mathbb{N}^\bullet \to \mathbb{X}$ are determined by their $\mathbb{N}^0$ and $\mathbb{N}^1$ components.*

Proof. Let $\varphi, \varphi' : (F,\alpha) \to (F',\alpha')$ be natural transformations between differential functors such that $\varphi_{\mathbb{N}^1} = \varphi'_{\mathbb{N}^1}$ and $\varphi_{\mathbb{N}^0} = \varphi'_{\mathbb{N}^0}$. We will show by induction that for all $k \geq 0$, $\varphi_{\mathbb{N}^k} = \varphi'_{\mathbb{N}^k}$. The base cases hold by assumption. In the inductive step, due to naturality of φ

$$\begin{tikzcd}
F(\mathbb{N}) \arrow[r, "\varphi_{\mathbb{N}}"] & F'(\mathbb{N}) \\
F(\mathbb{N}^k) \arrow[u, "F(\pi_0)"] \arrow[r, "\varphi_{\mathbb{N}^k}"] \arrow[d, "F'(\pi_0)"'] & F'(\mathbb{N}^k) \arrow[u, "F'(\pi_0)"'] \arrow[d, "F'(\pi_1)"] \\
F'(\mathbb{N}^{k-1}) \arrow[r, "\varphi_{\mathbb{N}^{k-1}}"] & F'(\mathbb{N}^{k-1})
\end{tikzcd}$$

commutes. Replacing φ with φ' produces an analogous commuting diagram. Therefore, since $\varphi_{\mathbb{N}^1} = \varphi'_{\mathbb{N}^1}$ and $\varphi_{\mathbb{N}^{k-1}} = \varphi'_{\mathbb{N}^{k-1}}$ by the inductive hypothesis, we see that $F'(\pi_0) \circ \varphi_{\mathbb{N}^k} = F'(\pi_0) \circ \varphi'_{\mathbb{N}^k}$ and $F'(\pi_1) \circ \varphi_{\mathbb{N}^k} = F'(\pi_1) \circ \varphi'_{\mathbb{N}^k}$.

Since F' preserves pullbacks over the terminal object, $F'(\mathbb{N}^k)$ is a pullback and thus in

$$\begin{tikzcd}
F(\mathbb{N}^k) \arrow[dr, shift left, "\varphi_{\mathbb{N}^k}"] \arrow[dr, shift right, "\varphi'_{\mathbb{N}^k}"'] \arrow[drr, bend left, "\varphi_{\mathbb{N}^1}"] \arrow[ddr, bend right, "\varphi_{\mathbb{N}^{k-1}}"'] & & \\
& F'(\mathbb{N}^k) \arrow[r, "F(\pi_0)"] \arrow[d, "F(\pi_1)"'] \arrow[dr, phantom, "\lrcorner", very near start] & F'(\mathbb{N}^1) \arrow[d] \\
& F'(\mathbb{N}^{k-1}) \arrow[r] & F'(\mathbb{N}^0)
\end{tikzcd}$$

the morphisms $\varphi_{\mathbb{N}^k}$ and $\varphi'_{\mathbb{N}^k}$ are determined by their compositions with $F'(\pi_0)$ and $F'(\pi_1)$. As these compositions coincide, the morphisms $\varphi_{\mathbb{N}^k}$ and $\varphi'_{\mathbb{N}^k}$ are equal. $\square$

4.5 Establishing the equivalence

In the previous section we mentioned that `Ind` is the inverse of $\texttt{Eval}_{\mathbb{N}} : \mathbb{D}\text{iffFun}(\mathbb{N}^\bullet, \mathbb{X}) \to \mathbb{D}\text{Bun}_{\text{add}}(\mathbb{X})$. In this section we will prove how exactly they are inverse to each other. This will lead to an equivalence between categories of differential bundles and categories of lax tangent functors.

Lemma 4.5.1. *Let $\mathbb{X}$ be a tangent category.*

(a) The composition of $\texttt{Eval}$ *and* $\texttt{Ind}$ *is the identity functor:*

$$\texttt{Eval} \circ \texttt{Ind} = 1_{\mathbb{D}\mathrm{Bun}_{\mathrm{add}}(\mathbb{X})}$$

(b) $\texttt{Eval}$ *is essentially surjective.*

(c) $\texttt{Eval}$ *is full.*

Proof. (a) We will check the equality of functors on objects and morphisms. Let $(E, M, q, \zeta, \sigma, \lambda)$ be an object, i.e. a differential bundle. Then $\texttt{Ind}(E)$ is a functor that sends $\mathbb{N}^0$ to M, $\mathbb{N}^1$ to E, $q_\mathbb{N}$ to q, $\zeta_\mathbb{N}$ to ζ, $\sigma_\mathbb{N}$ to σ and $\lambda_\mathbb{N}$ to λ. Thus applying the functor $\texttt{Eval}$ which is evaluation on $(\mathbb{N}, \mathbb{N}^0, q_\mathbb{N}, \zeta_\mathbb{N}, \sigma_\mathbb{N}, \lambda_\mathbb{N})$ returns $(E, M, q, \zeta, \sigma, \lambda)$.

For an additive morphism (f, g) of differential bundles, $\texttt{Ind}(f, g)$ is a natural transformation φ such that $\varphi_\mathbb{N} = f$ and $\varphi_{\mathbb{N}^0} = g$. Now $\texttt{Eval}$ is the functor that evaluates on the $\mathbb{N}$ and $\mathbb{N}^0$ components, thus the result is again (f, g).

(b) and (c) follow from (a).

□

Lemma 4.5.2. $\texttt{Eval}$ *is faithful.*

Proof. Let φ and φ' be natural transformations between differential functors such that $\texttt{Eval}(\varphi) = \texttt{Eval}(\varphi')$. This means that $\varphi_\mathbb{N} = \varphi'_\mathbb{N}$ and $\varphi_{\mathbb{N}^0} = \varphi_{\mathbb{N}^0}$. Now it follows from Proposition 4.4.11 that $\varphi = \varphi'$ which means that $\texttt{Eval}$ is faithful. □

Theorem 4.5.3. *The functors* $\texttt{Eval}$ *and* $\texttt{Ind}$ *form equivalences*

$$\mathbb{D}\mathrm{Bun}_{\mathrm{add}}(\mathbb{X}) \cong \mathbb{D}\mathrm{iffFun}(\mathbb{N}^\bullet, \mathbb{X}) \qquad \mathbb{D}\mathrm{Bun}_{\mathrm{lin}}(\mathbb{X}) \cong \mathbb{D}\mathrm{iffFun}_{\mathrm{lin}}(\mathbb{N}^\bullet, \mathbb{X})$$

Proof. Lemma 4.5.1 and Lemma 4.5.2 together show that Eval is essentially surjective, full and faithful. This shows there is an equivalence. Let $\texttt{Eval}^{-1}$ be a pseudo-inverse of $\texttt{Eval}$ (since $\texttt{Eval}$ is an equivalence, it exists). Then

$$\texttt{Ind} \circ \texttt{Eval} \cong \texttt{Eval}^{-1} \circ \texttt{Eval} \circ \texttt{Ind} \circ \texttt{Eval} = \texttt{Eval}^{-1} \circ \texttt{Eval} \cong 1_{\mathbb{D}\mathrm{iffFun}(\mathbb{N}^\bullet, \mathbb{X})}$$

It follows that $\texttt{Eval}$ and $\texttt{Ind}$ are pseudo-inverses to each other. Restricting this equivalence to the subcategory of linear morphisms is possible due to the linear versions in Corollary 4.3.11 and Theorem 4.4.10. □

This finally is the characterization of Differential bundles as functors from $\mathbb{N}^\bullet$.

Corollary 4.5.4. `Eval` *and* `Ind` *also form equivalences*

$$\mathbb{D}\mathrm{Obj}_{\mathrm{add}}(\mathbb{X}) \cong \mathbb{D}\mathrm{iffFun}^{\mathrm{strong}}(\mathbb{N}^\bullet, \mathbb{X}) \qquad \mathbb{D}\mathrm{Obj}_{\mathrm{lin}}(\mathbb{X}) \cong \mathbb{D}\mathrm{iffFun}^{\mathrm{strong}}_{\mathrm{lin}}(\mathbb{N}^\bullet, \mathbb{X})$$

Proof. As $\mathbb{D}\mathrm{iffFun}^{\mathrm{strong}}(\mathbb{N}^\bullet, \mathbb{X})$ is the full subcategory of functors that preserve the terminal object, it is equivalent to the full subcategory of differential bundles over the terminal object, i.e. differential objects. □

This Corollary generalizes Proposition 5.9 from [6], which states that the category of differential objects with linear morphisms is equivalent to the category of differential functors and linear natural transformations. Corollary 4.5.4 generalizes the correspondence to the case of additive morphisms as well, and Theorem 4.5.3 generalizes the correspondence to differential bundles.

4.6 Concluding remarks

Differential bundles appear in the tangent category literature in a number of places. In some cases, the approach taken by other authors is different than the approach taken here. We review these contributions, highlighting the intuition about how these different approaches reference the same structure.

Relation to Cockett-Cruttwell differential bundles

In [29], Cockett and Cruttwell characterize differential bundles as fibrations. Concretely, [29, Proposition 5.12] states that a differential bundle $E \to M$ can be seen as a differential object in the fiber $\mathrm{bun}_D(\mathbb{X})[M]$ over M of the fibration $\mathrm{bun}_D(\mathbb{X}) \to \mathbb{X}$ where $\mathrm{bun}_D(\mathbb{X})$ is a full subcategory of the arrow category. For the definition of this subcategory a transerse system and a display system are chosen, which is necessary to control the existence of certain pullbacks which are preserved by the tangent structure. These include the tangent pullbacks, but are more general. This approach was further developed in [32].

From this perspective, a differential bundle is a differential object in (i.e. a strong differential functor from $\mathbb{N}^\bullet$ into) a certain full subcategory of the arrow category. The intricacies about the pullbacks are specified in the display system in the target of this functor.

The difference between the approach of [29, 32] and our current approach is that the equivalence of Theorem 4.5.3 uses general, not strong, differential functors that do not preserve the terminal object.

The existence of any necessary pullbacks comes from their existence in $\mathbb{W}\mathrm{eil}$, as well as the fact that Cartesian lineators have certain pullbacks associated to them. Preservation of pullbacks is part of the data of differential functors, not part of the target category. This is a marked difference in perspective from that of [29, 32].

For future work we propose to investigate the relation of our characterization with differential functors to display systems. In particular, one could ask, which differential functors into a tangent category with a display system correspond to display differential bundles. The key intricacy is that not every morphism in the image of $F_E : \mathbb{N}^\bullet \to \mathbb{X}$ needs to lie in the display system, but certain morphisms need to.

Relation to Ching's classification

There is another approach to differential bundles in tangent categories in [18]. In [18, Corollary 9], Ching defines differential bundles as triples of morphisms which correspond to $(q : E \to M, \zeta : M \to E, \lambda : E \to T(E))$ from Definition 2.3.1. These must fulfill certain properties. The additive structure of Ching's differential bundles comes from the natural transformation σ in the tangent structure. Ching proves that this definition of differential bundles is equivalent to the one in [29], therefore ours is also equivalent to Ching's.

The approach in [18] is very axiomatic in nature, and does not use the structure of $\mathbb{N}^\bullet$ specifically. However, the differential structure in $\mathbb{N}^\bullet$ is already given by the tangent structure together with product projections, as in Lemma 4.2.22. In [18], Ching's approach to differential bundles is to generalize this property from $\mathbb{N}^\bullet$ to any arbitrary differential bundle axiomatically in any tangent category, rather than transporting this structure from $\mathbb{N}^\bullet$ using differential functors.

Relation to tangent infinity categories

In current work in progress Arro and Ching further generalize differential bundles to tangent infinity categories. Tangent infinity categories were defined in [6] generalizing the classification of tangent categories by Leung and Garner. The definition of differential bundles in tangent infinity categories is a generalization of Theorem 4.5.3. This result characterizes differential bundles as functors from a structure category. In [6] when studying differential objects in tangent infinity categories, the structure category $\mathbb{N}^\bullet$ is generalized with a certain structure category E_∞ of spans of finite sets in order to define differential objects in a tangent infinity category. Analogously one now can define differential bundles in a tangent infinity category as differential functors from E_∞ into a tangent infinity category. One advantage our $\mathbb{W}$eil-actegory approach over the axiomatic tangent category approaches of [18, 29, 32] is that our use of $\mathbb{W}$eil-actegories compares nicely to the way tangent infinity categories are constructed. We could have characterized differential bundles using differential functors in the axiomatic tangent category setting rather than the $\mathbb{W}$eil-actegory setting, but chose to use the Garner-Leung setting for tangent actegories exactly for this reason.

Future work in connections

In [22] differential bundles were further used to describe connections, a key feature of geometry and mathematical physics. This led to one of the approaches to describe dynamical systems in [23] which rely on differential bundles in other ways.

Classically, there are two different notions of connections which, under certain circumstances, are equivalent. A Koszul connection (also known as covariant derivative) is a differential operator sending a section of a vector bundle and a tangent vector to the directional derivative of the section along the tangent vector. An Ehresmann connection is a horizontal sub-bundle of a fiber bundle. In general, in the definition of an Ehresmann connection, the fibre bundle is not required to be a vector bundle. In the special case that it is a vector bundle, there is a classical equivalence between Koszul connections and Ehresmann connections.

Both Koszul and Ehresmann connections on vector bundles were generalized in [22] as vertical and horizontal connections on differential bundles which are the categorical generalization of vector bundles. Under certain circumstances they are equivalent, in general they are not though. We would like to investigate connections via the characterization of differential bundles in this chapter, and hope that this might advance this line of inquiry by a functorial characterization of bundles with connections. This will allow us to understand the behaviour of connections under many categorical constructions. For example, if we have a characterization that states that connections on differential bundles are equivalent to certain natural isomorphisms between certain differential functors, then we could be able to obtain an equivalence between connections on a differential bundle and connections of a corresponding differential bundle in the opposite category. Since the Kleisli construction is functorial, the Kleisli category could be another example where connections could be preserved.

Future work in dynamical systems

Another construction this characterization of differential bundles can be helpful with, are dynamical systems.

Classically, dynamical systems are defined in [74] to consists of a notion of how things can be, called the states, and a notion of how things will change given how they are, called the dynamics. Often the dynamics is described by a set of differential equations.

In [23], Cockett, Cruttwell and Lemay define a dynamical system in a tangent category, generalizing systems of differential equations.

While, in general the solution of a dynamical system in [23] only requires a curve object, they obtain more results (like the existence of an exponential function) when assuming a differential curve object.

We would like to investigate differential curve objects and dynamical systems in general via the characterization of differential bundles in this chapter. The functorial perspective in this chapter could make it easier to find out which curve objects are differential curve objects.

Chapter 5

Lie groups in tangent join restriction categories

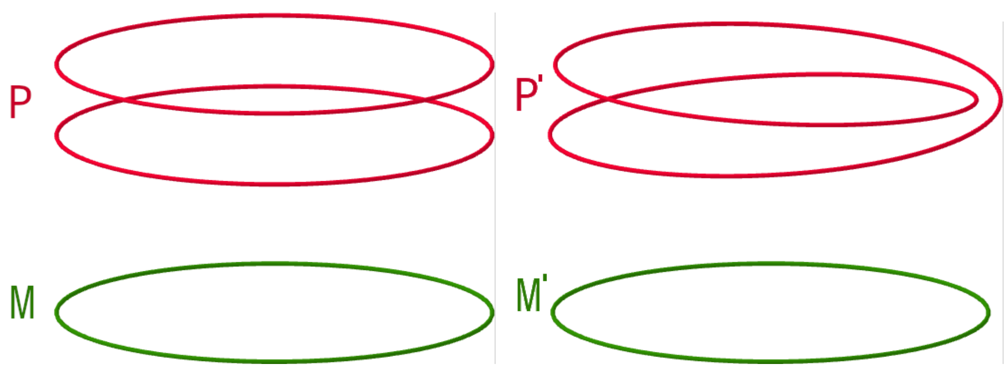

Figure 5.1: Principal bundles are locally a product of the base space and a group, but globally they might not be. Both the left and the right side the total space (red) is locally a product of the base space S^1 with two element group C_2, but globally the right hand side is not a product space [2]

In this chapter we construct a categorification of the concepts of Lie Groups and Principal bundles in differential geometry. We generalize Lie Groups as group objects in a tangent category and show that they generally fulfill several properties which are well-known for Lie Groups. We generalize principal bundles in differential geometry using restriction categories which allow to describe atlases with partial isomorphisms.

This chapter is based on [31], a preprint written by myself and my cosupervisor Robin Cockett, with small corrections and without the background section of tangent categories (since this background was already given in Chapter 2). Therefore, like in [31], we use diagrammatic composition order, i,e, we denote the composition of $f : A \to B$ and $g : B \to C$ as $fg : A \to C$.

[2]Figure created using GeoGebra 3D

5.1 Introduction

Tangent categories [20, 81] are categories equipped with an endofunctor and certain natural transformations which capture the behaviour of the tangent bundle in the category of smooth manifolds. They are a minimal setting for studying abstract differential geometry and, indeed, many aspects of differential geometry have been successfully reformulated in tangent categories. For example, vector fields, vector bundles [29, 67], connections [22], de Rham cohomology [35], and differential equations [23], all have been reformulated in tangent categories.

An important result in differential geometry which is studied in this chapter is the correspondence between Lie groups and Lie algebras, described for example in [39]. One direction of this correspondence is the observation that a Lie group – a group object in the category of manifolds and smooth maps – induces a Lie algebra by considering the tangent space over the identity. This is called the Lie algebra of the Lie group and it is classically used as a description of a Lie groups local structure.

The chapter starts by introducing join restriction categories. Then we continue with a general description of the Lie algebra of a group object in a tangent category. The construction of the Lie algebra requires that the pullback of the projection of the tangent bundle of the group over the group unit exists. This relatively mild assumption suffices to allow one to prove the classical result that differential bundle of a group object is "trivial" in the sense of being the product of the Lie algebra and the group object, see Theorem 5.3.3.

Classically there is an alternate description of the Lie algebra of a Lie group as the space of left-invariant vector fields of the Lie group (see Section II.4.11 of [71] or in Chapter 6 of [90]). In Proposition 5.3.18, we show that this correspondence holds generally for group objects in any tangent category.

Interestingly, we do not need to assume the existence of negatives in the tangent bundles: negatives are a consequence of the group structure and the existence of the required pullback. Thus, these classical results from differential geometry hold in great generality in any tangent category with the appropriate pullbacks.

Similar results for the tangent bundle of a group object in a tangent category have been developed in [2], however this chapter takes a different direction by exploring the tangent bundle of group objects and principal bundles in restriction categories.

The second main objective of the chapter is to develop the abstract theory of principal bundles. These, as we explain, can be described in any join restriction category which permits the construction of manifolds. In classical differential geometry, given a Lie group G, a principal G-bundle is a map of manifolds $E \to M$ that is locally given by the projection $U \times G \to U$ for some open subset $U \subset M$, and where the gluing between the different local pieces is mediated by the action of the group itself.

A principal G-bundle over a base space M is a total space E with a projection $E \xrightarrow{q} M$. Examples of

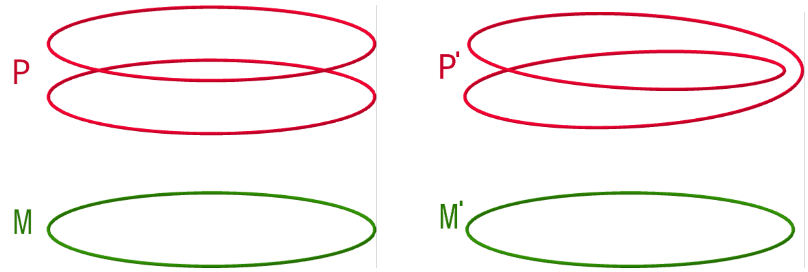

Figure 5.2: Two examples of principal C_2-bundles over S^1

principal G-bundles include the trivial bundle, $M \times G \xrightarrow{\pi_0} M$ and also spaces which are twisted over the base as in the second example in Figure 5.2.

Locally, structure, such as that of G-bundles, may be expressed as a requirement on partial maps and, as such, one can use the theory of restriction categories [24] to explore this sort of structure. Restriction categories provides an algebraic formulation of partial map categories. A partial map is a map that is defined only on a subset of its domain. These occur frequently and fundamentally in both analysis and geometry. For example, the function $f(x) = \frac{1}{x}$ is not defined on all real numbers: it is a partial map $f : \mathbb{R} \to \mathbb{R}$.

Restriction categories were systematically developed in [24], albeit the ideas themselves had a long provenance with origins in semigroup theory. Menger and his students [70] developed the basic idea of describing partial maps by what is here called a restriction. Subsequently, although independently, Di Paolo and Heller [38] used partiality to in developing a categorical approach to recursion theory. They based their formulation of partiality on the behavior of the product. Robinson and Rosolini [80] abstracted this notion of partiality based on the partial product by introducing p-categories. It were Hans-Jürgen Hoehnke and Marco Grandis, however who, in introducing what they called a "fractal category" in [52] and an "e-cohesive category" in [46], gave the first fully general formulation of a restriction category. Significantly for the development below, Grandis also developed the manifold construction for join restriction categories.

Join restriction categories were further studied in [27] and then in Xiuzhan Guo's thesis [47]. Join restriction categories are restriction categories in which morphisms that coincide on the intersection of their domains can be extended to a morphism defined on the union of their domains. This property allows the abstract construction of manifolds using atlases and gluings. Join restriction categories and their manifolds will be the basic setting in which we formulate the notion of a principal bundle.

The main objective of this work is to establish the theory of principal bundles in tangent categories. To achieve this we shall end up working in tangent join restriction categories (see Section 5.2.4).

The first results in this chapter are about the tangent space of a group object in a Cartesian tangent category, significantly Theorem 5.3.3 establishes the triviality of the tangent bundle $T(G)$. Table 5.1, then describes all the structure maps of $T(G)$ in terms of the local triviality structure. Theorem 5.3.16 establishes the interaction of tangent vector addition and group multiplication, and Theorem 5.3.21 shows the external Lie Algebra structure on the tangent space.

The next main results pertain to principal bundles in restriction categories and restriction tangent categories, as defined in Definition 5.4.14. Theorems 5.4.15, 5.4.42 and 5.4.43 show that they fulfill the desired properties. Proposition 5.4.31 and Proposition 5.4.35 show that they recover the classical notions from differential geometry and point-set topology.

Acknowledgements. For their ideas, comments, questions and discussions we would like to thank Geoff Vooys, Geoffrey Cruttwell and Rory Lucyshyn-Wright. Thanks go also to Kalin Krishna for pointing out the related work in [2]. Special thanks goes to Kristine Bauer for her extensive writing support. Florian Schwarz was funded by the Alberta Innovates Graduate Student Scholarship. Robin Cockett was partially funded by NSERC.

5.2 Join restriction categories

This section recalls the definition of join restriction categories. These will be the basis for our description of both fibre and principal bundles, see section 5.4.

Restriction categories, as defined in [24],[25] and [26], are a generalization of categories of partial maps. An important structure one can have in a restriction category is joins, which can be used to construct piecewise defined functions.

5.2.1 Restriction categories

Restriction categories as introduced in [24], capture the structure of partial maps. Here we review the definition and some results on restriction categories which will be needed in the sequel.

Definition 5.2.1 ([24], section 2.1.1, [47], Definition 3.12 and [47], Lemma 1.6.3)**.**

*(i) A **restriction category** is a category with a structure that assigns to each morphism $f : A \to B$ a morphism $\bar{f} : A \to A$ such that:*

$$\begin{array}{llll} [\mathbf{R.1}] & \bar{f}f = f & [\mathbf{R.3}] & \overline{\bar{g}f} = \bar{g}\bar{f} \\ [\mathbf{R.2}] & \bar{g}\bar{f} = \bar{f}\bar{g} & [\mathbf{R.4}] & f\bar{h} = \overline{fh}f \end{array}$$

*(ii) Two parallel morphisms $f, g : X \to Y$ in a restriction category are **compatible**, written $f \smile g$ if $\bar{f}g = \bar{g}f$*

(iii) *Two parallel morphisms in a restriction category are partially ordered by* $f \leq g$ *if and only if* $\bar{f}g = f$.

(iv) *A functor* $F : \mathbb{X} \to \mathbb{Y}$ *between restriction categories* $\mathbb{X}$ *and* $\mathbb{Y}$ *is a* ***restriction functor*** *if it preserves the restriction structure, i.e.* $F(\bar{f}) = \overline{F(f)}$ *for all morphisms* f *in* $\mathbb{X}$

Lemma 5.2.2. *Let* f *and* g *be morphisms in a restriction category, then*

(i)	$\bar{f}\bar{f} = \bar{f}$	*(ii)*	$\bar{\bar{f}} = \bar{f}$
(iii)	$\overline{f\bar{g}} = \overline{fg}$	*(iv)*	$\bar{f} \smile \bar{g}$
(v)	$f \leq g \Rightarrow f \smile g.$		

The first point means that the restriction of a morphism is idempotent, thus, morphisms of the form $\bar{f}$ are called **restriction idempotent**. The second point, $\bar{\bar{g}} = \bar{g}$, means that a repeated restriction is the same as a single restriction.

Equations (i)-(iii) are proven as part of Lemma 2.1 in [24] and (iv) and (v) are proven as part of Lemma 3.1.3 in [47]. We will still prove them here, since the proofs demonstrate how to work with restriction categories.

Proof.

(i) $\bar{f}\bar{f} \overset{[\mathbf{R.3}]}{=} \overline{\bar{f}f} \overset{[\mathbf{R.1}]}{=} \bar{f}.$

(ii) This follows from **[R.3]** if $f = 1$.

(iv) $\overline{f\bar{g}} \overset{[\mathbf{R.4}]}{=} \overline{\overline{f\bar{g}}f} \overset{[\mathbf{R.3}]}{=} \overline{f\bar{g}}\,\bar{f} \overset{[\mathbf{R.2}]}{=} \overline{\bar{f}fg} \overset{[\mathbf{R.3}]}{=} \overline{\bar{f}fg} \overset{[\mathbf{R.1}]}{=} \overline{fg}.$

(iv) $\overline{f\bar{g}} \overset{[\mathbf{R.4}]}{=} \overline{\overline{f\bar{g}}f}[\mathbf{R}_{;}^{\mathbf{.3}}][\mathbf{R.2}]|{=}\overline{\bar{f}fg} \overset{[\mathbf{R.3}]}{=} \overline{\bar{f}fg} \overset{[\mathbf{R.1}]}{=} \overline{fg}$ and thus $\overline{f\bar{g}} = \overline{fg}$. This implies that $\bar{\bar{f}}\bar{g} \overset{(i)}{=} \bar{f}\bar{g} \overset{[\mathbf{R.2}]}{=} \bar{g}\bar{f} \overset{(i)}{=} \bar{f}\bar{\bar{g}}$ and thus $\bar{f} \smile \bar{g}$.

(v) $\bar{f}g \overset{[\mathbf{R.1}]}{=} \bar{f}\bar{g}g \overset{[\mathbf{R.2}]}{=} \bar{g}\bar{f}g \overset{f \leq g}{=} \bar{g}f$ and thus $f \smile g$.

$\square$

Remark 5.2.3. *The motivating example of a restriction category is the category* $\mathbb{P}\mathrm{arSet}$ *of sets and partial maps. The objects are sets and the morphisms* $A \to B$ *are set maps of the form* $U \to B$ *where* $U \subset A$ *is a subset of* A*. Given a partial map* $f : A \to B$*, its restriction idempotent* $\bar{f}$ *is defined as the inclusion* $U \hookrightarrow A$ *of the domain of definition* U *into the domain* A *of* f*. We often interpret a restriction idempotent to describe the subset* $U \subset A$*. In this example*

(i) $\bar{f}g$ *is the restriction of* g *to* f*'s domain of definition,*

(ii) *and* $\overline{fg}$ *describes the preimage* $f^{-1}(V)$ *of* g*'s domain of definition* V *under* f.

With these interpretations in mind, the properties [R.1]-[R.4] are just common properties of partial maps.

As shown in section 2.1.3 of [24], restriction categories generalize partial map categories. The corresponding notions of being a total morphism and a partial isomorphism need to be described:

Definition 5.2.4.

*(i) A **total morphism** is a morphism $f : A \to B$ such that $\bar{f} = 1_A$*

*(ii) A **partial isomorphism** is a morphism $f : A \to B$ such that there is a morphism $f^* : B \to A$ such that $ff^* = \bar{f}$ and $f^*f = \bar{f^*}$. The morphism f^* is called the **partial inverse** of f.*

Partial inverses are unique and will be denoted by f^*.

Given a restriction category $\mathbb{X}$ one can take its full subcategory of total morphisms and obtain a category $\mathtt{Tot}(\mathbb{X})$. On the other hand, given a category $\mathbb{Y}$ one can see it as a restriction category $\mathtt{Triv}(\mathbb{X})$ by defining $\bar{f} = 1$ for all morphisms f.

Proposition 5.2.5. *The assignments* $\mathtt{Tot} : \mathbb{R}\mathrm{esCat} \to \mathbb{C}\mathrm{at}$ *and* $\mathtt{Triv} : \mathbb{C}\mathrm{at} \to \mathbb{R}\mathrm{esCat}$ *are functors and* $\mathtt{Tot}$ *is right-adjoint to the inclusion* $\mathtt{Triv}$*. Thus the 1-category* $\mathbb{C}\mathrm{at}$ *of categories is a coreflective subcategory of* $\mathbb{R}\mathrm{esCat}$*.*

Proof. In order to obtain functors we first need to define $\mathtt{Tot}$ and $\mathtt{Triv}$ on morphisms. Let $F : \mathbb{X} \to \mathbb{Y}$ be a restriction functor. Then as restriction functors preserve totals we can define $\mathtt{Tot}(F) := F|_{\mathtt{Tot}(X)} : \mathtt{Tot}(\mathbb{X}) \to \mathtt{Tot}(\mathbb{Y})$. Since $\mathtt{Tot}(F)$ it is a restriciton of F to a subcategory, $\mathtt{Tot}$ is automatically functorial.

Similarly, given a functor $F : \mathbb{X} \to \mathbb{Y}$ of ordinary categories, we can see it as a restriction functor $\mathtt{Triv}(F) : \mathtt{Triv}(\mathbb{X}) \to \mathtt{Triv}(\mathbb{Y})$. The functor $\mathtt{Triv}$ is automatically functorial as it does not change the underlying assignment of objects and morphisms.

Next we show the adjunction by showing that there is an isomorphism between the hom-sets

$$\mathbb{R}\mathrm{esCat}(\mathtt{Triv}(\mathbb{X}), \mathbb{Y}) \cong \mathbb{C}\mathrm{at}(\mathbb{X}, \mathtt{Tot}(\mathbb{Y}))$$

that is natural in functors $\mathbb{X} \to \mathbb{X}'$ and restriction functors $\mathbb{Y} \to \mathbb{Y}'$. Given a restriction functor $F : \mathtt{Triv}(\mathbb{X}) \to \mathbb{Y}$, this functor preserves total morphisms. As every morphism in $\mathtt{Triv}(\mathbb{X})$ is total this means that every morphism in the image of the functor is total. Thus the assignment of objects and morphisms that is F forms a functor $\varphi(F) : \mathbb{X} \to \mathtt{Tot}(\mathbb{Y})$ that sends an object X of $\mathbb{X}$ to $F(X)$ and a morphism f of X to $F(f)$.

Since the function $\varphi : \mathbb{R}\mathrm{esCat}(\mathtt{Triv}(\mathbb{X}), \mathbb{Y}) \to \mathbb{C}\mathrm{at}(\mathbb{X}, \mathtt{Tot}(\mathbb{Y}))$ does not change the assignments of objects and morphisms, it is automatically natural and injective. It is surjective because every functor $\mathbb{X} \to \mathtt{Tot}(\mathbb{Y})$

can be seen as a restriction functor. $\square$

5.2.2 Limits and products

Since restriction categories are enriched in partial orders, it turns out to be natural to consider lax limits:

Definition 5.2.6 ([26], section 4.4)**.** *Let $\mathbb{X}$ be a restriction category, $\mathbb{A}$ a finite category and $S : \mathbb{A} \to \mathbb{X}$ a diagram.*

(i) A lax cone of the diagram S is an object L of $\mathbb{X}$ together with (for every object C of $\mathbb{A}$) a morphism $q_C : L \to S(C)$ such that for every morphism $c \in \mathrm{Hom}_{\mathbb{A}}(C, D)$, $S(c)q_C \leq q_D$.

(ii) A restriction limit is a lax cone (L, p_C) with total morphisms p_C such that for every other cone (M, q_C) there is an unique morphism $f : M \to L$ such that for all objects C of $\mathbb{A}$, $fp_C = eq_C$ holds, where $e = \prod_{C \in \mathbb{A}_0} \bar{q}_C$.

An example for this definition is the following.

Example 5.2.7. *Let $\mathbb{X}$ be the restriction category with topological spaces as objects and partial continuous maps defined only on closed subsets as morphisms. Then the pullback of the maps $i_{[0,1]}, i_{[-1,0]} : \mathbb{R} \to \mathbb{R}$ given by the inclusion of $[0,1]$ and the inclusion of $[-1,0]$ is the object $\{0\}$ with the projections p_0, p_1 embedding it into $\mathbb{R}$ as zero.*

$$\begin{tikzcd}
\mathbb{R} \arrow[dr, dashed] \arrow[drr, "q_0 = i_0"] \arrow[ddr, "q_1 = i_{[0,1]}"'] & & \\
& \{0\} \arrow[r, "p_0 = i_0"'] \arrow[d, "p_1 = i_0"] & \mathbb{R} \arrow[d, "i_{[0,1]}"] \\
& \mathbb{R} \arrow[r, "i_{[-1,0]}"'] & \mathbb{R}
\end{tikzcd}$$

Now let us consider another cone $(\mathbb{R}, q_0, q_1)$ with $q_0 : \mathbb{R} \to \mathbb{R}$ sending 0 to 0 and not defined anywhere else and $q_1 : \mathbb{R} \to \mathbb{R}$ embedding the interval $[0,1]$. Then the unique map $f : \mathbb{R} \to 0$ fulfilling the conditions is the map sending 0 to 0 and that is nowhere else defined.

This is an example where $fp_1 \neq q_1$ as q_1 is defined on $[0,1]$ whereas fp_1 is defined only on $\{0\}$. However, as e is the restriction to $\{0\}$, $fp_1 = q_1 e$ holds.

In order to have a better grasp on restriction pullbacks, we will expand that definition in the special case of pullbacks of total morphisms. We will heavily rely on this Lemma in the proof of Theorem 5.4.43.

Lemma 5.2.8. *Let $A \xrightarrow{f} B \xleftarrow{g} C$ be a cospan of total morphisms in a restriction category. Let X be an object and let $p_A : X \to A$ and $p_C : X \to C$ be total morphisms such that the diagram*

$$\begin{tikzcd} X \arrow[r, "p_A"] \arrow[d, "p_C"'] & A \arrow[d, "f"] \\ C \arrow[r, "g"'] & B \end{tikzcd}$$

commutes, i.e. $p_A f = p_C g$. Then the following are equivalent:

(i) For any object Z with morphisms $r_A : Z \to A$, $r_C : Z \to C$ such that the diagram

$$\begin{tikzcd} Z \arrow[drr, bend left, "r_A"] \arrow[ddr, bend right, "r_C"'] & & \\ & X \arrow[r, "p_A"] \arrow[d, "p_C"'] & A \arrow[d, "f"] \\ & C \arrow[r, "g"'] & B \end{tikzcd}$$

commutes (i.e. $r_A f = r_C g$), there is an unique morphism $\phi : Z \to X$ with $\phi p_A = r_A$ and $\phi p_C = r_C$.

(ii) The cone $(X, (q_A, q_A f, q_C))$ is the restriction pullback of the cospan $A \xrightarrow{f} B \xleftarrow{g} C$.

The key difference between Lemma 5.2.8(i) and Definition 5.2.6 is that in Lemma 5.2.8(i) equalities hold whereas in Definition 5.2.6 cones commute only up to inequality and $f p_C$ differs from q_C by a restriction idempotent. Thereby condition (i) is easier and closer to the classical definition of a limit.

Proof. The implication $(ii) \Rightarrow (i)$ follows from Definition 5.2.6 since, due to the totality of f and g, $\bar{r}_A = \bar{r}_C$ and thus $\phi p_A = \bar{r}_C r_A = r_A$.

For the implication $(i) \Rightarrow (ii)$, suppose (i) and a lax cone $(Z, (q_A, q_B, q_C))$. Since f is total and $(Z, (q_A, q_B, q_C))$ is a cone, we know that $q_A f \leq q_B$ which implies $\bar{q}_A \leq \bar{q}_B$ and analogously $\bar{q}_C \leq \bar{q}_B$. Define $r_A = \bar{q}_C q_A$ and $r_C = \bar{q}_A q_C$. Since $(Z, (q_A, q_B, q_C))$ was a lax cone, they fulfill that

$$r_A f = \bar{q}_C q_A f = \bar{q}_C \bar{q}_A q_B = \bar{q}_A q_C g = r_C g.$$

Thus, by the assumption (i), there is an unique morphism $\phi : Z \to X$ with $\phi p_A = r_A$ and $\phi p_C = r_C$. From this it follows that ϕ is an unique morphism $Z \to X$ fulfilling

$$\phi p_A = \bar{q}_C q_A = \bar{q}_C \bar{q}_B \bar{q}_A q_A, \quad \phi p_A f = \bar{q}_C q_A f = \bar{q}_C \bar{q}_B \bar{q}_A q_B \quad \text{and} \quad \phi p_C = \bar{q}_A q_C = \bar{q}_C \bar{q}_B \bar{q}_A q_C.$$

Thus the cone $(X, (q_A, q_A f, q_C))$ fulfills Definition 5.2.6 proving the implication $(i) \Rightarrow (ii)$. $\square$

Using restriction limits we can define additional structures that we will use in section 5.4. A **Cartesian restriction category** is a restriction category that has all restriction products and a restriction terminal object 1. A **restriction tangent category** is a restriction category that has a tangent category structure where all the limits are restriction limits. It will be defined in more detail in Definition 5.2.14.

5.2.3 Joins, atlases and gluings

Defined in [47], a **join restriction category** $\mathbb{X}$ is a restriction category such that for each $A, B \in X$ and each set $S \subset \mathbb{X}(A, B)$ of compatible morphisms there is a join (least upper bound, or supremum)

$$\bigvee_{s \in S} s \in \mathbb{X}(A, B)$$

which fulfills the following properties

(i) $\overline{\bigvee_{s\in S} s} = \bigvee_{s\in S} \bar{s}$

(ii) $g(\bigvee_{s\in S} s) = \bigvee_{s\in S}(gs)$

(iii) it is a join (supremum) with respect to the partial ordering $\leq$, i.e. $f_i \leq g \forall \iota \Rightarrow \bigvee f_i \leq g$

Remark 5.2.9. *We will sometimes draw commutative diagrams in restriction categories. When we say that a diagram commutes, that means that the compositions are equal as morphisms in the restriction category. In particular, their domains of definition coincide. For example, when the diagram*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[d, "g"] \\ C \arrow[r, "k"'] & D \end{tikzcd}$$

commutes, it means that $fg = hk$, implying that both $fg \smile hk$ and $\overline{fg} = \overline{hk}$.

In classical differential geometry, manifolds are described by atlases which consist of local neighbourhoods and instructions for how to glue them together. Now in order to describe objects that are built piecewise out of multiple components in a join restriction category, we define atlases to consist of objects $(U_i)_{i\in I}$ as components and morphisms u_{ij} as instructions how to glue U_i and U_j together. This is known as the manifold construction that was introduced by Grandis in [46] and specified for join restriction categories by Cockett and Cruttwell in [20].

Definition 5.2.10 ([46],[20])**.** *In a join restriction category, an **atlas** is a collection of objects $\{U_i\}_{i\in I}$ and morphisms $\{u_{ij}\}_{i,j\in I}$ with $u_{ij} : U_i \to U_j$, fulfilling* *(i)* $u_{ii}u_{ij} = u_{ij} : U_i \to U_j$,

(ii) $u_{ij}u_{jk} \leq u_{ik} : U_i \to U_k$, *and*

(iii) $u_{ij}u_{ji} = \overline{u_{ij}} : U_i \to U_i$.

Definition 5.2.11 ([46],[20])**.** *An atlas morphism* $(U_i, u_{ij}) \to (V_k, v_{kl})$ *is a family of morphisms* $A_{ik} : U_i \to V_k$ *such that*

(i) $u_{ii}A_{ik} = A_{ik} : U_i \to V_k$,

(ii) $A_{ik}v_{kk} = A_{ik} : U_i \to V_k$,

(iii) $u_{ij}A_{jk} \leq A_{ik} : U_i \to V_k$,

(iv) $A_{ik}v_{kl} \leq A_{il} : U_i \to V_l$, *and*

(v) $A_{ik}v_{kl} = \overline{A_{ik}}A_{il} : U_i \to V_l$.

Condition 5.2.11.(iv) is actually a direct consequence of condition 5.2.11.(v). A trivial example of an atlas is obtained by choosing an object $U = U_0$, taking the index-set to be $\{0\}$ and $u_{00} = 1_U$. We will denote this atlas as $\mathrm{At}(1_U)$

An atlas is a specification of how to glue things together. The gluing construction from [46] and [20] makes this possible.

Definition 5.2.12. *Let* (U_i, u_{ij}) *be an atlas in a join restriction category* $\mathbb{X}$*. Then a* ***gluing*** *of it is an object* G *together with an atlas morphism* $g : (U_i, u_{ij}) \to \mathrm{At}(1_G)$ *that fulfills an universal property: for any object* X *and atlas morphism* $f : (U_i, u_{ij}) \to \mathrm{At}(1_X)$ *there is an unique morphism* $\check{f} : G \to X$ *of* $\mathbb{X}$ *such that* $f_i = g_i\check{f}$ *and if there is another atlas morphism* $f' : (U_i, u_{ij}) \to \mathrm{At}(1_X)$ *fulfilling* $f_i \leq f'_i$*, then* $\check{f} \leq \check{f}'$*.*

Equivalently but more explicitly, the gluing can be characterized by the components of the atlas morphism g. Since the target has only one object, the components A_{ik} (as in Definition 5.2.11) can be written as g_i with only one index.

Remark 5.2.13. *A gluing for* $\{u_{ij}\}$ *is an object* G_U *with a family of partial isomorphisms* $g_i : U_i \to G_U$ *(with partial inverses* g_i^**) such that*

(i) $u_{ij}g_j \leq g_i$,

(ii) $u_{ij} = g_i g_j^*$, *and*

(iii) $1_{G_U} = \bigvee_i g_i^* g_i$.

In fact condition (i) is redundant and follows from condition (ii) since $u_{ij}g_j = g_i g_j^* g_j = g_i \bar{g}_j \leq g_i$.

5.2.4 Tangent join restriction categories

The category of smooth manifolds and smooth maps was the motivating example for tangent categories. This category has total maps as morphisms. The category of smooth manifolds and smooth partial maps, defined on open subsets, is a restriction category. It is the motivating example of tangent restriction categories which encode both the local structure and the tangent structure of manifolds. Tangent join restriction categories combine the tangent category structure from Chapter 2 with the join restriction category structure from Section 5.2.1. This definition was given in [20].

Definition 5.2.14 (Definition 6.14 of [20])**.**
*A **tangent restriction structure** for a join restriction category $\mathbb{X}$ consists of a restriction preserving functor $T : \mathbb{X} \to \mathbb{X}$ and associated total transformations such that*

- *for each $M \in \mathbb{X}$, we have total morphisms $p_M : TM \to M$, restriction pullbacks $T_n(M)$ and total morphisms $+_M : T_2M \to TM, 0_M : M \to TM$ such that for each $f : M \to N$, the pair (Tf, f) is an additive bundle morphism;*
- *for each $n, k \in \mathbb{N}, T^n$ preserves the restriction pullbacks of k copies of p.*
- *there is a total natural transformation $T \xrightarrow{\ell} T^2$ such that for each M, the pair $(\ell_M, 0_M)$ is an additive bundle morphism from $(Tp : T^2M \to TM)$ to $(p_T : T^2M \to TM)$;*
- *there is a total natural transformation $T^2 \xrightarrow{c} T^2$ such that for eacch M the pair $(c_M, 1)$ is an additive bundle morphism from $(Tp : T^2M \to TM)$ to $(p_T : T^2M \to TM)$;*
- *we have $c^2 = 1, \ell c = \ell$ and the following diagrams commute:*

$$\begin{tikzcd} T \arrow[r, "\ell"] \arrow[d, "\ell"'] & T^2 \arrow[d, "T(\ell)"] \\ T^2 \arrow[r, "\ell_T"'] & T^3 \end{tikzcd} \qquad \begin{tikzcd} T^3 \arrow[r, "T(c)"] \arrow[d, "c_T"'] & T^3 \arrow[r, "c_T"] & T^3 \arrow[d, "T(c)"] \\ T^3 \arrow[r, "T(c)"'] & T^3 \arrow[r, "c_T"'] & T^3 \end{tikzcd} \qquad \begin{tikzcd} T^2 \arrow[r, "\ell_T"] \arrow[d, "c"'] & T^3 \arrow[r, "T(c)"] & T^3 \arrow[d, "c_T"] \\ T^2 \arrow[rr, "T(\ell)"'] & & T^3 \end{tikzcd}$$

- *the diagram*

$$\begin{tikzcd} T_2M \arrow[r, "\langle \pi_0 \ell, \pi_1 0_T \rangle T(+)"] \arrow[d, "\pi_0 p = \pi_1 p"'] & T^2M \arrow[d, "T(p)"] \\ M \arrow[r, "0_M"'] & TM \end{tikzcd}$$

  is a restriction pullback.

*If $\mathbb{X}$ has restriction products and T preserves them, then $(\mathbb{X}, T)$ is a **Cartesian tangent restriction category**.*

The motivating example for a Cartesian tangent join restriction category is the category of smooth manifolds and partial smooth maps. We will analyze this example further in section 5.4.3.

If $(\mathbb{X}, T)$ is any tangent category it can be turned into a tangent restriction category by defining $\bar{f} = 1_A$ for all morphisms $f : A \to B$ in $\mathbb{X}$.

If $\mathbb{X}$ is any restriction category it can be turned into a tangent restriction category by choosing the tangent structure $T := 1_{\mathbb{X}}$ to be the identity.

Proposition 5.2.15. *Let $(\mathbb{X}, T)$ be a tangent restriction category. Then $(\mathtt{Tot}(\mathbb{X}), \mathtt{Tot}(T))$ is an ordinary tangent category.*

Proof. Let $(\mathbb{X}, T, p, 0, +, \ell, c)$ be a tangent restriction category. We proceed by checking that its subcategory of total maps, $(\mathrm{Tot}(\mathbb{X}), \mathrm{Tot}(T), p, 0, +, \ell, c)$, fulfills the axioms in Definition 2.2.4. As T was a restriction functor, $\mathrm{Tot}(T)$ is an endofunctor $\mathrm{Tot}(\mathbb{X}) \to \mathrm{Tot}(\mathbb{X})$. Since $p, 0, +, \ell, c$ were total morphisms in $\mathbb{X}$ they are morphisms in $\mathrm{Tot}(\mathbb{X})$. For every object $X \in \mathrm{Obj}(\mathbb{X}) = \mathrm{Obj}(\mathrm{Tot}(\mathbb{X}))$ pullback powers of $\mathrm{Tot}(T)(X) \xrightarrow{p} X$ exist and are given by $\mathrm{Tot}(T)_n(X) = T_n(X)$ because restriction limits of total morphisms are limits in $\mathrm{Tot}(\mathbb{X})$ according to section 4.4 of [26]. All the equations hold in $\mathrm{Tot}(\mathbb{X})$ as they hold in $\mathbb{X}$. The universality of the vertical lift holds because restriction limits of total morphisms in $\mathbb{X}$ become limits in $\mathrm{Tot}(\mathbb{X})$. □

Interestingly we did not need to ask for the tangent bundle endofunctor to be compatible with joins, only with restrictions. The reason is the following lemma which says that compatibility with the restriction implies compatibility with the join for a tangent structure. We will use it later in Section 5.4.5.

Lemma 5.2.16.

(i) Let $(\mathbb{X}, T, p, 0, \ell, c)$ be a Cartesian tangent join restriction category. Then for any parallel morphisms f, g in $\mathbb{X}$, $f \leq g$ if and only if $T(f) \leq T(g)$.

(ii) Let $(\mathbb{X}, T, p, 0, \ell, c)$ be a Cartesian tangent join restriction category. Then for any parallel morphisms f, g in $\mathbb{X}$, $f \smile g$ if and only if $T(f) \smile T(g)$.

(iii) Let $(\mathbb{X}, T, p, 0, \ell, c)$ be a Cartesian tangent join restriction category and let $(f_i)_{i \in I}$ be a compatible family of morphisms in $\mathbb{X}$. Then

$$\bigvee_{i \in I} T(f_i) = T\left(\bigvee_{i \in I} f_i\right).$$

Proof.

(i) If $f \leq g$, that means that $\bar{f}g = f$. Thus, applying T to the equation we obtain that

$$\overline{T(f)}T(g) = T(\bar{f})T(g) = T(\bar{f}g) = T(f)$$

which means that $T(f) \leq T(g)$. Conversely, if $T(f) \leq T(g)$, that means that $\overline{T(f)}T(g) = T(f)$. Due to naturality of p, the equation $f = 0T(h)p$ holds for any morphism h. Thus we obtain that

$$\bar{f}g = 0T(\bar{f}g)p = 0\overline{T(f)}T(g)p = 0T(f)p = f$$

which means that $f \leq g$.

(ii) This is analogous to (i).

(iii) Due to (i), $f_i \leq \bigvee_{i\in I} f_i$ implies that $F(f_i) \leq F(\bigvee_{i\in I} f_i)$. Thus, since the join is a supremum, this means that

$$\bigvee_{i\in I} F(f_i) \leq F\left(\bigvee_{i\in I} f_i\right).$$

Next we will use that the projection $p : T \to 1_{\mathbb{X}}$ is a natural transformation with total components, thus post-composition with it does not change the restriction idempotent. Thus,

$$\overline{F\left(\bigvee_{i\in I} f_i\right)} = \overline{F\left(\bigvee_{i\in I} f_i\right)p} = \overline{p\bigvee_{i\in I} f_i} = \bigvee_{i\in I} \overline{pf_i} = \bigvee_{i\in I} \overline{F(f_i)p} = \overline{\bigvee_{i\in I} F(f_i)}$$

holds. This implies the required equation.

$\square$

Note that the inequality $\bigvee_{i\in I} F(f_i) \leq F\left(\bigvee_{i\in I} f_i\right)$. holds for any restriction preserving functor, while equality requires the projection natural transformation.

5.3 The tangent bundle of a group object

In classical Lie theory, the tangent bundle of a Lie group $G = (G, \cdot, u)$ is trivial in the sense that $T(G) = G\times\mathfrak{g}$, where $\mathfrak{g} = T(G)_u$ is the Lie algebra of G. In this section we will show that the tangent bundle of a group object in any tangent category is also trivial. This recovers the classical result, but is more general, since it now holds in many different categories.

While Section 5.4 is about join restriction categories, this section does not assume join restriction structures. The results pertain to Cartesian tangent categories in general. Let us start by stating the definition of a group object:

Definition 5.3.1. *A **group object** (G, m, u, ι) in a Cartesian category $\mathbb{X}$ is an object G together with morphisms $m : G \times G \to G$, $u : 1 \to G$ and $\iota : G \to G$ (where 1 is the terminal object) such that* *(i)* $(m \times 1_G)m = (1_G \times m)m \quad : \quad G \times G \times G \to G$

(ii) $\langle !u, 1_G \rangle m = 1_G = \langle 1_G, !u \rangle m \quad : \quad G \to G$

(iii) $\langle \iota, 1_G \rangle m = !u = \langle 1_G, \iota \rangle m \quad : \quad G \to G,$

where $! : A \to 1$ *is the unique morphism to the terminal object and* $\langle f, g \rangle$ *is the induced morphism into a pullback or product. In the special case of morphisms between products, we use the notation* $f \times g := \langle \pi_0 f, \pi_1 g \rangle$.

If $\mathbb{X}$ *is a Cartesian restriction category, we demand that the morphisms* m, u *and* ι *are total.*

While a group object consists of G, m, u and ι we often denote the group object (G, m, u, ι) by G for brevity. The concept of a group object is used in most areas of mathematics. In the category of sets and functions, groups are exactly the group objects. In the category of algebraic varieties and algebraic maps, algebraic groups are exactly the group objects. In the category of topological spaces with continuous maps, topological groups are exactly the group objects. In the category of smooth manifolds with smooth maps, Lie groups are exactly the group objects.

Lemma 5.3.2. *If* G *is a group object in a Cartesian tangent category* $(\mathbb{X}, T)$, $T(G)$ *is also group object.*

Proof. Since $(\mathbb{X}, T)$ is a Cartesian tangent category, $\langle T(\pi_0), T(\pi_1) \rangle : T(G \times G) \to T(G) \times T(G)$ is an isomorphism. The multiplication is $T(G) \times T(G) \overset{\langle T(\pi_0), T(\pi_1) \rangle}{\cong} T(G \times G) \xrightarrow{T(m)} T(G)$, the unit is given by $1 \overset{0_1}{\cong} T(1) \xrightarrow{T(u)} T(G)$, and the inverse is given by $T(G) \xrightarrow{T(\iota)} T(G)$. These data satisfy the required identities because T is functorial. $\square$

5.3.1 Trivializing the tangent bundle

The purpose of this section is to define an isomorphism between $T(G)$ and $G \times T(G)_u$. Here $T(G)_u$ is the pullback

$$\begin{tikzcd}
T(G)_u \arrow[r, "p_u^*"] \arrow[d, "!"'] \arrow[dr, phantom, "\lrcorner", very near start] & T(G) \arrow[d, "p"] \\
1 \arrow[r, "u"'] & G
\end{tikzcd}$$

which we require to exist.

While the first component of the isomorphism is simply the projection $p : T(G) \to G$, the second component is defined using the universal property of the pullback defining $T(G)_u$. The idea of the second component is to start with a tangent vector $v_g \in T(G)$ over the basepoint g and use the group multiplication with g^{-1} to move the tangent vector v_g over the unit. Concretely, the multiplication $T(m)$ multiplies $p\iota 0$ of a tangent vector to the tangent vector, as shown in the diagram below.

$$
\begin{tikzcd}
& T(G) \arrow[dr, "\langle p,1_{T(G)}\rangle"] \arrow[dddl] \arrow[dd, dashed, "\langle p\iota 0,1\rangle T(m)"] \\
&& G \times T(G) \arrow[dr, "\iota\times 1_{T(G)}"] \\
& T(G)_u \arrow[dl] \arrow[drr] \arrow[d, phantom, "\lrcorner"] && G \times T(G) \arrow[dr, "0\times 1_{T(G)}"] \\
& {} &&& T(G)\times T(G) \arrow[dl, "T(m)"] \\
1 \arrow[dr, "u"'] &&& T(G) \arrow[dll, "p"] \\
& G
\end{tikzcd}
$$

Together they form the morphism

$$\phi = \langle p, \langle !, \langle p\iota 0, 1\rangle T(m)\rangle\rangle : T(G) \to G \times T(G)_u.$$

Theorem 5.3.3. *Suppose G is a group object in a Cartesian tangent category such that the pullback $T(G)_u$ exists. Then $T(G) \cong G \times T(G)_u$. The isomorphism is given as*

$$\phi = \langle p, \langle !, \langle p\iota 0, 1\rangle T(m)\rangle\rangle : T(G) \to G \times T(G)_u$$

and its inverse is

$$\phi^{-1} = (0 \times p_u^*)T(m) : G \times T(G)_u \to T(G).$$

Proof. The morphism $\langle p\iota 0, 1\rangle T(m) : T(G) \to T(G)$ satisfies

$$\langle p\iota 0, 1\rangle T(m)p = \langle p\iota 0, 1\rangle pm = \langle p\iota 0p, p\rangle m = p\langle \iota, 1\rangle m = p!0 = !0$$

and therefore it induces a morphism $\langle !, \langle p\iota 0, 1\rangle T(m)\rangle : T(G) \to T(G)_u$. The composition $p! = !$, because it is the unique morphism from $T(G)$ to the terminal object 1. The candidate for its inverse is

$$(0 \times p_u^*)T(m) : G \times T(G)_u \to T(G)$$

where p_u^* is the pullback of $p : T(G) \to G$ along $u : 1 \to G$ defining $T(G)_u$:

$$\begin{tikzcd}
T(G)_u \arrow[r, "p_u^*"] \arrow[d, "!"'] & T(G) \arrow[d, "p"] \\
1 \arrow[r, "u"'] & G
\end{tikzcd}$$

Now we need to check that they are inverse to each other by checking that both ways of composing yield the identity morphism. First, we have

$$\begin{aligned}
\langle p, \langle !, \langle p\iota 0, 1\rangle T(m)\rangle\rangle(0 \times p_u^*)T(m) &= \langle p0, \langle p\iota 0, 1\rangle T(m)\rangle T(m) \\
\text{(by associativity)} \qquad &= \langle\langle p0, p\iota 0\rangle T(m), 1\rangle T(m) \\
&= \langle p\langle 1, \iota\rangle m0, 1\rangle T(m) \\
&= \langle p!u0, 1\rangle T(m) \\
&= 1,
\end{aligned}$$

where the last equality holds because $u0 = 0T(u)$ by naturality of 0 and this is the unit for the multiplication $T(m)$ from Lemma 5.3.2. On the other hand,

$$\begin{aligned}
& (0 \times p_u^*)T(m)\langle p, \langle !, \langle p\iota 0, 1\rangle T(m)\rangle\rangle \\
&= \langle (0 \times p_u^*)T(m)p \ , \ \langle !, \langle (0 \times p_u^*)T(m)p\iota 0, (0 \times p_u^*)T(m)\rangle T(m)\rangle\rangle \\
\text{(by naturality of } p\text{)} \qquad &= \langle (0p \times p_u^* p)m \ , \ \langle !, \langle (0p \times p_u^* p)m\iota 0, (0 \times p_u^*)T(m)\rangle T(m)\rangle\rangle \\
&= \langle (1{\times}!u)m \ , \ \langle !\langle (1{\times}!u)m\iota 0, (0 \times p_u^*)T(m)\rangle T(m)\rangle\rangle \\
&= \langle \pi_0 \ , \ \langle !, \langle \pi_0\iota 0, (0 \times p_u^*)T(m)\rangle T(m)]\rangle \\
\text{(by associativity)} \qquad &= \langle \pi_0 \ , \ \langle !, \langle\langle \pi_0\iota 0, \pi_0 0\rangle T(m), \pi_1 p_u^*\rangle T(m)\rangle\rangle \\
&= \langle \pi_0 \ , \ \langle !, \langle \pi_0 !u0, \pi_1 p_u^*\rangle T(m)\rangle\rangle \\
&= \langle \pi_0 \ , \ \langle !, \langle \pi_0 !u0, \pi_1 p_u^*\rangle T(m)\rangle\rangle \\
&= \langle \pi_0, \pi_1\rangle = 1.
\end{aligned}$$

Here again the last equality holds as $u0 = 0T(u)$ by naturality of 0 and this is the unit for the multiplication $T(m)$ from Lemma 5.3.2.

As the first component of the isomorphism $\langle p, \langle !, \langle p\iota 0, 1\rangle T(m)\rangle\rangle$ is p, it follows directly that

$$\langle p, \langle !, \langle p\iota 0, 1\rangle T(m)\rangle\rangle \pi_0 = p$$

and therefore p corresponds to π_0. □

5.3.2 Structure maps of the tangent bundle of a group object

We will now continue to analyze the behaviour of G and $T(G)_u$. In particular, G acts on $T(G)_u$ in a way that recovers the adjoint action of a Lie group on its Lie algebra in differential geometry. In the classical case in which G is a Lie group, it is given by sending a group element $g \in G$ and a vector $v \in T(G)_u$ to $g \cdot v \cdot g^{-1}$. This provides intuition for what follows.

Recall that Definition 5.3.1 guarantees that the multiplication $m : G \times G \to G$ is associative. Thus, we denote for $k \in \mathbb{N}^+$ the unique morphism from G^k to G multiplying all components together as $m_k : G^k \to G$. One expression of m_k as a composition is $(m \times 1_G \times ... \times 1_G)...m$, but any other composition of m that leads to a morphism $G^k \to G$ is the same morphism.

Definition 5.3.4.

$$\mathrm{Ad}_\ell : G \times T(G)_u \to T(G)_u$$

is defined as the morphism

$$\langle !, \langle \pi_0 0_G, \pi_1 p_u^*, \pi_0 \iota 0\rangle T(m_3)\rangle$$

which is the induced morphism into the pullback

$$\begin{tikzcd}
G \times T(G)_u \arrow[drr, "\langle \pi_0 0, \pi_1 p_u^*, \pi_0 \iota 0\rangle T(m_3)"] \arrow[ddr, "!"'] & & \\
& T(G)_u \arrow[r, "p_u^*"'] \arrow[d, "!"'] \arrow[dr, phantom, "\lrcorner"] & T(G) \arrow[d, "p"] \\
& 1 \arrow[r, "u"'] & G
\end{tikzcd},$$

where $m_3 = (m \times 1)m : G \times G \times G \to G$ is the multiplication of three group elements in G.

*The **adjoint right group action** is the adjoint action with the inverse of a group element, i.e. it is defined by*

$$\mathrm{Ad}_r = \langle !, \langle \pi_1 \iota 0, \pi_0 p_u^*, \pi_1 0\rangle T(m_3)\rangle : T(G)_u \times G \to T(G)_u.$$

Here the ℓ and r in the index keep track which of them is a left- and which of them is a right-action. The actions Ad_r and Ad_ℓ are the generalizations of gvg^{-1} and $g^{-1}vg$ in the classical case.

In order to see that Definition 5.3.4 is well-defined we need to check that it actually induces a morphism into the pullback, that is $\langle \pi_0 0, \pi_1 p_u^*, \pi_0 \iota 0\rangle T(m_3)p = !u$:

$$\begin{aligned}
&\langle \pi_0 0, \pi_1 p_u^*, \pi_0 \iota 0\rangle T(m_3)p \\
\overset{\text{naturality of } p}{=}\,& \langle \pi_0 0p, \pi_1 p_u^* p, \pi_0 \iota 0p\rangle m_3 \\
\overset{0p=1\ ,\ p_u^* p=!u}{=}\,& \langle \pi_0, !u, \pi_0 \iota\rangle m_3 \\
\overset{\iota \text{ is inversion}}{=}\,& !u.
\end{aligned}$$

Definition 5.3.5. *In a Cartesian category $\mathbb{X}$, given an object A and a group object G, a **left G-action** on A is a morphism $a : G \times A \to A$ such that the diagrams*

$$\begin{tikzcd}
A \arrow[r, "u\times 1_A"] \arrow[dr] & G \times A \arrow[d, "a"] \\
& A
\end{tikzcd}
\qquad
\begin{tikzcd}
G\times G\times A \arrow[r, "m\times 1_A"] \arrow[d, "1_G\times a"'] & G \times A \arrow[d, "a"'] \\
G\times A \arrow[r, "a"'] & A
\end{tikzcd}$$

*commute. A **right G-action** on A is a morphism $a : A \times G \to A$ such that the diagrams*

$$\begin{tikzcd}
A \arrow[r, "1_A\times u"] \arrow[dr] & A \times G \arrow[d, "a"] \\
& A
\end{tikzcd}
\qquad
\begin{tikzcd}
A\times G\times G \arrow[r, "1_A\times m"] \arrow[d, "a\times 1_G"'] & A \times G \arrow[d, "a"'] \\
A\times G \arrow[r, "a"'] & A
\end{tikzcd}$$

commute.

Lemma 5.3.6 (Left action)**.** *The morphism Ad_ℓ is a left group action. Analogously Ad_r is a right group action.*

The proof that Ad_l is a left group action and Ad_r is a right group action follows from the associativity of m.

In this section we will use the trivialization from Theorem 5.3.3 to express the natural transformations in the tangent structure of Definition 2.2.4 and the group structure morphisms of Definition 5.3.1 on $T(G)$ through structures on $G \times T(G)_u$. The result is summarized in Table 5.1.

An observation from Table 5.1 is that the tangent structure acts on the $T(G)_u$ component and not on the G component. The group structure involves the G component as well.

Lines 1,2,3 and 7 of the table (pertaining to $+$, 0, p and u) follow very directly from the definition of the isomorphism ϕ in Theorem 5.3.3. We show these equations in Proposition 5.3.7.

Operation on $T(G)$	Operation on $G \times T(G)_u$
$+ : T_2G \to T(G)$	$1 \times +_u : G \times T(G)_u \times T(G)_u \to G \times T(G)_u$
$0 : G \to T(G)$	$\langle 1, !0_u \rangle : G \to G \times T(G)_u$
$p : T(G) \to G$	$\pi_0 : G \times T(G)_u \to G$
$\ell : T(G) \to TT(G)$	$\langle \pi_0, !0_u, !0_u, \pi_1 \rangle : G \times T(G)_u \to G \times T(G)_u \times T(G)_u \times T(G)_u$
$c : TT(G) \to TT(G)$	$\langle \pi_0, \pi_2, \pi_1, \pi_3 \rangle : G \times T(G)_u^3 \to G \times T(G)_u^3$
$Tm : T(G) \times T(G) \to T(G)$	$(1 \times s \times 1)(m \times T(m)_u) : G \times T(G)_u \times G \times T(G)_u \to G \times T(G)_u$
$Tu : 1 \to T(G)$	$\langle u, 0_u \rangle : 1 \to G \times T(G)_u$
$T\iota : T(G) \to T(G)$	$\langle \pi_0 \iota, \mathrm{Ad}_\ell T(\iota)_u \rangle : G \times T(G)_u \to G \times T(G)_u$

Table 5.1: Structure morphisms on $T(G)$ and their translation to the local coordinates $G \times T(G)_u$. The meanings of 0_u, $+_u$ and s are discussed in Proposition 5.3.7 and Lemma 5.3.8

.

The next entry of Table 5.1, pertaining to ℓ, is shown in Proposition 5.3.13. The following entry, pertaining to c is shown in in Proposition 5.3.14. The following entry, pertaining to $T(m)$, is shown in Proposition 5.3.9. The last entry, pertaining to $T(\iota)$, is shown in Proposition 5.3.10.

In all the propositions throughout the rest of this section, the vertical arrows are isomorphisms built from ϕ that translate the operations on $T(G)$ into operations on $G \times T(G)_u$. The horizontal arrows are the morphisms in Table 5.1.

Proposition 5.3.7. *The diagrams*

$$\begin{tikzcd} G \times T(G)_u \times T(G)_u \arrow[r, "1_G \times +_u"] \arrow[d, "\cong"', "\langle\langle\pi_0,\pi_1\rangle\phi^{-1},\langle\pi_0,\pi_2\rangle\phi^{-1}\rangle"] & G \times T(G)_u \arrow[d, "\cong"', "\phi^{-1}"] \\ T_2G \arrow[r, "+"'] & TG \end{tikzcd} \quad (5.1) \qquad \begin{tikzcd} G \arrow[r, "\langle 1_G, !u0_u\rangle"] \arrow[d, "1_G"', "\cong"] & G \times T(G)_u \arrow[d, "\cong"', "\phi^{-1}"] \\ G \arrow[r, "0"'] & TG \end{tikzcd} \quad (5.2)$$

$$\begin{tikzcd} G \times T(G)_u \arrow[r, "\pi_0"] \arrow[d, "\phi^{-1}"', "\cong"] & G \arrow[d, "\cong"', "1_G"] \\ TG \arrow[r, "p"'] & G \end{tikzcd} \quad (5.3) \qquad \begin{tikzcd} 1 \arrow[r, "\langle u, u0_u\rangle"] \arrow[d, "0_1"', "\cong"] & G \times T(G)_u \arrow[d, "\cong"', "\phi^{-1}"] \\ T(1) \arrow[r, "T(u)"'] & TG \end{tikzcd} \quad (5.4)$$

commute where $\phi^{-1} : G \times T(G)_u \to T(G)$ is the isomorphism from Theorem 5.3.3, $0_u := \langle !, u0_G \rangle : 1 \to T(G)_u$ is the zero morphism into $T(G)_u$ induced by $0_G : G \to T(G)$ and

$$+_u = \langle !, \langle p_u^*, p_u^* \rangle + \rangle : T(G)_u \times T(G)_u \to T(G)_u$$

is the addition on $T(G)_u$ induced by $+$.

In this proposition the vertical arrows are isomorphisms and express the group action in terms of the tangent space at the unit.

Here, and and in the following proofs when we will show that a diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[d, "g"] \\ C \arrow[r, "k"'] & D \end{tikzcd}$$

commutes, we will refer to hk as the "left path" and to fg as the "right path".

Proof. Diagram 5.1 commutes because its right path $(1_G \times +_u)\phi^{-1}$ is

$$(1_G \times +_u)\langle 0 \times p_u^* \rangle T(m) = (0 \times (p_u^* \times p_u^*)+)T(m) = \langle\langle \pi_0 0, \pi_1 p_u^* \rangle, \langle \pi_0 0, \pi_2 p_u^* \rangle\rangle T_2(m)+,$$

which is equal to the left path $\langle\langle \pi_0, \pi_1 \rangle \phi^{-1}, \langle \pi_0, \pi_2 \rangle \phi^{-1} \rangle +$. Here we used the naturality of $+$ for the first and second equation and the definition of ϕ^{-1} as $\langle \pi_0 0G, \pi_1 p_u^* \rangle T(m)$ to conclude that it equals to the left path.

Diagram 5.2 commutes because its right path is

$$\langle 1_G, !u0_u \rangle (0 \times p_u^*)T(m) = \langle 0, !u0 \rangle T(m) = 0 = 1_G 0$$

since $u0$ is the unit of the group object $T(G)$.

Diagram 5.3 commutes because its left path is

$$\langle \pi_0 0, \pi_1 p_u^* \rangle T(m) p = \langle \pi_0 0 p, \pi_1 p_u^* p \rangle m = \langle \pi_0, \pi_1 !u \rangle m = \pi_0.$$

Diagram 5.4 commutes because the right path is

$$\langle u, u0_u \rangle \langle \pi_0 0, \pi_1 p_u^* \rangle T(m) = \langle u0, u0 \rangle T(m) = u0 = 0T(u).$$

The vertical arrow 0_1 is an isomorphism because $\mathbb{X}$ is a Cartesian tangent category. The vertical arrow $\langle \pi_0 0, \pi_1 p_u^* \rangle T(m)$ is an isomorphism as we proved in Theorem 5.3.3. Therefore its pullback square is also an isomorphism. $\square$

In order to prove that the remaining lines in the table are correct, the following Lemma is useful. It identifies $G \times T_u G$ with $T_u G \times G$ in a way that is compatible with the group multiplication and thereby tells us how to move group elements past each other.

Lemma 5.3.8 (Switching order)**.**

(i) The diagram

$$\begin{tikzcd}
T(G)_u \times G \arrow[rr, "\langle \pi_1, \mathrm{Ad}_r\rangle"] \arrow[d, "p_u^* \times 0"'] & & G \times T(G)_u \arrow[d, "0 \times p_u^*"] \\
T(G) \times T(G) \arrow[dr, "Tm"'] & & T(G) \times T(G) \arrow[dl, "Tm"] \\
& T(G) &
\end{tikzcd}$$

commutes.

(ii) The composition $T(G)_u \times G \xrightarrow{\langle p_u^*, 0\rangle T(m)} T(G) \xrightarrow{\cong} G \times T(G)_u$ *is* $\langle \pi_1, \mathrm{Ad}_r\rangle$.

(iii) The morphism $s := \langle \pi_1, \mathrm{Ad}_r\rangle : T(G)_u \times G \to G \times T(G)_u$ *is an isomorphism and its inverse is* $s^{-1} := \langle \mathrm{Ad}_\ell, \pi_0\rangle$.

Proof.

(i) Putting everything together the right path in the diagram is

$$\langle \pi_1, \mathrm{Ad}_r\rangle(0 \times p_u^*)T(m) = \langle \pi_1 0, \mathrm{Ad}_r p_u^*\rangle T(m)$$
$$= \langle \pi_1 0, \pi_1 \iota 0, \pi_0 p_u^*, \pi_1 0\rangle T(m_4) = \langle \pi_0 p_u^*, \pi_1 0\rangle T(m).$$

which is exactly the left path in the diagram.

(ii) follows now by composing with the inverse of the right vertical arrows since the isomorphism between $T(G)$ and $G \times T(G)_u$ is

$$(0 \times p_u^*)T(m) : G \times T(G)_u \to T(G).$$

(iii) is a straightforward calculation:

$$\langle \pi_1, \mathrm{Ad}_r\rangle\langle \mathrm{Ad}_\ell, \pi_0\rangle = \langle \pi_0, \mathrm{Ad}_r\langle \mathrm{Ad}_\ell, \pi_0\rangle\rangle = \langle \pi_0, \langle !, \langle \pi_0 \iota 0, \mathrm{Ad}_\ell p_u^*, \pi_0 0\rangle T(m_3)\rangle\rangle$$
$$= \langle \pi_0, \langle !, \langle \pi_0 \iota 0, \langle \pi_0 0, \pi_1 p_u^*, \pi_0 \iota 0\rangle T(m_3), \pi_0 0\rangle T(m_3)\rangle\rangle$$
$$= \langle \pi_0, \langle !, \langle \pi_0 \iota 0, \pi_0 0, \pi_1 p_u^*, \pi_0 \iota 0, \pi_0 0\rangle T(m_5)\rangle\rangle$$
$$= \langle \pi_0, \langle !, \pi_1 p_u^*\rangle\rangle = \langle \pi_0, \pi_1\rangle = 1_{G \times T(G)_u}.$$

The composition in the inverse composition is similar:

$$\langle \mathrm{Ad}_\ell, \pi_0\rangle\langle \pi_1, \mathrm{Ad}_r\rangle = \langle\langle !, \langle \pi_1 0, \langle \pi_1 \iota 0, \pi_0 p_u^*, \pi_1 0\rangle T(m_3), \pi_1 \iota 0\rangle T(m_3), \pi_1\rangle$$
$$= \langle\langle !, \pi_0 p_u^*\rangle, \pi_1\rangle = \langle \pi_0, \pi_1\rangle = 1_{G \times T(G)_u}$$

$\square$

The diagram in Lemma 5.3.8 means that the morphism $s := \langle \pi_1, \mathrm{Ad}_r \rangle$ is the isomorphism that identifies $T(G)_u \times G$ with $G \times T(G)_u$ in a way that is compatible with the group multiplication.

Proposition 5.3.9 (The multiplication)**.** *The diagram*

$$\begin{tikzcd}
G \times T(G)_u \times G \times T(G)_u \arrow[r, "1\times s\times 1"] \arrow[d, "(\phi^{-1})^2"'] & G \times G \times T(G)_u \times T(G)_u \arrow[r, "m\times T(m)_u"] & G \times T(G)_u \arrow[d, "\phi^{-1}"] \\
T(G) \times T(G) \arrow[rr, "Tm"] & & T(G)
\end{tikzcd}$$

commutes, where $T(m)_u : T(G)_u \times T(G)_u \to T(G)_u$ *is induced by the multiplication* $Tm : T(G) \times T(G) \to T(G)$*.*

Proof. The left path in the square is, using associativity,

$$\langle 0, p_u^*, 0, p_u^* \rangle T(m_4).$$

The right path in the square is

$$\begin{aligned}
(1 \times s \times 1)(m \times T(m)_u)\langle 0, p_u^* \rangle Tm &= (1 \times s \times 1)\langle \langle \pi_0, \pi_1 \rangle m0, \langle \pi_2, \pi_3 \rangle T(m)_u p_u^* \rangle T(m) \\
&= (1 \times s \times 1)\langle \langle \pi_0 0, \pi_1 0 \rangle T(m), \langle \pi_2 p_u^*, \pi_3 p_u^* \rangle T(m) \rangle T(m) \\
&= (1 \times s \times 1)\langle 0 \times 0 \times p_u^* \times p_u^* \rangle T(m_4)
\end{aligned}$$

Now it follows from part (ii) of Lemma 5.3.8 that this equals

$$= \langle 0, p_u^*, 0, p_u^* \rangle T(m_4)$$

which was the left path. $\square$

We can also formulate the group inverse $T(\iota)$ for $T(G)$ in terms of $G \times T(G)_u$.

Proposition 5.3.10 (The inverse)**.** *Suppose G is a group object in a tangent category and $T(G)_u$ exists. Then the following diagram commutes:*

$$\begin{tikzcd}
G \times T(G)_u \arrow[r, "\langle \pi_0 \iota, \mathrm{Ad}_\ell T(\iota)_u \rangle"] \arrow[d, "\phi^{-1}"'] & G \times T(G)_u \\
T(G) \arrow[r, "T(\iota)"'] & T(G) \arrow[u, "\phi"']
\end{tikzcd}$$

commutes.

Proof. In the first component the diagram commutes because

$$(0 \times p_u^*)T(m)T(\iota)p = (0p, p_u^* p)m\iota = \pi_0\iota.$$

In the second component we obtain:

$$\begin{aligned}(0 \times p_u^*)T(m)T(\iota)\langle !, \langle p\iota 0, 1\rangle T(m)\rangle &= \langle !, \langle (0 \times p_u^*)T(m)T(\iota)p\iota 0, (0 \times p_u^*)T(m)T(\iota)\rangle T(m)\rangle \\ &= \langle !, \langle (0 \times p_u^*)T(m), (0p \times p_u^* p)m\iota 0\rangle T(m)T(\iota)\rangle \\ &= \langle !, \langle \pi_0 0, \pi_1 p_u^*, \pi_0\iota 0\rangle T(m_3)T(\iota)\rangle = \mathrm{Ad}_l T(\iota)_u\end{aligned}$$

Here we used that $p_u^* p = !u$ which is the unit of the multiplication m. $\square$

It is known from Theorem 4.15 of [20] that $T(G)_u$ is a differential object. The following Lemma 5.3.11 recalls what that means.

Lemma 5.3.11. *For $f : X \to T(T(G)_u)$, let*

$$\{f\} : X \to T(G)_u$$

denote the unique morphism such that

$$f = \langle \{f\}\ell_u, fp0\rangle T(+_u),$$

where $\ell_u : T(G)_u \to T(T(G)_u)$ and $+_u : T(G)_u \times T(G)_u \to T(G)_u$ are the morphisms induced by $\ell : T(G) \to TT(G)$ and $+ : T_2G \to T(G)$. Then the morphism

$$T(T(G)_u) \xrightarrow{\langle p, \{1\}\rangle} T(G)_u \times T(G)_u$$

is an isomorphism with inverse

$$T(G)_u \times T(G)_u \xrightarrow{(0 \times \ell_u)T(+_u)} T(T(G)_u)$$

The morphism $\{f\}$ exists because of [29, Lemma 2.10] which applies because $fp_u^* : X \to T(T(G))$ equalizes $T(p)$ and $pp0$.

Proof. We show that they form an isomorphism by showing that their compositions are the identity. The first composition is

$$\langle p, \{1\}\rangle(0 \times \ell_u)T(+_u) = \langle 1p0, \{1\}\ell_u\rangle T(+_u) = 1$$

by definition of $\{\bullet\}$ The other composition is

$$\begin{aligned}(0 \times \ell_u)T(+_u)\langle p, \{1\}\rangle &= \langle (0 \times \ell_u)T(\sigma)p, \{(0 \times \ell_u)T(+_u)\}\rangle \\ &= \langle (0p \times \ell_u p)+_u, \{\langle \pi_0 0, \pi_1 \ell_u\rangle T(+_u)\}\rangle \\ &= \langle (1 \times !0_u)+_u, \langle \{\pi_0 0\}, \{\pi_1 \ell_u\}\rangle +_u\rangle \\ &= \langle \pi_0, \pi_0 !0_u, \pi_1\rangle +_u \\ &= \langle \pi_0, \pi_1\rangle = 1\end{aligned}$$

where $0_u : 1 \to T(G)_u$ is induced by $u0 : 1 \to T(G)$. There we used that $\{0\} = !0_u$ and $\{\ell_u\} = 1$ from Lemma 2.12 of [29]. $\square$

Remark 5.3.12. *We provided an explicit isomorphism $T(T(G)_u) \cong T(G)_u \times T(G)_u$. However, one could alternatively simply apply Theorem 4.15 of* [20] *which shows that $T(G)_u$ is a differential object. According to Definition 4.8 of* [20] *this implies that $T(T(G)_u) \cong T(G)_u \times T(G)_u$ and therefore proves that there is an isomorphism $T(T(G)_u) \cong T(G)_u \times T(G)_u$.*

Proposition 5.3.13 (The lift)**.** *Suppose G is a group object in a tangent category and the pullback $T(G)_u$ exists and is preserved by T. Then the following diagram commutes:*

$$\begin{tikzcd}
G \times T(G)_u \arrow[r, "{\langle \pi_0, !0_u, !0_u, \pi_1\rangle}"] \arrow[dd, "\phi^{-1}"', "\cong"] & G \times T(G)_u \times T(G)_u \times T(G)_u \\
 & T(G) \times TT(G)_u \arrow[u, "\cong", "\phi \times \langle p, \{1\}\rangle"'] \\
T(G) \arrow[r, "\ell"'] & TT(G) \arrow[u, "\cong", "T(\phi)"']
\end{tikzcd}$$

Here $\phi = \langle p, \langle !, \langle p\iota 0, p_u^* \rangle Tm\rangle\rangle$ is the isomorphism from Theorem 5.3.3.

This shows that the inclusion in the first and last components corresponds to the vertical lift ℓ, similar to the formulas in Examples 2.2.7 and 2.2.11.

Proof. As we know that $\ell T(\phi)\pi_0 = \ell T(p) = p0$, it follows that the first two components are π_0 and 0_u. The third component is zero because p is natural, $\phi^{-1}p = \pi_0$, and therefore

$$\phi^{-1}\ell T(\phi)\pi_1 p = \phi^{-1}\ell p\phi\pi_1 = \pi_0\langle 1, 0\rangle\pi_1 = !0_u.$$

The fourth component is given by the following calculation using the naturality of p,0 and ℓ:

$$\begin{aligned}
\phi^{-1}\ell T(\phi)\pi_1\{1\} &= \phi^{-1}\ell T(\langle p, \langle !, \langle p\iota 0, 1\rangle T(m)\rangle\rangle)\pi_1\{1\} \\
&= \phi^{-1}\langle !, \langle \ell T(p)T(\iota)T(0), \ell\rangle T^2(m)\rangle\{1\} \\
&= \phi^{-1}\langle !, \langle p0T(\iota)T(0), \ell\rangle T^2(m)\rangle\{1\} \\
&= \phi^{-1}\langle !, \langle p\iota 0\ell, \ell\rangle T^2(m)\rangle\{1\} \\
&= \phi^{-1}\langle !, \langle p\iota 0, 1\rangle T(m)\ell\rangle\{1\} \\
&= \phi^{-1}\langle !, \langle p\iota 0, 1\rangle T(m)\rangle\ell_u\{1\} \\
&= \phi^{-1}\langle !, \langle p\iota 0, 1\rangle T(m)\rangle = \phi^{-1}\phi\pi_1 = \pi_1
\end{aligned}$$

Here $\ell_u\{1\} = \{\ell_u\} = 1$ because $(T(G)_u, 1, !, +_u, 0_u, \ell_u)$ is a differential bundle ($T(G)_u$ is a differential object) and thus [20, Lemma 2.12 (vii)] shows it. ☐

Proposition 5.3.14 (The canonical flip)**.** *Suppose G is a group object in a Cartesian tangent category for which the pullback $T(G)_u$ exists and is preserved by T. Then the diagram*

$$\begin{tikzcd}
G \times T(G)_u \times T(G)_u \times T(G)_u \arrow[r, "\langle \pi_0, \pi_2, \pi_1, \pi_3\rangle"] & G \times T(G)_u \times T(G)_u \times T(G)_u \\
T(G) \times TT(G)_u \arrow[u, "\phi\times\langle p, \{1\}\rangle", "\cong"'] & T(G) \times TT(G)_u \arrow[u, "\cong", "\phi\times\langle p, \{1\}\rangle"'] \\
TT(G) \arrow[u, "T\phi", "\cong"'] \arrow[r, "c"'] & TT(G) \arrow[u, "\cong", "T\phi"']
\end{tikzcd} \tag{5.5}$$

commutes, where ϕ is the isomorphism from Theorem 5.3.3.

This shows that in our coordinates the canonical flip just flips the middle two components, analogous to the formula in Example 2.2.11 and the local formula in Example 2.2.7.

Proof. For this proof we will actually use Theorem 5.3.15, the fact that $+_u$ and $T(m)_u$ coincide and are commutative. Since we don't use the canonical flip when proving Theorem 5.3.15 this is logically sound. We will show the following diagram commutes

$$\begin{tikzcd}
G \times T(G)_u \times T(G)_u \times T(G)_u \arrow[rr, "\langle \pi_0,\pi_2,\pi_1,\pi_3\rangle"] \arrow[d, "\phi^{-1}\times(0\times\ell_u)T(+_u)"', "\cong"] & & G \times T(G)_u \times T(G)_u \times T(G)_u \arrow[d, "\cong"', "\phi^{-1}\times(0\times\ell_u)T(+_u)"] \\
T(G)\times TT(G)_u \arrow[d, "T\phi^{-1}"', "\cong"] & & T(G)\times TT(G)_u \arrow[d, "\cong"', "T\phi^{-1}"] \\
TT(G) \arrow[rr, "c"'] & & TT(G).
\end{tikzcd} \qquad (5.6)$$

Because the vertical arrows in Diagram 5.6 are the inverses of the vertical arrows in Diagram 5.5, this is equivalent to the statement we need to show.

First we will write out the vertical arrows of Diagram 5.6 in components

$$\begin{aligned}
&((\phi^{-1}) \times (0 \times \ell_u)T(+_u))T(\phi^{-1}) \\
=&\langle\langle \pi_0 0, \pi_1 p_u^*\rangle T(m), \langle \pi_2 0, \pi_3 \ell_u\rangle T(+_u)\rangle \langle \pi_0 T(0), \pi_1 T(p_u^*)\rangle T^2(m) \\
=&\langle\langle \pi_0 0, \pi_1 p_u^*\rangle T(m)T(0), \langle \pi_2 0, \pi_3 \ell_u\rangle T(+_u)T(p_u^*)\rangle T^2(m)
\end{aligned}$$

Now we use Theorem 5.3.15 to replace $+_u$ with $T(m)_u$ and use $T(m)_u p_u^* = p_u^* T(m)$, $\ell_u T(p_u^*) = p_u^* \ell$ and naturality of 0 to obtain

$$\begin{aligned}
&\langle\langle \pi_0 0, \pi_1 p_u^*\rangle T(m)T(0), \langle \pi_2 0, \pi_3 \ell_u\rangle T(T(m)_u)T(p_u^*)\rangle T^2(m) \\
=&\langle\langle \pi_0 0 T(0), \pi_1 p_u^* T(0)\rangle T^2(m), \langle \pi_2 p_u^* 0, \pi_3 p_u^* \ell\rangle T^2(m)\rangle T^2(m) \\
=&\langle \pi_0 0 T(0), \pi_1 p_u^* T(0), \pi_2 p_u^* 0, \pi_3 p_u^* \ell\rangle T^2(m_4),
\end{aligned}$$

where $m_4 : G^4 \to G$ is the multiplication of four group elements, unique by associativity. Now if we follow the left path of Diagram 5.6 we have

$$\begin{aligned}
((\phi^{-1}) \times (0 \times \ell_u)T(+_u))T(\phi^{-1})c &= \langle \pi_0 0 T(0), \pi_1 p_u^* T(0), \pi_2 p_u^* 0, \pi_3 p_u^* \ell\rangle T^2(m_4)c \\
&= \langle \pi_0 0 T(0)c, \pi_1 p_u^* T(0)c, \pi_2 p_u^* 0c, \pi_3 p_u^* \ell c\rangle T^2(m_4) \\
&= \langle \pi_0 0 T(0), \pi_1 p_u^* 0, \pi_2 p_u^* T(0), \pi_3 p_u^* \ell\rangle T^2(m_4) \\
&= \langle \pi_0 0 T(0), \pi_1 0 T(p_u^*), \pi_2 T_u(0) T(p_u^*), \pi_3 p_u^* \ell\rangle T^2(m_4) \\
&= \langle \pi_0 0 T(0), \pi_2 T_u(0) T(p_u^*), \pi_1 0 T(p_u^*), \pi_3 p_u^* \ell\rangle T^2(m_4) \\
&= \langle \pi_0 0 T(0), \pi_2 p_u^* T(0), \pi_1 p_u^* 0, \pi_3 p_u^* \ell\rangle T^2(m_4) \\
&= \langle \pi_0, \pi_2, \pi_1, \pi_3\rangle ((\phi^{-1}) \times (0 \times \ell_u)T(+_u))T(\phi^{-1})
\end{aligned}$$

which is the right path through Diagram 5.6. When going from the fourth to fifth line we used that $T(m)_u$ is commutative and therefore

$$\langle ap_u^*, bp_u^*\rangle T(m) = \langle a,b\rangle T(m)_u p_u^* = \langle b,a\rangle T(m)_u p_u^* = \langle bp_u^*, ap_u^*\rangle T(m).$$

$\square$

This concludes the verification of all the structures summarized in Table 5.1, with the exception of showing that $+_u = T(m)_u$, which we will verify in Section 5.3.3 below.

5.3.3 Comparing addition and multiplication

There are two monoid structures on $T(G)_u$, one of them is the group multiplication and it is a group (i.e. has an inverse). The other one is the addition of tangent vectors and it is commutative. In this section we show that they are in fact the same monoid structure and therefore both of them are abelian group structures. This implies that the two monoid structures (addition of tangent vectors and group multiplication) on $T(G)$ coincide up to conjugation with ϕ. Theorem 5.3.15, the central result of this subsection will make this explicit.

Let $\mathbb{X}$ be a Cartesian tangent category and G a group object in $\mathbb{X}$ such that the pullback $T(G)_u$ exists. Then there are two monoid structures on $T(G)_u$, one of them is given by $T(m)_u$, the morphism induced by

$$\begin{tikzcd}
 & & T(G)\times T(G) \arrow[drr, "T(m)"] & & \\
T(G)_u \times T(G)_u \arrow[urr, "p_u^*\times p_u^*"] \arrow[rr, dashed, "T(m)_u"] \arrow[drr, "!"'] & & T(G)_u \arrow[rr, "p_u^*"] \arrow[d, "!"'] \arrow[drr, phantom, "\lrcorner"] & & T(G) \arrow[d, "p"] \\
 & & 1 \arrow[rr, "u"'] & & G
\end{tikzcd}.$$

The other one is $+_u$, the morphism induced by

$$\begin{tikzcd}
 & & T_2G \arrow[drr, "+_G"] & & \\
T(G)_u \times T(G)_u \arrow[urr, "p_u^*\times_G p_u^*"] \arrow[rr, dashed, "+_u"] \arrow[drr, "!"'] & & T(G)_u \arrow[rr, "p_u^*"] \arrow[d, "!"'] \arrow[drr, phantom, "\lrcorner"] & & T(G) \arrow[d, "p"] \\
 & & 1 \arrow[rr, "u"'] & & G
\end{tikzcd},$$

where $p_u^* \times_G p_u^*$ stands for $\langle \pi_0 p_u^*, \pi_1 p_u^*\rangle$. In the following theorem, we show that these operations coincide and are commutative. The unit has a convenient expression as $0_u = \langle !, u0_G\rangle = \langle !, 0_1 T(u)\rangle : 1 \to T(G)_u$.

Theorem 5.3.15. *The two binary operations on $T(G)_u$ satisfy the following:*

(i) They coincide, $+_u = T(m)_u$.

(ii) They are commutative, $T(m)_u = \langle \pi_1, \pi_0 \rangle T(m)_u$.

(iii) Their unit is 0_u.

(iv) Their inverse is $T(\iota)_u$.

Proof. The morphism $+_u$ is commutative, since it is induced by the commutative monoid addition $+_G$ on $T_2(G)$. The morphism 0_u is the unit for $T(m)_u$ because the unit for $T(m)$ is $0T(u) = u0$ and $T(m)_u$ is induced by $T(m)$. The morphism 0_u is the unit for $+_u$ because 0_G is the unit for $+_G$. Since $+_G : T_2G \to T(G)$ is natural the morphism $+_u : T(G)_u^2$ is also natural. That implies the naturality diagram of $+_u$ with respect to the morphism $T(m)_u : T(G)_u \times T(G)_u \to T(G)_u$, the diagram

$$\begin{tikzcd}
T(G)_u^2 \times TuG^2 \arrow[rr, "(T(m)_u)^2"] \arrow[d, "+_u \times +_u"'] & & T(G)_u^2 \arrow[d, "+_u"] \\
T(G)_u \times T(G)_u \arrow[rr, "T(m)_u"'] & & T(G)_u
\end{tikzcd},$$

commutes. Now an Eckmann–Hilton style argument implies that the two operations coincide:

$$T(m)_u \overset{\text{neut.}}{=} \langle \langle \pi_0, 0_u \rangle +_u, \langle 0_u, \pi_1 \rangle +_u \rangle T(m)_u \overset{\text{nat.}}{=} \langle \langle \pi_0, 0_u \rangle T(m)_u, \langle 0_u, \pi_1 \rangle T(m)_u \rangle +_u \overset{\text{neut.}}{=} +_u.$$

Since the two monoid structures coincide and $T(\iota)_u$ is an inverse for $T(m)_u$, it is an inverse for both of them. ☐

From this we will conclude in the following theorem that the additions on $T(G)$ have a negation. This result is surprising, as we did not assume that the addition of tangent vectors has negatives. However, for the tangent vectors of the group object G it is defined. It is only defined on $T(G)$, it is not a natural transformation.

Theorem 5.3.16. *Let $\mathbb{X}$ be a Cartesian tangent category and let G be a group object in $\mathbb{X}$ such that the pullback $T(G)_u$ exists.*

(i) Then the addition $+_G : T_2G \to T(G)$ is explicitly given by the composition

$$T_2G \xrightarrow{\phi_2^{-1}} G \times T(G)_u \times T(G)_u \xrightarrow{(1_G \times T(m)_u)} G \times T(G)_u \xrightarrow{\phi} T(G)$$

where $\phi : G \times T(G)_u \to T(G)$ is the isomorphism from theorem 5.3.3.

(ii) The addition $+_G : T_2(G) \to T(G)$ has a negation that is given by the composition

$$T(G) \xrightarrow{\phi^{-1}} G \times T(G)_u \xrightarrow{(1_G \times T(\iota)_u)} G \times T(G)_u \xrightarrow{\phi} T(G).$$

(iii) The addition $+_{T(G)} : T_2^2 G \to T^2 G$ has a negation.

Proof. The addition $+_G : T_2 G \to T(G)$ can be explicitly described as the composition

$$G \times T(G)_u \times T(G)_u \xrightarrow{\phi_2} T_2 G \xrightarrow{+_G} T(G) \xrightarrow{\phi^{-1}} G \times T(G)_u$$

where $\phi : G \times T(G)_u \to T(G)$ is the isomorphism from Theorem 5.3.3. The second component of this composition can be explicitly calculated as the following (the first component is just p, as $+$ preserves the basepoint):

$$\begin{aligned}
\phi_2 +_G \phi^{-1}\pi_1 &= \langle\langle \pi_0 0, \pi_1 p_u^* \rangle T(m), \langle \pi_0 0, \pi_2 p_u^* \rangle T(m)\rangle + \langle !, \langle p\iota 0, 1\rangle T(m)\rangle \\
&= \langle\langle \pi_0 0, \pi_1 p_u^* \rangle, \langle \pi_0 0, \pi_2 p_u^* \rangle\rangle +^2 T(m)\langle !, \langle p!0, 1\rangle T(m)\rangle \\
&= \langle\langle \pi_0 0, \pi_0 0\rangle +, \langle \pi_1 p_u^*, \pi_2 p_u^* \rangle + \rangle T(m)\langle !, \langle p\iota 0, 1\rangle T(m)\rangle \\
&= \langle \pi_0 0, \langle \pi_1 p_u^*, \pi_2 p_u^* \rangle T(m)\rangle T(m)\langle !, \langle p\iota 0, 1\rangle T(m)\rangle \\
&= \langle !, \langle\langle \pi_0, !u\rangle m\iota 0, \langle \pi_0 0, \langle \pi_1 p_u^*, \pi_2 p_u^* \rangle T(m)\rangle T(m)\rangle T(m)\rangle \\
&= \langle !, \langle \pi_1 p_u^*, \pi_2 p^* u\rangle T(m)\rangle \\
&= \langle \pi_1, \pi_2 \rangle T(m)_u
\end{aligned}$$

This calculation now tells us that $\phi_2 +_G \phi^{-1} = 1_G \times T(m)_u$ Pre- and postcomposing this equality with the isomorphism ϕ results in

$$+_G = \phi_2^{-1}(1_G \times T(m)_u)\phi,$$

the equality we needed to show for (i). This explicit formula shows directly that

$$- := \phi^{-1}(1 \times T(\iota)_u)\phi$$

defines a negation on $T(G)$ implying part (ii) of the theorem.

It follows that T^2G has negatives with respect to $+_{T(G)}$ since $T(G)$ is a group object and

$$\begin{tikzcd}
T(G)_u \times T(G)_u \arrow[r, "\langle \pi_0, \pi_1 \rangle"] \arrow[d, "!"'] \arrow[dr, phantom, "\lrcorner", very near start] & T_2G \arrow[r, "\nu"'] \arrow[d, "\pi_0 p"'] \arrow[dr, phantom, "\lrcorner", very near start] & T^2G \arrow[r, "c"] \arrow[d, "T(p)"'] \arrow[dr, phantom, "\lrcorner", very near start] & T^2G \arrow[d, "p"'] \\
1 \arrow[r, "u"'] & G \arrow[r, "0"'] & T(G) \arrow[r, "1"'] & T(G)
\end{tikzcd}$$

is a pullback composed of three pullbacks. This negation can be explicitly described by using the fact that the canonical flip is an additive bundle morphism and thus $+ = c_2T(+)c : T_2^2G \to T^2G$. From this formula we can see that if $- : T(G) \to T(G)$ is a negation with respect to $+ : T(G) \to T(G)$, then $cT(-)c : T^G \to T^2G$ is a negation with respect to $+ : T^2G \to T^2G$, proving part (iii). $\square$

The negation $-$ is only defined for the tangents of the group object G, it is not a natural transformation. However the negation is compatible with multiplication and addition in the following sense.

Lemma 5.3.17. *The diagrams*

$$\begin{tikzcd}
T(G) \times T(G) \arrow[r, "- \times -"] \arrow[d, "T(m)"'] & T(G) \times T(G) \arrow[d, "T(m)"] \\
T(G) \arrow[r, "-"'] & T(G)
\end{tikzcd}
\qquad
\begin{tikzcd}
T_2G \arrow[r, "- \times_G -"] \arrow[d, "+"'] & T_2G \arrow[d, "+"] \\
T(G) \arrow[r, "-"'] & T(G)
\end{tikzcd}$$

commute.

Proof. In the diagram

$$\begin{tikzcd}
T(G) \times T(G) \arrow[r, "- \times -"] \arrow[d, "\phi^{-1} \times \phi^{-1}"'] & T(G) \times T(G) \arrow[d, "\phi^{-1} \times \phi^{-1}"] \\
G \times T(G)_u \times G \times T(G)_u \arrow[r, "1 \times T(\iota)_u \times 1 \times T(\iota)_u"] \arrow[d, "(1 \times s \times 1)(m \times T(m)_u)"'] & G \times T(G)_u \times G \times T(G)_u \arrow[d, "(1 \times s \times 1)(m \times T(m)_u"] \\
G \times T(G)_u \arrow[r, "1 \times T(\iota)_u"'] \arrow[d, "\phi"'] & G \times T(G)_u \arrow[d, "\phi"] \\
T(G) \arrow[r, "-"'] & T(G)
\end{tikzcd}$$

the top and bottom square commute by the definition of $-$. The commutativity of the central square is that $(g^{-1}vg)^{-1}w = gv^{-1}g^{-1}w$. The first diagram commutes by the definition of $-$. Then the second diagram commutes because $+ = \phi_2(1 \times T(m)_u)\phi^{-1}$. $\square$

5.3.4 Left-invariant vector fields

A **left-invariant vector field** is a section $\xi : G \to T(G)$ of $p : T(G) \to G$ such that the diagram

$$\begin{tikzcd} G \times G \arrow[r, "m"] \arrow[d, "0\times\xi"'] & G \arrow[d, "\xi"] \\ T(G) \times T(G) \arrow[r, "T(m)"'] & T(G) \end{tikzcd}$$

commutes. As shown in Proposition 5.3.19, the zero vector field is an example of a left-invariant vector field.

Classically, the vector space of left-invariant vector fields is an alternate description of a Lie group's Lie algebra. In this case, one can prove that the vector space of left-invariant vector fields is isomorphic to $T(G)_u$. Unfortunately, we were not able to fully generalize this result to the general tangent category setting. At least, it is true for the set of left-invariant vector fields and the set of elements, i.e. morphisms $1 \to T(G)_u$.

Technically, the word "elements" is used here with two meanings, morphisms $1 \to T(G)_u$ and the classical meaning of elements of the underlying set of the manifold. These two meanings correspond to each other by identifying a morphism $1 \to T(G)_u$ with its image (remembering that 1 has the one point-set as its underlying set).

Proposition 5.3.18 (Left-invariant vector fields)**.** *The set of left invariant vector fields $\xi : G \to T(G)$ is isomorphic to the set of elements $V : 1 \to T(G)_u$ of $T(G)_u$.*

Intuitively the isomorphism is given by evaluating a vector field at the unit.

Proof. The morphism $\mathbb{X}_{\mathrm{Linv}}(G, T(G)) \to \mathbb{X}(1, T(G)_u)$ is given by $\xi \mapsto V_\xi := \langle 1_1, u\xi\rangle : 1 \to T(G)_u$.

The morphism $\mathbb{X}(1, T(G)_u) \to \mathbb{X}_{\mathrm{Linv}}(G, T(G))$ is given by sending $V : 1 \to T(G)_u$ to the composition of

$$G \cong G \times 1 \xrightarrow{1_G \times V} G \times T(G)_u \xrightarrow{0\times p_u^*} T(G) \times T(G) \xrightarrow{T(m)} T(G),$$

i.e $V \mapsto \xi_V := (0 \times V p_u^*)T(m)$.

These two constructions are inverse to each other because

$$\xi_{V_\xi} = (0 \times \langle 1_1, u\xi\rangle p_u^*)T(m) = (0 \times u\xi)T(m) = (1 \times u)(0 \times \xi)T(m) = (1 \times u)m\xi = \xi$$

and

$$V_{\xi_V} = \langle 1_1, u(0 \times V p_u^*)T(m)\rangle = \langle 1_1, (u0 \times V p_u^*)T(m)\rangle = \langle 1_1, V p_u^*\rangle = V$$

where the second to last equality holds true as $u0 = 0T(u)$ is the unit of the multiplication $T(m)$ by Lemma 5.3.2. $\square$

This gives a new perspective on Theorem 5.3.3. Theorem 5.3.3 says that the tangent space $T(G)$ is $G \times T(G)_u$. Classically, an element of $T(G)$ is a vector $v \in T(G)_u$ which is "shifted" by an action of $g \in G$. Now Proposition 5.3.18 says that furthermore, we can think of v as a left-invariant vector field ξ_v rather than a vector, namely the left-invariant vector field that is given by "shifting" the vector $v \in T(G)_u$ to any group element $g \in G$. The inverse is given by sending (g, ξ) to the vector $\xi(g) \in T(G)$.

Given two vector fields $\xi, \zeta : G \to T(G)$, we call $\langle \xi, \zeta \rangle +$ the addition of ξ and ζ. We call $\xi-$ the negation of ξ. Together with the zero vector field this turns the set of all vector fields into an abelian group. The following Proposition 5.3.19 shows that the left-invariant vector fields are a subgroup.

Proposition 5.3.19. *Let $\mathbb{X}$ be a Cartesian tangent category and let G be a group object such that the pullback $T(G)_u$ exists. Then*

(i) The zero vector field is left-invariant,

(ii) The addition of two left-invariant vector fields is left-invariant, and

(iii) The negation of a left-invariant vector field is left invariant.

Proof.

(i) The zero vector field is left-invariant because

$$\begin{tikzcd} G \times G \arrow[r, "m"] \arrow[d, "0\times 0"'] & G \arrow[d, "0"] \\ T(G) \times T(G) \arrow[r, "Tm"'] & T(G) \end{tikzcd}$$

commutes by the naturality of 0.

(ii) In order to show that the addition of the two left-invariant vector fields ξ and ζ is left-invariant, we need to show that

$$\begin{tikzcd} G \times G \arrow[r, "m"] \arrow[d, "0\times\langle\xi,\zeta\rangle+"'] & G \arrow[d, "\langle\xi,\zeta\rangle+"] \\ T(G) \times T(G) \arrow[r, "Tm"'] & T(G) \end{tikzcd}$$

commutes. It commutes because

$$\begin{aligned}(0 \times \langle \xi, \zeta \rangle +)Tm =&(\langle 0, 0\rangle + \times \langle \xi, \zeta \rangle +)Tm\\ =&\langle (0 \times \xi)Tm, (0 \times \zeta)Tm \rangle +\\ =&\langle m\xi, m\zeta \rangle +\\ =&m\langle \xi, \zeta \rangle + .\end{aligned}$$

(iii) The negation preserves left-invariance because in the diagram

$$\begin{tikzcd} G \times G \arrow[r, "m"] \arrow[d, "0\times\xi"'] & G \arrow[d, "\xi"] \\ T(G) \times T(G) \arrow[r, "Tm"'] \arrow[d, "-\times-"'] & T(G) \arrow[d, "-"] \\ T(G) \times T(G) \arrow[r, "Tm"'] & T(G) \end{tikzcd}$$

the upper part commutes because of left-invariance, the lower part commutes because of Lemma 5.3.17 and the whole diagram shows the left-invariance of $\xi-$ since $0- = 0$.

□

5.3.5 Defining the Lie bracket

As the tangent bundle $T(G)$ has negatives it is possible to define a Lie-bracket on sections. For two vector fields $w_1, w_2 : G \to T(G)$, [20, Definition 3.14] defines the Lie bracket as

$$[w_1, w_2] = \{\langle w_1 T(w_2), w_2 T(w_1) c- \rangle +\}$$

where the brackets turn a morphism $f : A \to T^2G$ equalizing $pp0$ and $T(p)$ to a morphism $\{f\} : A \to T(G)$ given by the composition of the induced morphism into the equalizer T_2G and the projection $\pi_0 : T_2G \to T(G)$. This means $\{f\}$ it is the unique morphism such that $f = \langle \{f\}\ell, fp0 \rangle T(+)$.

We will now use this to define a Lie bracket on the left-invariant vector fields mimicking classical differential geometry where the left-invariant vector fields form the Lie-Algebra of a Lie group.

Lemma 5.3.20. *If the vector fields $w_1, w_2 : G \to T(G)$ are left-invariant, then $[w_1, w_2]$ is left-invariant.*

Proof. We need to show that $m[w_1, w_2] = (0 \times [w_1, w_2])T(m)$ using the fact that w_1 and w_2 are left-invariant, i.e. $mw_1 = (0 \times w_1)T(m)$.

$$
\begin{aligned}
& m[w_1, w_2] = \\
& m\{\langle w_1 T(w_2), w_2 T(w_1) c - \rangle +\} \\
=& \{\langle m w_1 T(w_2), m w_2 T(w_1) c - \rangle +\} && f\{g\} = \{fg\} \\
=& \{\langle (0 \times w_1) T(m) T(w_2), (0 \times w_2) T(m) T(w_1) c - \rangle +\} && \text{left-invariance} \\
=& \{\langle (0 \times w_1)(0 \times T(w_2)) T^2(m), (0 \times w_2)(0 \times T(w_1)) T^2(m) c - \rangle +\} && \text{left-invariance} \\
=& \{\langle (0 \times w_1)(0 \times T(w_2)), (0 \times w_2)(0 \times T(w_1)) c - \rangle + T^2(m)\} && \text{Lemma 5.3.17} \\
=& \{00 \times \langle w_1 T(w_2), w_2 T(w_1) c - \rangle + T^2(m)\} && \text{composition} \\
=& \{00 \times \langle w_1 T(w_2), w_2 T(w_1) c - \rangle +\} T(m) && \text{[20, Lemma 2.14]} \\
=& (0 \times [w_1, w_2]) T(m) && \text{[20, Lemma 2.14]}
\end{aligned}
$$

□

This defines a Lie-bracket on the set of left invariant vector fields. Using Proposition 5.3.18 we may also consider the Lie-bracket to be a pairing on $\mathbb{X}(1, T(G)_u)$:

$$[\bullet, \bullet]_{1 \to T(G)_u} : \mathbb{X}(1, T(G)_u) \times \mathbb{X}(1, T(G)_u) \to \mathbb{X}(1, T(G)_u)$$

That is a pairing on the elements of $T(G)_u$, i.e. the morphisms from the terminal object 1 into $T(G)_u$. We call this the external Lie Algebra.

Theorem 5.3.21. *Let G be a group object in a Cartesian tangent category $\mathbb{X}$ for which the pullback $T(G)_u$ exists. Then the set $\mathbb{X}(1, T(G)_u)$ is a Lie-Algebra over the ring of integers $\mathbb{Z}$.*

Proof. Instead of $\mathbb{X}(1, T(G)_u)$, we use the set of left-invariant vector fields in this proof. By Proposition 5.3.18 this is isomorphic to $\mathbb{X}(1, T(G)_u)$. The addition $+$, the zero section 0 and the negation from Theorem 5.3.16 (ii) induce an abelian group structure on the set of left-invariant vector fields which induces a $\mathbb{Z}$-module structure. It follows from Lemma 5.3.20 that the bracket is an operation on the set of left-invariant vector fields. It follows from Theorem 3.17 of [20] that the Lie-bracket is bilinear with respect to this $\mathbb{Z}$-module structure. The Jacobi identity has been proved by in Theorem 4.2 of [21]. Thereby the set of left-invariant vector fields is a Lie Algebra. □

Theorem 7.5 of [2] is a related result, though with different assumptions and a different perspective.

A natural question to ask is whether this external Lie algebra structure comes from an internal Lie-bracket on $T(G)_u$: Is there an antisymmetric morphism fulfilling the Jacobi identity $[\bullet,\bullet]_{T(G)_u} : T(G)_u \times T(G)_u \to T(G)_u$ such that $[\chi,\xi]_{1\to T(G)_u} = \langle \chi,\xi\rangle[\bullet,\bullet]_{T(G)_u}$? We doubt that there is such an internal Lie-bracket this Lie-Algebra structure comes from. The reason for this is that the external Lie-bracket is defined on left-invariant vector fields and this only gives a Lie-bracket that is a set map. This does not relate to the type of objects in an arbitrary Cartesian tangent category. In the well known cases of smooth manifolds, it does work.

Corollary 5.3.22. *Let G be a Lie group (a group object in the category of smooth manifolds). Then its tangent space $T(G)_u$ at the unit has an internal Lie-bracket $T(G)_u \times T(G)_u \to T(G)_u$ that is bilinear with respect to the addition of tangent vectors.*

We would like to apply it analogously to the category of smooth varieties. However, there is still some work to be done on describing the tangent structure for the category of smooth varieties.

The key property in the two setups of Corollary 5.3.22 is that an arbitrary linear map of elements is a morphism in the category and that two such morphisms coincide if they coincide on elements.

Proof. The result follows because the category of smooth manifolds is a tangent category in which tangent spaces exist. In this category the group objects are Lie groups. The Lie bracket, defined on points, is a linear map. Since it is linear, it is a smooth map $T(G)_u \otimes T(G)_u \to T(G)_u$ and therefore the Lie bracket is a smooth map between manifolds.

□

Corollary 5.3.22 is the recovered classical result from differential geometry. In classical differential geometry the Lie algebra determines the Lie group locally via the exponential map. We doubt that this is the case in full generality because in general there is no exponential morphism in an arbitrary tangent category.

5.4 Principal bundles

Physical systems that have an inherent symmetry or redundancy are often described as principal G-bundles where G is the system's symmetry group. For example, the electromagnetic interaction between charged particles naturally displays symmetry of the complex unit circle, $U(1)$. The details of this example can be found in [75, Chapter 10]. The electromagnetic field has an associated $U(1)$-principal bundle. This bundle is important in electromagentism - the electromagnetic potential is the expression in local coordinates of a connection on this bundle. Physically inspired examples such as this one make it clear that principal bundles are important.

The central idea of principal bundles is that locally they are the Cartesian product $M \times G$ of a base-manifold M and the symmetry group G, while globally they may have a nontrivial geometry. The purpose of this section is to generalize this geometric construction to an arbitrary join restriction category. The structure of join restriction categories allows for a particularly nice realization of fibre bundles in which local trivializations are expressed as partial isomorphisms.

5.4.1 Fibre Bundles

In differential geometry, given manifolds M and F, a fibre bundle over M with typical fibre F is a manifold E with a smooth map $q : E \to M$ such that for every $p \in M$ there is an open neighbourhood $U \ni p$ such that $q^{-1}(U)$ is diffeomorphic to $U \times F$. In this section we generalize this definition into the setting of join-restriction categories.

In the following definition it will be useful to recall from Definition 5.2.4 that a total morphism is a morphism $f : A \to B$ in a restriction category fulfilling $\bar{f} = 1_A$ and that a morphism $g : A \to B$ is called a partial isomorphism if it has a partial inverse, denoted g^*. Recall in addition from Remark 5.2.9 that a diagram commutes if the compositions along all paths are equal. Two partial maps are equal if both their values and their domains coincide, i.e. $f = g$ if both $f \smile g$ and $\bar{f} = \bar{g}$ hold.

Definition 5.4.1. *Let M and F be objects in a join restriction category $\mathbb{X}$.*

*(i) a **fibre bundle** over M with typical fibre F in a join restriction category $\mathbb{X}$ is an object E with a total morphism $q : E \to M$ and partial isomorphisms $\alpha_i : E \to M \times F$, called **local trivializations**, such that the diagram*

$$\begin{tikzcd} E \arrow[r, "\alpha_i"] \arrow[d, "\bar{\alpha}_i"'] & M \times F \arrow[d, "\pi_0"] \\ E \arrow[r, "q"'] & M \end{tikzcd} \tag{5.7}$$

commutes and $\bigvee_{i \in I} \bar{\alpha}_i = 1_E$ and there exists a restriction idempotent $e_i : M \to M$ such that $\bar{\alpha}_i^* = e_i \times 1_F$.

(ii) A morphism between bundles $(E, M, F, q, (\alpha_i)_{i\in I}) \to (E', M', F', q', (\alpha'_k)_{k\in K})$ is a pair of morphisms $\Phi : E \to E', \phi : M \to M'$ such that

$$\begin{tikzcd} E \arrow[r, "\Phi"] \arrow[d, "q"'] & E' \arrow[d, "q'"] \\ M \arrow[r, "\phi"'] & M' \end{tikzcd}$$

commutes.

(iii) A fibre bundle $(E, M, q, (\alpha_i)_{i\in I})$ is called ***totally fibred*** *if, for all $i \in I$,*

$$\bar{\alpha}_i = \overline{qe_i}.$$

The morphisms Φ and ϕ forming the principal bundle morphisms are not total, in general. Fibre bundles and fibre bundle morphisms, as in Definition 5.4.1, form a join restriction category $\mathbb{F}\mathrm{ib}\mathrm{Bun}(\mathbb{X})$. The restrictions structure is given by $\overline{(\Phi, \phi)} = (\bar{\Phi}, \bar{\phi})$ and the join is $\bigvee_i (\Phi_i, \phi_i) = (\bigvee_i \Phi_i, \bigvee_i \phi_i)$.

The condition for a bundle to be totally fibred will be relevant in Section 5.4.3. Remember from Remark 5.2.3,(ii) that the restriction idempotent of a composition can be interpreted as a preimage. For intuition we consider the category $\mathbb{P}\mathrm{ar}\mathrm{Mfld}$ of smooth manifolds and smooth partial maps defined on open subsets, a full definition (including the restriction structure) will be given in Definition 5.4.25. In $\mathbb{P}\mathrm{ar}\mathrm{Mfld}$, total fibredness describes that the local trivialization $\alpha_i : E \to M \times F$ is not just defined on an arbitrary open subset $V \subset E$, but that, in fact, it is of the form $q^{-1}(U)$ for some $U \subset M$.

Remark 5.4.2. *An equivalent characterization of a fibre bundle that is totally fibred is that the diagram*

$$\begin{tikzcd} E \arrow[r, "\alpha_i"] \arrow[d, "q"'] & M \times G \arrow[d, "\pi_0"] \\ M \arrow[r, "e_i"'] & M \end{tikzcd}$$

commutes. This shows how total fibredness forces the morphism α_i to be total on fibres and the partiality to be contained in the base-space component.

The formal reason why the diagram in Remark 5.4.2 holds, is that Diagram 5.7 commutes, that in the totally fibred case $\bar{\alpha}_i = \overline{qe_i}$ and that with this the left path in the diagram simplifies to $q\bar{e}_i$. It is an equivalent condition since taking the restriction idempotent of $\alpha_i \pi_0 = qe_i$ gives exactly the totally fibred condition $\bar{\alpha}_i = \overline{qe_i}$.

Recall from Definition 5.2.10 that an atlas is an index set I together with objects $(U_i)_{i\in I}$ and morphisms $u_{ij} : U_i \to U_j$ for each $i, j \in I$. The object $M \times F$ together with a collection of morphisms $u_{ij} = \alpha_i^* \alpha_j : M \times F \to M \times F$, form an atlas, indexed by the index set I of $(\alpha_i)_{i\in I}$ and given by the composition of the partial inverses α_i^* and the local trivializations α_j. The atlases produced by this construction are the same as the atlases which are characterized in Definition 5.4.3. Theorem 5.4.9, which states that if gluings exist, bundle atlases are equivalent to fibre bundles, will provide a proof of this statement. First, we need some preliminary definitions and properties.

Definition 5.4.3. *Let M and F be objects in a join restriction category $\mathbb{X}$. Then a **bundle atlas** is an atlas $(u_{ij} : M \times F \to M \times F)_{i,j \in I}$ such that*

- *for all $i, j \in I$, $u_{ij} = \langle \pi_0, u_{ij}\pi_1 \rangle$, and*
- *for $i = j$, $u_{ii} = e_i \times 1_F$ for a restriction idempotent $e_i = \bar{e}_i$.*

Here the expression $u_{ij} = \langle \pi_0, u_{ij}\pi_1 \rangle$ does not mean that $u_{ij}\pi_0 = \pi_0$ or that u_{ij} is total in some first component sense. In fact, due to the definition of restriction limits in Definition 5.2.6(ii), $\langle f, g \rangle \pi_0 = \bar{g} f$ which in our case means $u_{ij}\pi_0 = \bar{u_{ij}}\pi_0$.

Recall the notion of a gluing from Definition 5.2.12. In the following lemma, we show that E is the gluing of a bundle atlas.

Lemma 5.4.4. *For a fibre bundle $(E, M, q, F, (\alpha_i)_{i \in I})$ in a join restriction category, E is a gluing of the atlas $u_{ij} = \alpha_i^* \alpha_j$. This atlas is a bundle atlas as defined in Definition 5.4.3.*

Proof. The partial isomorphisms $\alpha_i^* : M \times F \to E$ fulfill the required properties of Definition 5.2.13 because the atlas u_{ij} was constructed from the partial isomorphisms α_i. The components g_j from Remark 5.2.13 are the inverses α_i^*. They fulfill 5.2.13(ii) by the definition of u_{ij}. They fulfill (iii) because $\bigvee_{i \in I} \bar{\alpha}_i = \bigvee_{i \in I} \alpha_i \alpha_i^* = 1_E$.

For any $i, j \in I$ is of the form $u_{ij} = (\pi_0, u_{ij}\pi_1)$ since

$$u_{ij}\pi_0 = \alpha_i^* \alpha_j \pi_0 = \alpha_i^* \bar{\alpha}_j q = \overline{\alpha_i^* \bar{\alpha}_j} \bar{\alpha}_i^* \alpha_i^* q = \overline{\alpha_i^* \alpha_j} \alpha_i^* \bar{\alpha}_i q = \bar{\alpha_i^*} \alpha_i^* \alpha_i \pi_0 = \bar{\alpha_i^*} \bar{\alpha}_i^* \pi_0.$$

For $i = j$, it is of the form $u_{ii} = e_i \times 1_F$ for an $e_i = \bar{e}_i$ as

$$u_{ii} = \alpha_i^* \alpha_i = \bar{\alpha}_i^* = e_i \times 1.$$

$\square$

Example 5.4.5. *A simple class of examples are the bundles that are given by the projection morphisms $\pi_0 : M \times F \to M$. In this case the structure is given by a single local trivialization:*

$$q = \pi_0 : M \times F \to M, \alpha_0 = 1_{M \times F}$$

This bundle's atlas is $\mathrm{At}(1_{M \times F})$, consisting of the object $M \times F$ and the identity morphism $1_{M \times F}$.

A well known nontrivial example is the Möbius strip in $\mathbb{P}$arMfld (the precise definition of $\mathbb{P}$arMfld, including the restriction structure, will be given in Definition 5.4.25).

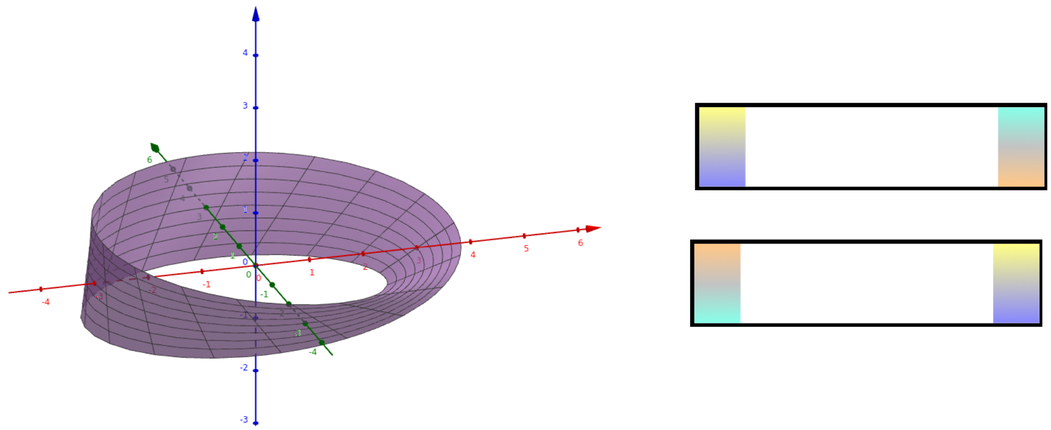

Figure 5.3: On the left is an embedding of the Möbius strip into $\mathbb{R}^3$. On the left is the atlas, out of which it can be glued together. The two stripes are the domains of u_{00} and u_{11} respectively. The colours show how the change of chart partial isomorphisms u_{ij} glue together parts of their domains.[4]

Example 5.4.6. *Let $E = \mathrm{Mo}$ be the Möbius strip. While defined as an abstract manifold, it was shown for example in [36] that it can be embedded into $\mathbb{R}^3$ as*

$$\left\{\left(2-\beta\sin\left(\frac{\alpha}{2}\right)\sin(\alpha), 2-\beta\sin\left(\frac{\alpha}{2}\right)\cos(\alpha), \beta\cos\left(\frac{\alpha}{2}\right)\middle| 0\leq\alpha\leq 2\pi, -1<\beta<1\right)\right\}\subset\mathbb{R}^3.$$

Let $M = S^1$ be the circle and $F = (-1, 1)$ the open interval. Then there is a map $E \to S^1$ given by the angle to the vertical axis. If you embed S^1 into $\mathbb{R}^2$ as

$$S^1 = \{(\sin(\gamma), \cos(\gamma)) \in \mathbb{R}^2 | \gamma \in [0, 2\pi]\} \subset \mathbb{R}^2,$$

this projection map sends a point $(x, y, z) \in \mathrm{Mo} \subset \mathbb{R}^3$ to

$$(x/\sqrt{x^2+y^2}, y/\sqrt{x^2+y^2}) \in S^1 \subset \mathbb{R}^2.$$

Let ε be in $(0, \pi/2)$, for example 1/2. The maps α_0 and α_1 are the partial isomorphisms that identify the left and the right half of the Möbius strip with $(-\varepsilon, \pi+\varepsilon) \times F$ and $(\pi-\varepsilon, 2\pi+\varepsilon) \times F$, both seen as a subset of $S^1 \times F$.

[4]The figure was created with GeoGebra3D (using a slightly tweaked formula), Gimp and LibreOffice.

The Möbius strip is defined by an atlas, $U_0 = U_1 = S^1 \times F$, where we describe the circle by its angle $S^1 = \mathbb{R}/\sim$ with $2\pi n \sim 0\, \forall n \in \mathbb{N}$. The change of chart maps u_{ij} are

$$u_{10}(x,y) = \begin{cases} (x,y) & \text{if } x \in (-\epsilon, \epsilon) \\ (x,-y) & \text{if } x \in (\pi-\epsilon, \pi+\epsilon) \end{cases}, \quad u_{00} = 1|_{(-\epsilon,\pi+\epsilon)\times F},$$

$$u_{01}(x,y) = \begin{cases} (x,y) & \text{if } x \in (-\epsilon, \epsilon) \\ (x,-y) & \text{if } x \in (\pi-\epsilon, \pi+\epsilon) \end{cases}, \text{ and } u_{11} = 1|_{(\pi-\epsilon,2\pi+\epsilon)\times F}.$$

This atlas can be seen as having a straight connection on the one side and a twisted connection on the other side. Its gluing is the Möbius strip Mo*, as depicted in Figure 5.3.*

If we are in a category where gluings exist, the implication of Lemma 5.4.4 becomes an one to one correspondence between atlases and bundles. In the following two lemmas we will establish more precisely what that means. First, we show that one can produce a fibre bundle from an atlas.

Lemma 5.4.7. *Let M and F be objects in a join restriction category $\mathbb{X}$. Let $u_{ij} : M \times F \to M \times F$ be a bundle atlas. Then if the gluing of the atlas $(M \times F, u_{ij})$ exists (see definition 5.2.12), it is a fibre bundle $E \xrightarrow{q} M$ over M with typical fibre F such that*

$$u_{ij} = \alpha_i^* \alpha_j$$

Proof. Let E be a gluing and let $(\alpha_i^*)_{i\in I} : (M \times F, u_{ij}) \to \mathrm{At}(1_E)$ be the resulting bundle morphism as in Definition 5.2.12. We must check that the α_i^* fulfill the conditions in Definition 5.4.1. Due to the remarks following Definition 5.2.12, the morphisms α_i^* are partial isomorphisms. We denote their partial inverses as α_i. In addition the remarks following Definition 5.2.12 guarantee that $\bigvee_i \alpha_i \alpha_i^* = 1$ and $u_{ij} = \alpha_i^* \alpha_j$. Due to the second condition in Definition 5.4.3, $\bar{\alpha}_i^* = \bar{u}_{ii} = (e_{ii} \times 1)$.

Since $f_i = \bar{u}_{ii}\pi_0 : M \times F \to M$ is an atlas morphism from $(M \times F, u_{ij})$ to the atlas $\mathrm{At}(1_M)$ there is an induced morphism $q : E \to M$ such that $f_i = \alpha_i^* q$. Precomposing $f_i = \alpha_i^* q$ with α_i ensures that Diagram 5.7 commutes. Taking the join, $q = \bigvee_i (\alpha_i \pi_0)$, thus $\bar{q} = \bigvee_i \bar{\alpha}_i = 1_E$. Thus q is total.

Therefore $(E, M, q, (\alpha_i)_{i\in I})$ is a fibre bundle. □

Next, we show that fibre bundle morphisms correspond to atlas morphisms of the form $a_{ik} = (\pi_0\phi, A_{ik}\pi_1)$. This is the complement of Lemma 5.4.7 on morphisms.

Lemma 5.4.8. *Let $\mathbb{X}$ be a join restriction category.*

(i) For each fibre bundle morphism

$$(\Phi, \phi) : (E, M, F, q, (\alpha_i)_{i \in I}) \to (E', M', F', q', (\alpha'_k)_{k \in K})$$

the morphisms

$$A_{ik} = \alpha_i^* \Phi \alpha'_k : M \times F \to M' \times F'$$

are atlas morphism whose first components are ϕ, i.e. atlas morphisms of the form $A_{ik} = (\pi_0 \phi, A_{ik} \pi_1)$ between the atlases given by $u_{ij} = \alpha_i^* \alpha_j$ and $u'_{kl} = \alpha'^*_k \alpha'_l$.

(ii) Let u_{ij} and u'_{kl} be fibre atlases which have gluings. For each atlas morphism

$$A_{ik} : (M \times F, u_{ij}) \to (M' \times F', u'_{kl})$$

of the form $(\pi_0 \phi, A_{ik} \pi_1)$, with u_{ij} and u'_{kl} there is an unique morphism between their gluings $\Phi : G_U \to G_{U'}$ such that (Φ, ϕ) is a fibre bundle morphism.

$$A_{ik} = \alpha_i^* \Phi \alpha_k : M \times F \to M' \times F'$$

Proof. (i) We have to check the 5 conditions in the definition of an atlas morphism. All of them follow directly from the definition $A_{ik} = \alpha_i^* \Phi \alpha'_k$.

$$\begin{aligned}
u_{ii} A_{ik} &= \alpha_i^* \alpha_i \alpha_i^* \Phi \alpha'_k = \alpha_i^* \Phi \alpha'_k = A_{ik} \\
A_{ik} u'_{kk} &= \alpha_i^* \Phi \alpha'_k \alpha'^*_k \alpha'_k = \alpha_i^* \Phi \alpha'_k = A_{ik} \\
u_{ij} A_{jk} &= \alpha_i^* \alpha_j \alpha_j^* \Phi \alpha'_k = \alpha_i^* \bar{\alpha}_j \Phi \alpha'_k \le \alpha_i^* \Phi \alpha'_k = A_{ik} \\
A_{ik} u'_{kl} &= \alpha_i^* \Phi \alpha'_k \alpha'^*_k \alpha'_l = \alpha_i^* \Phi \bar{\alpha}'_k \alpha'_l \le \alpha_i^* \Phi \alpha'_k \\
A_{ik} u'_{kl} &= \alpha_i^* \Phi \bar{\alpha}'_k \alpha'_l = \overline{\alpha_i^* \Phi \alpha'_k} \alpha_i^* \Phi \alpha'_l = \bar{A}_{ik} A_{il}
\end{aligned}$$

Since (Φ, ϕ) is a fibre bundle morphism,

$$A_{ik} \pi_0 = \alpha_i^* \Phi \alpha'_k \pi_0 \le \alpha_i^* \Phi q' = \alpha_i^* q \phi \le \pi_0 \phi$$

Thus $A_{ik} = \bar{A}_{ik} (\pi_0 \phi, A_{ik} \pi_1) = (\pi_0 \phi, A_{ik} \pi_1)$.

(ii) Let G_U be the gluing of u_{ij} and $G_{U'}$ the gluing of u_{kl}. Because of the previous lemma, they form a fibre bundle. Since A_{ik} is an atlas morphism from u_{ij} to u'_{kl} and α_k^* form an atlas morphism u'_{kl} to $\mathrm{At}(1_{G_U})$, their composition has target $\mathrm{At}(E')$. Therefore, by the universal property of the gluing G_U, there is an unique morphism $\Phi : G_U \to G_{U'}$ fulfilling

$$\alpha_i^*\Phi = A_{ik}\alpha_k^*$$

Thus

$$\alpha_i^*\Phi\bar{\alpha}_k q' = A_{ik}\alpha_k^*\bar{\alpha}_k q' = A_{ik}\alpha_k^*\alpha_k\pi_0 = \overline{A_{ik}\alpha_k^*}A_{ik}\pi_0 = \overline{A_{ik}\alpha_k^*}\pi_0\phi$$

Precomposing with α_i gives:

$$\bar{\alpha}_i\Phi\bar{\alpha}_k q' = \overline{\alpha_i A_{ik}\alpha_k^*}\alpha_i\pi_0\phi$$

As $\alpha_i\pi_0 = \bar{\alpha}_i p$ this becomes

$$\overline{\bar{\alpha}_i\Phi\alpha_k}\Phi q' = \overline{\alpha_i A_{ik}\alpha_k^*}q\phi$$

Because $\alpha_i^*\Phi = A_{ik}\alpha_k^*$, we obtain

$$\overline{\alpha_i A_{ik}\alpha_k^*}\Phi q' = \overline{\alpha_i A_{ik}\alpha_k^*}\pi_0\phi$$

and taking the join over all i and k gives that $\Phi p' = p\phi$.

□

Lemma 5.4.7 and Lemma 5.4.8 imply together that there is an equivalence of categories between fibre bundles and atlases that fulfill special conditions. Recall from the beginning of section 5.4.1 that we call the category of fibre bundles with fibre bundle morphisms $\mathrm{FibBun}(\mathbb{X})$. We also define the category $\mathrm{BunAtlas}(\mathbb{X})$ which has

- as objects atlases fulfilling the conditions of Lemma 5.4.7
- as morphisms fibre bundle morphisms fulfilling the conditions of Lemma 5.4.8.

Theorem 5.4.9 (Classification of fibre bundles)**.** *Let $\mathbb{X}$ be a join restriction category with gluings. There is an equivalence of categories between FibBun($\mathbb{X}$) and BunAtlas($\mathbb{X}$)*

Proof. We consider the functor sending an atlas in BunAtlas to its gluing. According to Lemmas 5.4.4 and 5.4.7 this is essentially surjective and according to Lemma 5.4.8 it is fully faithful. □

5.4.2 G-bundles and principal G-bundles

Now we will take a group object G which, by Definition 5.3.1, is an object with total multiplication $m : G \times G \to G$, total unit $u : 1 \to G$ and total inverse $\iota : G \to G$. We use this group object to determine the structure of a fibre bundle by acting on the fibre. Having such an action, the fibre is a G-object:

Definition 5.4.10. *Let G be a group object in a Cartesian category $\mathbb{X}$.*

*(i) A G-**object** is an object F together with a left G-action, i.e. a total morphism $a : G \times F \to F$ fulfilling*

$$\begin{tikzcd} G \times G \times F \arrow[r, "1_G \times a"] \arrow[d, "m \times 1_F"'] & G \times F \arrow[d, "a"] \\ G \times F \arrow[r, "a"'] & F \end{tikzcd} \qquad \begin{tikzcd} F \arrow[r, "u \times 1_F"] \arrow[dr, "1_F"'] & G \times F \arrow[d, "m"] \\ & F \end{tikzcd}$$

*(ii) A **morphism of G-objects** from (F, a_F) to (E, a_E) is a morphism $f : F \to E$ in the underlying restriction category, such that*

$$\begin{tikzcd} G \times F \arrow[r, "1_G \times f"] \arrow[d, "a_F"'] & G \times E \arrow[d, "a_G"] \\ F \arrow[r, "f"'] & E \end{tikzcd}$$

commutes.

(iii) With the composition, unit and restriction of the underlying category, $G - \mathbb{O}\mathrm{bj}(\mathbb{X})$ is the category of G-objects and morphisms of G-objects in a given restriction category $\mathbb{X}$.

This definition is analogous to the beginning of Section 5.10 in [71]. In topology and differential geometry, G-spaces are commonly used and they denote G-objects in $\mathbb{T}\mathrm{op}$ and $\mathbb{S}\mathrm{mMan}$ respectively.

Example 5.4.11. *Given a topological group G, a G-space is a topological space Y equipped with a continuous action map $a : G \times Y \to Y$, often denoted like $a(g, y) =: g \triangleright y$ such that $1 \triangleright y = y$, and $(g \cdot g') \triangleright y = g \triangleright (g' \triangleright y)$ for all $g, g' \in G$ and $y \in Y$. In the category* Top *of topological spaces and continuous maps, G-spaces are exactly G-objects.*

An important example is the group object itself.

Example 5.4.12. *A group object G is an example of a G-object. The action is given by*

$$a = m : G \times G \to G.$$

Now we are going to use these in a Cartesian join restriction category. There we will always demand for the morphisms u, m, ι, a to be total.

Definition 5.4.13. *Let G be a group object and M any other object in a join restriction category $\mathbb{X}$. Then a G-**atlas** on M consists of partial morphisms $\tau_{ij} : M \to G$ such that*

$$(\tau_{jk}, \tau_{ij})m \leq \tau_{ik} \qquad \tau_{ii} \leq !u \qquad \tau_{ji} = \tau_{ij}\iota$$

Definition 5.4.14. *Let G be a group object in a Cartesian join restriction category and M another object. Then*

*(i) A G-**bundle** over M with typical fibre F consist of a G-object F, a fibre bundle $E \to M$ with fibre F, and a G-atlas τ_{ij} on M such that*

$$u_{ij} = \alpha_i^* \alpha_j = (\pi_0, (\pi_0 \tau_{ij}, \pi_1)a) : M \times F \to M \times F$$

*(ii) A **principal** G-**bundle** over M is a G-bundle with fibre G and the multiplication from the left $a = m : G \times G \to G$ as the action.*

In Definition 5.4.17 we will define morphisms in a way so that we will obtain a join-restriction category of G-bundles and a join-restriction category of principal G-bundles.

First we construct a G-action on a principal bundle:

Theorem 5.4.15 (Total right action on principal bundles)**.** *Let $P \to M$ be a principal G-bundle in a join restriction category $\mathbb{X}$. Then there exists a total morphism (a right action)*

$$r : P \times G \to P$$

such that

$$(r \times 1)r = (1 \times m)r : P \times G \times G \to P, \quad (1 \times e)r = 1_P \quad \textit{and} \quad rq = \pi_0 q : P \times G \to M.$$

Proof. In order to define the morphism r we define the partial morphisms

$$r_i = (\alpha_i \times 1_G)(1_M \times m)\alpha_i^* : P \times G \to G.$$

This means, r_i is the following composition:

$$P \times G \xrightarrow{\alpha_i \times 1_G} M \times G \times G \xrightarrow{1_M \times m} M \times G \xrightarrow{\alpha_i^*} P$$

If the join $r = \bigvee_{i\in I} r_i$ is defined, it will automatically fulfill all the required properties, as its components fulfill it. It will be a total morphism as

$$\begin{aligned}
\bar{r}_i &= \overline{(\alpha_i \times 1)(1 \times m)\alpha_i^*} \\
&= \overline{(\alpha_i \times 1_G)(1_M \times m)\bar{\alpha}_i^*} \\
&= \overline{(\alpha_i \times 1_G)(1_M \times m)(e_i \times 1_G)} \\
&= \overline{(\alpha_i \times 1_G)(e_i \times 1_{G\times G})(1 \times m)} \\
&= \overline{(\alpha_i \times 1_G)(e_i \times 1_{G\times G})} \\
&= \overline{(\alpha_i \times 1_G)(\bar{\alpha}_i^* \times 1)} \\
&= \overline{(\alpha_i \times 1_G)(\alpha_i^* \times 1)} \\
&= \overline{(\bar{\alpha}_i \times 1_G)} \\
&= \bar{\alpha}_i \times 1_G
\end{aligned}$$

and $\bigvee_i (\bar{\alpha}_i^* \times 1_G) = \bigvee_i \bar{\alpha}_i^* \times 1_G = 1_P \times 1_G$, and thus

$$\overline{\bigvee r_i} = \bigvee \bar{r}_i = \bigvee (\bar{\alpha}_i \times 1) = 1_{P\times G}.$$

Let's now prove that the join exists, i.e. that $r_i \smile r_j$. For this we need to show that $\bar{r}_i r_j = \bar{r}_j r_i$. Observe that

$$\begin{aligned}
\bar{r}_i r_j &= \overline{(\alpha_i \times 1_G)(1_M \times m)\alpha_i^*}(\alpha_j \times 1_G)(1_M \times m)\alpha_j^* \\
&= \overline{(\bar{\alpha}_i \times 1_G)(\alpha_i \times 1_G)(1_M \times m)\alpha_i^*}\,\overline{(\alpha_j \times 1_G)}(\alpha_j \times 1_G)(1_M \times m)\alpha_j^* \\
&= \overline{(\bar{\alpha}_j \times 1_G)(\alpha_i \times 1_G)(1_M \times m)\alpha_i^*}(\bar{\alpha}_i\alpha_j \times 1_G)(1_M \times m)\alpha_j^* \\
&= \overline{(\alpha_j u_{ji} \times 1_G)(1_M \times m)\alpha_i^*}(\bar{\alpha}_i\alpha_j \times 1_G)(1_M \times m)\alpha_j^* \\
&= \overline{(\alpha_j \times 1_G)(\pi_0, (\pi_0\tau_{ji}, \pi_1)m, \pi_2)(1_M \times m)\alpha_i^*}(\bar{\alpha}_i\alpha_j \times 1_G)(1_M \times m)\alpha_j^*.
\end{aligned}$$

By associativity this equals

$$\begin{aligned}
&= \overline{(\alpha_j \times 1_G)(\pi_0, \pi_0\tau_{ji}, (\pi_1, \pi_2)m)(1_M \times m)\alpha_i^*}(\bar{\alpha}_i\alpha_j \times 1_G)(1_M \times m)\alpha_j^* \\
&= \overline{(\alpha_j \times 1_G)(1 \times m)((\pi_0, \pi_0\tau_{ji}) \times 1_G)(1_M \times m)\alpha_i^*}(\bar{\alpha}_i\alpha_j \times 1_G)(1_M \times m)\alpha_j^*.
\end{aligned}$$

As the multiplication with τ_{ji} gives $u_{ji} = \alpha_j^* \alpha_i$ this equals

$$
\begin{aligned}
&= \overline{(\alpha_j \times 1_G)(1 \times m)\alpha_j^* \alpha_i \alpha_i^*}(\bar{\alpha}_i \alpha_j \times 1_G)(1_M \times m)\alpha_j^* \\
&= \overline{(\alpha_j \times 1_G)(1 \times m)\alpha_j^* \bar{\alpha}_i}(\alpha_i u_{ij} \times 1_G)(1_M \times m)\alpha_j^* \\
&= \overline{(\alpha_j \times 1_G)(1 \times m)\alpha_j^* \bar{\alpha}_i}(\alpha_i \times 1_G)((1, \tau_{ij}) \times 1_G \times 1_G)(1_M \times m \times 1_G)(1_M \times m)\alpha_j^*.
\end{aligned}
$$

By associativity this is

$$
\begin{aligned}
&= \overline{(\alpha_j \times 1_G)(1 \times m)\alpha_j^* \bar{\alpha}_i}(\alpha_i \times 1_G)((1_M, \tau_{ij}) \times m)(1_M \times m)\alpha_j^* \\
&= \overline{(\alpha_j \times 1_G)(1 \times m)\alpha_j^* \bar{\alpha}_i}(\alpha_i \times 1_G)(1_M \times m)\alpha_i^* \alpha_j \alpha_j^* \\
&= \overline{(\alpha_j \times 1_G)(1 \times m)\alpha_j^* \bar{\alpha}_i}(\alpha_i \times 1_G)(1_M \times m)\alpha_i^* \bar{\alpha}_j \\
&\leq \overline{(\alpha_j \times 1_G)(1 \times m)\alpha_j^*}(\alpha_i \times 1_G)(1_M \times m)\alpha_i^* \\
&= \bar{r}_j r_i.
\end{aligned}
$$

Thus $\bar{r}_i r_j \leq \bar{r}_j r_i$. As i and j were arbitrary, we also obtain $\bar{r}_i r_j \geq \bar{r}_j r_i$. Thus they are equal. □

It is not possible to define such a right G action for arbitrary G-bundles because F does not have a right action, in general. Since it is defined using the multiplication m and the partial isomorphisms α_i, it behaves like the multiplication when composed with the partial isomorphisms α_i, as the following Corollary shows.

Corollary 5.4.16. *Let $P \to M$ be a principal bundle in a join restriction category $\mathbb{X}$. Let $r : P \times G \to P$ be the action constructed in Theorem 5.4.15. Then the equations*

$$
\begin{aligned}
r\alpha_i &= (\alpha_i \times 1_G)(1_M \times m) : P \times G \to M \times G \\
(\alpha_i^* \times 1_G)r &= (1_M \times m)\alpha_i^* : M \times G \times G \to P
\end{aligned}
$$

hold.

Proof. We will only prove the first equation, the second equation works analogously.

$$\begin{aligned}
r\alpha_i &= \bigvee_{j\in I}(\alpha_j \times 1_G)(1_M \times m)\alpha_j^*\alpha_i \\
&= \bigvee_{j\in I}(\alpha_j \times 1_G)(1_M \times m)u_{ji} \\
&= \bigvee_{j\in I}(\alpha_j \times 1_G)(1_M \times m)\langle \pi_0, \langle \pi_0\tau_{ji}, \pi_1\rangle m\rangle \\
&= \bigvee_{j\in I}(\alpha_j \times 1_G)\langle \pi_0, \langle \pi_0\tau_{ji}, \langle \pi_1, \pi_2\rangle m\rangle m \\
&= \bigvee_{j\in I}(\alpha_j \times 1_G)\langle \pi_0, \langle\langle \pi_0\tau_{ji}, \pi_1\rangle m, \pi_2\rangle m && \text{by associativity} \\
&= \bigvee_{j\in I}(\alpha_j \times 1_G)(u_{ji} \times 1_G)(1_M \times m) \\
&= \bigvee_{j\in I}(\bar{\alpha}_j\alpha i \times 1_G)(1_M \times m) \\
&= (\alpha i \times 1_G)(1_M \times m) && \text{since } \bigvee_{j\in I}\bar{\alpha}_j = 1_P
\end{aligned}$$

□

Now let us define morphisms for G-bundles and principal G-bundles.

Definition 5.4.17. *For a fixed group-object G, **a morphism of principal G-bundles** is a morphism of fibre bundles*

$$(\Phi, \phi) : (P, M) \to (P', M')$$

such that

$$r_P\Phi = (\Phi \times 1_G)r_{P'}$$

i.e. the right action commutes with the morphism.

In a join-restriction category $\mathbb{X}$, the principal G-bundles and morphisms of principal G-bundles form the restriction category $\mathbb{P}\mathrm{Bun}_G(\mathbb{X})$.

We will now relate this to the way the principal bundle is built out of an atlas.

Proposition 5.4.18. *Let G be a group object in a join-restriction category $\mathbb{X}$. Let (P, M) and (P', M') be totally fibred principal G-bundles. Giving a morphism of principal G-bundles $(\Phi, \phi) : (P, M) \to (P', M')$ is equivalent to giving*

- *the morphism $\phi : M \to M'$ fulfilling $q\phi = q\phi \bigvee_{k\in K} e_k$*

- *and morphisms $T_{ik} : M \to G$ such that*
  - *(i)* $T_{ik} \geq (T_{jk}, \tau_{ij})m$,
  - *(ii)* $T_{ik} \geq (\phi\tau'_{lk}, T_{il})m$,
  - *(iii)* $\bar{T}_{ik}T_{ih} = (\phi\tau_{kh}, T_{ik})m$ *(this actually implies the second point), and*
  - *(iv)* $\overline{T_{ik}} = \overline{e_i\phi e_k}$.

The composition is given by

$$(\phi, T_{ik})(\tilde{\phi}, \tilde{T}_{kl}) = (\phi\tilde{\phi}, \bigvee_k (T_{kl}, T_{ik})m)$$

and the restriction is given by

$$\overline{(\phi, T_{ik})} = (\bar{\phi}, \bar{\phi}\tau_{ij}).$$

The idea of this is that Φ is locally given by the left-multiplication with T_{ik}.

Proof. Let Φ, ϕ be a principal bundle morphism from P to P'. Then Φ is a morphism between gluings and therefore induces an atlas morphism $A_{ik} : M \times G \to M' \times G$ between the atlas u_{ij} of P and the atlas u_{kl} of P'. Explicitly $A_{ik} = \alpha_i^*\Phi\alpha_k$. Since the diagram

$$\begin{tikzcd}
M \times G \arrow[r, "\alpha_i^*"] \arrow[d, "\pi_0"'] & P \arrow[r, "\Phi"] \arrow[d, "1_P"'] & P' \arrow[r, "\alpha_k"] \arrow[d, "1_{p'}"'] & M \times G' \arrow[d, "\pi_0"'] \\
M \times G \arrow[r, "="] & P \arrow[r, "\phi"] & P' \arrow[r, "="] & M' \times G
\end{tikzcd}$$

commutes in the sense that the lower path $\geq$ the upper path we know that $A_{ik}\pi_0 \leq \pi_0\phi$. Thus A_{ik} can be written in the form $A_{ik} = (\pi_0\phi, \tilde{A}_{ik})$, with $\tilde{A}_{ik} : M \times G \to G$. Since the right action commutes with the morphism, we have that

$$(\tilde{A}_{ik} \times 1_G)m = (1_M \times m)\tilde{A}_{ik}.$$

But because of the unit property of the group object G, we have that

$$\tilde{A}_{ik} = (1_M \times u \times 1_G)(1_M \times m)\tilde{A}_{ik} = (1_M \times u \times 1_G)(\tilde{A}_{ik} \times 1_G)(1_M \times m) = (T_{ik} \times 1_G)(1_M \times m)$$

where $T_{ik} = (1_M \times u)\tilde{A}_{ik} : M \to G$ fulfills Properties (i),(ii) and (iii), because $A_{ik} = (\phi, (T_{ik} \times 1_G)m)$ is an atlas morphism. In order to show Property (iv) we will first see that according to the construction above

$$T_{ik} = (1_M \times u)\tilde{A}_{ik} = (1_M \times u)A_{ik}\pi_1 = (1_M \times u)\alpha_i^*\Phi\alpha_k\pi_1$$

holds. Using this we can calculate

$$
\begin{aligned}
\overline{T_{ik}} &= \overline{(1_M \times u)\alpha_i^* \Phi \alpha_k \pi_1} \\
&= \overline{(1_M \times u)\alpha_i^* \Phi \alpha_k \pi_0} \\
&= \overline{(1_M \times u)\alpha_i^* \Phi \bar{\alpha}_k q'} \\
&= \overline{(1_M \times u)\alpha_i^* \Phi q' e_k} && \text{since } (E', M') \text{ is totally fibred} \\
&= \overline{(1_M \times u)\alpha_i^* q \phi e_k} \\
&= \overline{(1_M \times u)\alpha_i^* \bar{\alpha}_i q \phi e_k} \\
&= \overline{(1_M \times u)\alpha_i^* \alpha_i \pi_0 \phi e_k} \\
&= \overline{(1_M \times u)(e_i \times 1_G)\pi_0 \phi e_k} \\
&= \overline{e_i \phi e_k}
\end{aligned}
$$

which proves property (iv). Since the bundle (P', M') was assumed to be totally fibred, the equation

$$q\phi \bigvee_{k \in K} e_k = \Phi q \bigvee_{k \in K} e_k = \Phi \bigvee_{k \in K} \bar{\alpha}_k q = \Phi q = q\phi$$

holds.

Conversely, let now ϕ and T_{ik} be given. Then because of properties (i),(ii),(iii) of T_{ik}, the morphisms $A_{ik} := (\phi, (T_{ik} \times 1_G)m)$ form an atlas morphism, thus they induce a morphism

$$\Phi = \bigvee_{i,k} \alpha_i \langle \pi_0 \phi, (T_{ik} \times 1_G)m \rangle \alpha_k^*$$

between the gluings. Next we will show that the diagrams

$$
\begin{array}{ccc}
P & \xrightarrow{\Phi} & P' \\
{\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q'} \\
M & \xrightarrow[\phi]{} & M'
\end{array}
\qquad
\begin{array}{ccc}
P \times G & \xrightarrow{\Phi \times 1_G} & P' \times G \\
{\scriptstyle r}\downarrow & & \downarrow{\scriptstyle r'} \\
P & \xrightarrow[\Phi]{} & P
\end{array}
$$

commute. Here r and r' denote the total right-actions from Theorem 5.4.15. We calculate

$$\begin{aligned}
\Phi q' &= \bigvee_{i,k} \alpha_i \langle \pi_0\phi, (T_{ik} \times 1_G)m\rangle \alpha_k^* q' \\
&= \bigvee_{i,k} \alpha_i \langle \pi_0\phi, (T_{ik} \times 1_G)m\rangle \alpha_k^* \bar{\alpha}_k q' \\
&= \bigvee_{i,k} \alpha_i \langle \pi_0\phi, (T_{ik} \times 1_G)m\rangle \bar{\alpha}_k^* \pi_0 \\
&= \bigvee_{i,k} \alpha_i \langle \pi_0\phi, (T_{ik} \times 1_G)m\rangle (e_k \times 1_G)\pi_0 \\
&= \bigvee_{i,k} \alpha_i \langle \pi_0\phi e_k, (T_{ik} \times 1_G)m\rangle \pi_0 \\
&= \bigvee_{i,k} \alpha_i \overline{\pi_0 T_{ik}} \pi_0 \phi e_k \\
&= \bigvee_{i,k} \alpha_i \pi_0 \overline{T_{ik}} \phi e_k \\
&= \bigvee_{i,k} q e_i \overline{T_{ik}} \phi e_k && \text{totally fibred} \\
&= \bigvee_{i,k} q \overline{T_{ik}} e_i \phi e_k \\
&= \bigvee_{i,k} q \overline{e_i \phi e_k} e_i \phi e_k \\
&= \bigvee_{i,k} q e_i \phi e_k \\
&= \bigvee_{i,k} \bar{\alpha}_i q \phi e_k && \text{totally fibred} \\
&= q\phi \bigvee_{k} e_k = q\phi.
\end{aligned}$$

In order to show that Φ is compatible with the right action r, we calculate

$$
\begin{aligned}
(\Phi \times 1_G)r' &= \bigvee_{i,k} \big(\alpha_i\langle \pi_0\phi, (T_{ik} \times 1_G)m\rangle\alpha_k^* \times 1_G\big)r' \\
&= \bigvee_{i,k} \big(\alpha_i\langle \pi_0\phi, \langle \pi_0 T_{ik}, \pi_1\rangle m\rangle \times 1_G\big)(\alpha_k^* \times 1_G)r' \\
&= \bigvee_{i,k} \big(\alpha_i\langle \pi_0\phi, \langle \pi_0 T_{ik}, \pi_1\rangle m\rangle \times 1_G\big)(1_M \times m)\alpha_k^* && \text{Corollary 5.4.16} \\
&= \bigvee_{i,k} \langle \pi_0\alpha_i\pi_0\phi, \langle\langle \pi_0\alpha_i\pi_0 T_{ik}, \pi_0\alpha_i\pi_1\rangle m, \pi_1\rangle m\rangle\alpha_k^* \\
&= \bigvee_{i,k} \langle \pi_0\alpha_i\pi_0\phi, \langle \pi_0\alpha_i\pi_0 T_{ik}, \langle \pi_0\alpha_i\pi_1, \pi_1\rangle m\rangle m\rangle\alpha_k^* && \text{associativity} \\
&= \bigvee_{i,k} (\alpha_i \times 1_G)(1_M \times m)\langle \pi_0\phi, (T_{ik} \times 1_G)m\rangle\alpha_k^* \\
&= \bigvee_{i,k} r\alpha_i\langle \pi_0\phi, (T_{ik} \times 1_G)m\rangle\alpha_k^* && \text{Corollary 5.4.16} \\
&= r\Phi
\end{aligned}
$$

The composition is given as the composition of atlas morphisms like in [20]. The restriction is determined by the formula for T_{ik} when substituting in $\bar{\Phi}$, i.e. we calculate

$$
\begin{aligned}
(1_M \times u)\alpha_i^*\bar{\Phi}\alpha_j\pi_1 &= (1_M \times u)\alpha_i^*\overline{q\phi}\alpha_j\pi_1 && \text{since } \bar{\Phi} = \overline{q\phi} \\
&= \overline{(1_M \times u)\alpha_i^* q\phi}(1_M \times u)\alpha_i^*\alpha_j\pi_1 \\
&= \overline{e_i\phi}(1_M \times u)u_{ij}\pi_1 && \text{since } \alpha_i^* q = \bar{\alpha}_i^*\pi_0 = \pi_0 e_i \\
&= \bar{\phi}e_i(1_M \times u)(\tau_{ij} \times 1_G)m && \text{since } u_{ij} = \langle \pi_0, (\tau_{ij} \times 1_G)m\rangle \\
&= \bar{\phi}e_i(\tau_{ij} \times u)m \\
&= \bar{\phi}e_i\tau_{ij} = \bar{\phi}\tau_{ij} && \text{since } e_i\tau_{ij} = \tau_{ij}
\end{aligned}
$$

□

We now generalize this characterization from Proposition 5.4.18 as the definition for morphisms between arbitrary (not just principal) G-bundles. The morphism $\psi : F \to F'$ allows us to define morphisms between fibre bundles with different fibres, as long as ψ preserves the group actions.

Definition 5.4.19. *Let $(E, M, F, q, \alpha_i, \tau_{ij})$ and $(E', M', F', q', \alpha_i', \tau_{kl}')$ be G-bundles. Then a **morphism of G-bundles** between them consists of*

- *A morphism $\phi : M \to M'$ fulfilling $q\phi = q\phi \bigvee_{k\in K} e_k$*

- *A morphism $\psi : F \to F'$ such that $(\psi \times 1_G)a' = a\psi$*

- *A collection of morphisms $T_{ik} : M \to G$ such that*

  (i) $T_{ik} \geq (T_{jk}, \tau_{ij})m$,

  (ii) $T_{ik} \geq (\phi\tau'_{lk}, T_{il})m$,

  (iii) $\bar{T}_{ik}T_{ih} = (\phi\tau_{kh}, T_{ik})m$, *and*

  (iv) $\overline{T_{ik}} = \overline{e_i\phi e_k}$.

- *A morphism $\Phi : E \to E'$ such that*

$$\begin{tikzcd}
M \times F \arrow[r, "\langle \pi_0\phi, (T_{ik} \times \psi)a' \rangle"] \arrow[d, "\alpha_i^*"'] & M' \times F' \\
E \arrow[r, "\Phi"'] & E' \arrow[u, "\alpha_k"']
\end{tikzcd}$$

commutes.

The composition of morphisms $(\phi, \psi, (T_{ij})_{i\in I, j\in J}, \Phi)$ and $(\phi', \psi', (T'_j)_{j\in J, k\in K}, \Phi)$ of G-bundles is given by the composition and matrix multiplication $(\phi\phi', \psi\psi', (\bigvee_{j\in J} T_{jk}T_{ij})_{i\in I, k\in K}, \Phi\Phi')$. The identity of morphisms of G-bundles is $(1_M, 1_F, (\tau_{ij})_{i,j\in I}, 1_E)$. The restriction of a morphism $(\phi, \psi, (T_{ij})_{i\in I, j\in J}, \Phi)$ of G-bundles is

$$\overline{(\phi, \psi, (T_{ij})_{i\in I, j\in J}, \Phi)} = (\bar{\phi}, \bar{\psi}, (\bar{\phi}\tau_{ij})_{i,j\in I}, \bar{\Phi}).$$

We denote the restriction category of G-bundles and morphisms of G-bundles as $\mathbb{B}\mathrm{un}_G(\mathbb{X})$.

Summarizing, we have the following categories which are all join restriction categories:

(i) $\mathbb{F}\mathrm{ibBun}(\mathbb{X})$: Objects are fibre bundles, morphisms are fibre bundle morphisms.

(ii) $G - \mathbb{O}\mathrm{bj}(\mathbb{X})$: Objects are G-objects, morphisms are morphisms commuting with the group action.

(iii) $\mathbb{B}\mathrm{un}_G(\mathbb{X})$: Objects are G-bundles, morphisms are the G-bundle morphisms from Definition 5.4.19.

(iv) $\mathbb{P}\mathrm{Bun}_G(\mathbb{X})$: Objects are principal G-bundles, morphisms are the G-bundle morphisms from Definition 5.4.17 .

In order to obtain the equivalence of Theorem 5.4.24, we use a slightly variation of of the condition for a G-bundle to be totally fibred. We first prove a lemma that will make it easier to characterize bundles which are totally fibred, since it only relies on the atlas $(\alpha_i^*\alpha_j)_{i,j\in I}$ and then generalize it.

Lemma 5.4.20. *In a join restriction category, a fibre bundle $(E, M, F, q, (\alpha_i)_{i\in I})$ is totally fibred if and only if for all $i \in I$, the equation $\overline{\alpha_i^* \alpha_j} = \overline{\pi_0 e_i e_j}$ holds.*

Proof. Suppose $(E, M, F, q, (\alpha_i)_{i\in I})$ is totally fibred. Then

$$\overline{\alpha_i^* \alpha_j} = \overline{\alpha_i^* q_0 e_j} = \overline{\bar{\alpha}_i^* \pi_0 e_j} = \overline{(1 \times e_i)\pi_0 e_j} = \overline{\pi_0 e_i e_j}.$$

Conversely, let $\overline{\alpha_i^* \alpha_j} = \overline{\pi_0 e_i e_j}$ hold for all $i, j \in I$. Then

$$\begin{aligned}
\bar{\alpha}_i &= \overline{\bigvee_{j\in I} \bar{\alpha}_j \alpha_i} && \text{since } \bigvee_{j\in I} \bar{\alpha}_j = 1_E \text{ by Definition 5.4.1} \\
&= \overline{\bigvee_{j\in I} \alpha_j \alpha_j^* \alpha_i} \\
&= \overline{\bigvee_{j\in I} \alpha_j \overline{\alpha_j^* \alpha_i}} \\
&= \overline{\bigvee_{j\in I} \alpha_j \overline{\pi_0 e_j e_i}} && \text{by the assumption} \\
&= \overline{\bigvee_{j\in I} \alpha_j \pi_0 e_i e_i} \\
&= \overline{\bigvee_{j\in I} \alpha_j \bar{\alpha}_j^* \pi_0 e_i} \\
&= \overline{\bigvee_{j\in I} \alpha_j \pi_0 e_i} \\
&= \overline{\bigvee_{j\in I} \bar{\alpha}_j q e_i} \\
&= \overline{q e_i}
\end{aligned}$$

which means that $(E, M, F, q, (\alpha_i)_{i\in I})$ is totally fibred. ◻

Definition 5.4.21. *Let $(E, M, F, q, (\alpha_i)_{i\in I}, (\tau_{ij})_{i,j\in I})$ be a G-bundle in a join restriction category. Then it is called **totally G-fibred** if $\bar{\tau}_{ij} = e_i e_j$ for all $i, j \in I$.*

The next lemma shows that, intuitively this is equivalent to being totally fibred, as long as there is a point in the fibre F.

Lemma 5.4.22. *Let $(E, M, F, q, (\alpha_i)_{i\in I}, (\tau_{ij})_{i,j\in I})$ be a G-bundle in a join restriction category.*

(i) If $(E, M, F, q, (\alpha_i)_{i\in I}, (\tau_{ij})_{i,j\in I})$ is totally G-fibred, then it is totally fibred.

(ii) If $\pi_0 : M \times F \to F$ is an epimorphism and $(E, M, F, q, (\alpha_i)_{i\in I}, (\tau_{ij})_{i,j\in I})$ is totally fibred, then it is totally G-fibred.

(iii) If $q : E \to M$ is an epimorphism and $(E, M, F, q, (\alpha_i)_{i\in I}, (\tau_{ij})_{i,j\in I})$ is totally fibred, it is totally G-fibred.

(iv) If there exists at least one point $x : 1 \to F$ and $(E, M, F, q, (\alpha_i)_{i\in I}, (\tau_{ij})_{i,j\in I})$ is totally fibred, then it is totally G-fibred.

(v) A principal G-bundle $(P, M, q, (\alpha_i)_{i\in I}, (\tau_{ij})_{i,j\in I})$ is totally G-fibred if and only if it is totally fibred.

Proof. (i) Suppose $\bar{\tau}_{ij} = e_i e_j$. Then

$$\overline{\alpha_i^* \alpha_j} = \overline{\langle \pi_0, \langle \pi_0 \tau_{ij}, \pi_1 \rangle a \rangle} = \overline{\pi_0 \tau_{ij}} = \overline{\pi_0 e_i e_j},$$

which implies that $(E, M, F, q, (\alpha_i)_{i\in I}, (\tau_{ij})_{i,j\in I})$ is totally fibred due to Lemma 5.4.20.

(ii) Suppose $\pi_0 : M \times F \to F$ is an epimorphism and $\overline{\alpha_i^* \alpha_j} = \overline{\pi_0 e_i e_j}$. Then

$$\pi_0 \overline{\tau_{ij}} = \overline{\pi_0 \tau_{ij}} \pi_0 = \overline{\alpha_i^* \alpha_j} \pi_0 = \overline{\pi_0 e_i e_j} \pi_0 = \pi_0 \overline{e_i e_j}$$

and since π_0 is an epimorphism this implies $\bar{\tau}_{ij} = e_i e_j$.

(iii) Suppose $q : E \to M$ is an epimorphism. We will show that $\pi_0 : M \times F \to M$ is an epimorphism and therefore (ii) will apply. Suppose $\pi_0 f = \pi_0 g$. Then

$$\bar{\alpha}_i q f = \alpha_i \pi_0 f = \alpha_i \pi_0 g = \bar{\alpha}_i q g.$$

Thus it follows that

$$qf = \bigvee_{i\in I} \bar{\alpha}_i q f = \bigvee_{i\in I} \bar{\alpha}_i q g = qg.$$

Since q was an epimorphism, it follows that $f = g$, proving that π_0 is an epimorphism.

(iv) If there is a point $x : 1 \to F$, then $\langle 1_M, !x \rangle : M \to M \times F$ is a section of π_0. Then π_0 is a retraction and therefore an epimorphism, so (iii) applies.

(v) A principal G-bundle has the group object G as its fibre and the unit $u : 1 \to G$ is a point in G. Thus (iv) applies.

□

We will now use this notion of totally G-fibred to define certain functors between categories of G-bundles. For this, let $\mathbb{B}\mathrm{un}_G^{\mathrm{tot}_G}$ denote the full subcategory of $\mathbb{B}\mathrm{un}_G$ consisting of totally G-fibred G-bundles. Let $\mathbb{P}\mathrm{Bun}_G^{\mathrm{tot}}$ denote the full subcategory of $\mathbb{P}\mathrm{Bun}_G$ consisting of totally fibred principal G-bundles.

Definition 5.4.23. *Given a join-restriction category $\mathbb{X}$ with gluings, we define the following functors. Note that they satify the definition of restriction functors (Definition 5.2.1 (iv)).*

(i) $\texttt{Fibre} : \mathbb{B}\mathrm{un}_G \to G - \mathbb{O}\mathrm{bj}$ *that sends a G-bundle to its fibre F and a morphism of G-bundles to its fibre component ψ.*

(ii) $\texttt{Principal}^{\mathrm{tot}_G} : \mathbb{B}\mathrm{un}_G^{\mathrm{tot}_G} \to \mathbb{P}\mathbb{B}\mathrm{un}_G^{\mathrm{tot}}$ *that sends a totally G-fibred G-bundle E to the principal bundle P_E that is the gluing of the atlas*

$$u_{ij} := (\tau_{ij} \times 1_G)(1_M \times m) : M \times G \to M \times G$$

which will be totally fibred due to Lemma 5.4.20 and since $\bar{u}_{ij} = \overline{\pi_0 \tau_{ij}} = \overline{\pi_0 e_i e_j}$. $\texttt{Principal}^{\mathrm{tot}_G}$ *sends a morphism $(\phi, \psi, T_{ik}, \Phi)$ to (ϕ, T_{ik}) which is, according to Proposition 5.4.18 a way of giving a principal bundle morphism.*

(iii) $\texttt{Build}^{\mathrm{tot}_G} : \mathbb{P}\mathbb{B}\mathrm{un}_G^{\mathrm{tot}} \times G - \mathbb{O}\mathrm{bj} \to \mathbb{B}\mathrm{un}_G^{\mathrm{tot}_G}$ *that sends a totally fibred principal G-bundle P and a G-object F to the G-bundle given as the gluing of the atlas*

$$u_{ij} := ((1_M, \tau_{ij}) \times 1_F)(1_M \times a) : M \times F \to M \times F.$$

where $(1_M, \tau_{ij}) : M \to M \times G$ gives a group element that acts through $a : G \times F \to F$, the G-action on F. It is a totally G-fibred G-bundle since its G-atlas is τ_{ij}, the same G-atlas as P had and P was totally G-fibred due to Lemma 5.4.22(v). For a morphism $((\phi, T_{ik}), (\psi))$ of $\mathbb{P}\mathbb{B}\mathrm{un}_G \times G$-Space, consider the atlas morphism given by

$$A_{ik} = ((\phi, T_{ik}) \times \psi)(1_{M'} \times a')$$

where $(\phi, T_{ik}) \times \psi) : M \times F \to M' \times G \times F'$ produces a group element that acts through $(1_{M'} \times a') : M' \times G \times F' \to M' \times F'$. This atlas morphism induces a morphism Φ between the gluings. The result of the functor is then the morphism of G-bundles given by $(\phi, \psi, T_{ik}, \Phi)$.

All this works because the collection $(T_{ik})_{i \in I, k \in J}$ in Definition 5.4.19 of a morphisms of G-bundles was required to fulfill the conditions of an atlas-morphism and of the equivalent characterization of principal bundle morphisms in Proposition 5.4.18.

Theorem 5.4.24. *The functors*

$$\mathbb{B}\mathrm{un}_G^{\mathrm{tot}_G} \xrightarrow{(\mathtt{Principal}^{\mathrm{tot}_G},\mathtt{Fibre})} \mathbb{P}\mathrm{Bun}_G^{\mathrm{tot}} \times G - \mathbb{O}\mathrm{bj}$$
$$\mathbb{B}\mathrm{un}_G^{\mathrm{tot}_G} \xleftarrow{\mathtt{Build}^{\mathrm{tot}_G}} \mathbb{P}\mathrm{Bun}_G^{\mathrm{tot}} \times G - \mathbb{O}\mathrm{bj}$$

form an equivalence of categories between $\mathbb{B}\mathrm{un}_G^{\mathrm{tot}_G}$ *and* $\mathbb{P}\mathrm{Bun}_G^{\mathrm{tot}} \times G - \mathbb{O}\mathrm{bj}$.

This means that all totally G-fibred G-bundles can already be described by their fibre and a principal bundle containing the bundle information.

Proof. In order to prove that these functors form an equivalence, we need to show that their composition is naturally isomorphic to the identity. Let $(P, F) \in \mathbb{P}\mathrm{Bun}_G^{\mathrm{tot}} \times G - \mathbb{O}\mathrm{bj}$ be given. Then $\mathtt{Build}^{\mathrm{tot}_G}(P, F)$ has fibre F and the same τ_{ij} as P. Thus

$$(\mathtt{Principal}^{\mathrm{tot}_G}, \mathtt{Fibre})(\mathtt{Build}^{\mathrm{tot}_G}(P, F)) = (\tilde{P}, F)$$

where $\tilde{P}$ is the gluing of the atlas $((\tau_{ij} \times 1_G)(1_M \times m))_{i,j\in I}$. But P is also a gluing of the same atlas. Thus there is an isomorphism $\varphi : \tilde{P} \to P$ with $\alpha_i = \varphi\tilde{\alpha}_i$.

Due to Proposition 5.4.18 this morphism is completely described by $(1_M, \tau_{ij})$. Let $(\phi, (T_{ik})_{i\in I,k\in J})$ describe any principal bundle morphism. Then, due to the composition of atlases, the compositions equal

$$(1_M, (\tau_{ij})_{i,j\in I}); (\phi, (T_{jk})_{j\in I,k\in J}) = (\phi, (T_{ik})_{i\in I,k\in J}) = (\phi, (T_{ik})_{i\in I,k\in J})(1_{M'}, (\tau_{kl})_{k,l\in J}).$$

Since $\mathtt{Build}^{\mathrm{tot}_G}; \mathtt{Principal}^{\mathrm{tot}_G}$ does not change what a morphism does on atlases, this proves that φ is natural. Thus $(\varphi \times 1_{G-\mathrm{Space}})$ is the required natural isomorphism.

In the other direction let E be a totally G-fibred G-bundle. Then $\mathtt{Build}^{\mathrm{tot}_G}(\mathtt{Fibre}(E), \mathtt{Principal}^{\mathrm{tot}_G}(E))$ is by definition the gluing of the atlas

$$u_{ij} := (\tau_{ij} \times 1_F)(1_M \times a) : M \times F \to M \times F$$

But so is E, so there is an isomorphism $\varphi_E : \mathtt{Build}^{\mathrm{tot}_G}(\mathtt{Fibre}(E), \mathtt{Principal}^{\mathrm{tot}_G}(E)) \to E$. Naturality again follows by considering the atlas description of φ_E, where it is just the identity and therefore natural. □

5.4.3 Examples

We will now explore how the principal bundles we defined relate to the classical notions of principal bundles in different aspects of geometry. Differential geometry, point-set topology and algebraic geometry each have their version of principal bundles. All these notions have in common that a principal bundle is locally isomorphic to the Cartesian product of a base space with a group.

We directly recover principal bundles from differential geometry and point-set topology. For algebraic geometry, future work is needed, but we strongly suspect that principal bundles in algebraic geometry can also be described using the language of restriction categories. A way to include Grothendieck topologies and, in particular, étale topology in restriction categories still needs to be further developed though.

Principal and fibre bundles in differential geometry

Here we will connect the notions defined above with the classical notions of fibre bundles and principal bundles from differential geometry.

Definition 5.4.25. *Let* $\mathbb{P}\mathrm{arMfld}$ *be the following join restriction category:*

(i) Objects are smooth manifolds

(ii) Morphisms from a manifold M *to a manifold* M' *are are smooth maps from an open subset* $U \subset M$ *to* M'*. Here the smooth structure on the open subset* U *is defined as a restriction of the smooth structure of* M*.*

(iii) Composition, identity and restrictions are analogous to the category of partial maps on sets. In particular the restriction of a map $f : A \to B$ *that is defined on* $U \subset A$ *is the inclusion* $U \to A$*. The composition gives a smooth map defined on an open subset since the composition of smooth maps is smooth, composing partial maps intersects domains of definition and finite intersections of open sets are open.*

This restriction category of manifolds with smooth partial maps is the category where the categorical constructions defined in Sections 5.4.1 and 5.4.2 correspond to constructions from classical differential geometry.

Definition 5.4.26. [71, Definition 20.1] *A* ***classical fibre bundle*** (E, q, M, F) *consists of smooth manifolds* E, M, F *and a smooth surjective mapping* $q : E \to M$ *; furthermore each* $x \in M$ *has an open neighbourhood*

U such that $E|_U := q^{-1}(U)$ is diffeomorphic to $U \times F$ via a fibre respecting diffeomorphism, α:

$$
\begin{tikzcd}
E|_U \arrow[rr, "\alpha"] \arrow[dr, "q"'] & & U \times F \arrow[dl, "\pi_0"] \\
& U &
\end{tikzcd}
$$

In [71, Definition 20.1], it is required that q is a surjective submersion. However, from the fact that $\pi_0 : U \times F \to U$ is a submersion, it follows automatically that q is a submersion. This is the notion of fibre bundles from classical differential geometry that Definition 5.4.1 is meant to generalize.

However, a fibre bundle as in Definition 5.4.1(i) is not the same as a classical fibre bundle. We need two extra conditions. Interestingly, the results of section 5.4.1 and 5.4.2 hold without assuming these extra conditions. Recall from Definition 5.4.1(iii) that a fibre bundle $(E, M, q, F, (\alpha_i)_{i\in I}, (e_i)_{i\in I})$ is totally fibred if $\overline{qe_i} = \bar{\alpha}_i$ for all $i \in I$.

Theorem 5.4.27. *Let M and F be smooth manifolds. A classical fibre bundle E over M with fibre F is exactly the same as a categorical fibre bundle $(E, M, q, F, (\alpha_i)_{i\in I}, (e_i)_{i\in I})$ in $\mathbb{P}$arMfld which*

- *is totally fibred, and*
- *for which that the projection $q : E \to M$ is an epimorphism.*

Proof. Suppose $(E, M, q, F, (\alpha_i)_{i\in I}, (e_i)_{i\in I})$ is a totally fibred categorical principal bundle in $\mathbb{P}$arMfld for which q is an epimorphism (i.e. surjective). Then for every $x \in M$ there is a $y \in q^{-1}(\{x\})$. Because $\bigvee_{i\in I} \bar{\alpha}_i = 1_E$, there exists an $i \in I$ such that $\alpha_i(y)$ is defined.

Recall from Remark 5.2.3 that restriction idempotents describe the subset that is their domain of definition.

Since $\bar{\alpha}_i^*$ is of the form $e_i \times 1_F$, it follows that the domain of definition of α_i^* is of the form $U \times F$ where U is the domain of definition of e_i. Thus the partial isomorphism α_i is an isomorphism of open subsets $V \xrightarrow{\cong} U \times F$ where V is the domain of definition for α_i.

Since $\bar{\alpha}_i$ describes V, $\bar{e}_i$ describes U and the restriction idempotent of a composition describes a preimage, $\bar{\alpha}_i = \overline{qe_i}$ implies $V = q^{-1}(U)$. Thus α_i is an isomorphism between $E|_U$ and $U \times F$. Since $y \in V = q^{-1}(U)$, it follows that $x \in U$.

We have thus constructed for every $x \in M$ an open neighbourhood $U \subset M$ and an isomorphism α_i : $q^{-1}(U) \to U \times F$ such that

$$
\begin{tikzcd}
E|_U \arrow[rr, "\alpha"] \arrow[dr, "q"'] & & U \times F \arrow[dl, "\pi_0"] \\
& U &
\end{tikzcd}
$$

commutes. Therefore (E, q, M, F) is a classical fibre bundle, showing that every totally fibred categorical fibre bundle with an epimorphism q is a classical fibre bundle.

Conversely, let $p : E \to M$ be a classical fibre bundle. Then we take the set M of points in the underlying manifold as our index-set. By definition of the classical fibre bundle, for every point $i \in M$ there is a partial isomorphism $\alpha_i : E \to M \times F$ with domain $p^{-1}(U)$. As the domain of α_i^* is $U \times F$, its restriction idempotent is of the form $\bar{\alpha}_i^* = e \times 1$. As every point $i \in M$ has an α_i defined in its preimage, the join satisfies $\bigvee_i \alpha_i = 1_E$. Thus every classical fibre bundle is a categorical fibre bundle. $\square$

Similarly we will prove that a classical principal bundle is exactly a totally fibred categorical principal bundle with surjective q-map in the category of manifolds.

We will consider the classical definitions from [71] or [90] (these are the same) and prove that they are equivalent to Definition 5.4.14.

Definition 5.4.28 (Definition 21.1 in [71])**.** *Let G be a Lie group and let (E, q, M, F) be a classical fibre bundle. A **classical G-bundle** structure on the fibre bundle (E, q, M, F) consists of the following data.*

(i) A left action $a : G \times F \to F$ of the Lie group on the standard fibre.

(ii) A fibre bundle atlas (U_i, α_i) whose transition functions (u_{ij}) act on F via the G-action. That is, there is a family of smooth mappings $(\tau_{ij} : U_i \cap U_j \to G)$ which satisfies the cocycle condition $\tau_{jk}(x)\tau_{ij}(x) = \tau_{ik}(x)$ for $x \in U_i \cap U_j \cap U_k$ and $\tau_{ii}(x) = u$, the unit in the group, such that $u_{ij}(x, f) = a(\tau_{ij}(x), f)$.

A fibre bundle with a G-bundle structure is called a G-bundle.

Comparing this with Definition 5.4.14 we see that a classical G-bundle structure is exactly the same as a G-bundle structure in $\mathbb{P}$arMfld.

Proposition 5.4.29. *A classical G-bundle is exactly the same as a G-bundle in the join-restriction category $\mathbb{P}$arMfld,*

- *that is totally fibred, and*
- *where the projection map $q : E \to M$ is surjective.*

This proposition follows from Theorem 5.4.27 since categorical G-bundles are defined from categorical fibre bundles the same way as classical G-bundles from classical fibre bundles.

Now we consider the definition of principal bundles in classical differential geometry.

Definition 5.4.30. *(Classical principal bundle) Let M be a manifold and let G be a Lie group. Then a classical principal bundle with structure group G over M is a classical G-bundle over M where the typical fibre is G endowed with the left multiplications as left action.*

Again we see that this is the same as part 2 of Definition 5.4.14.

Proposition 5.4.31. *A classical principal G-bundle is exactly the same as a principal G-bundle in the join-restriction category* $\mathbb{P}$arMfld,

- *that is totally fibred, and*
- *where the projection map $q : E \to M$ is surjective.*

Principal bundles in point-set topology

In this section we will look at the definition of a principal bundle in point-set topology and prove that it coincides with the notion developed here.

A **right-G-object** is a topological space P with a right-group action $a : P \times G \to P$. Given two right-G-objects P, P', a **G-equivariant** map is a smooth map $f : P \to P'$ such that

$$\begin{tikzcd} P \times G \arrow[r, "f \times 1_G"] \arrow[d, "a"'] & P' \times G \arrow[d, "a'"] \\ P \arrow[r, "f"'] & P' \end{tikzcd}$$

commutes. In [73] a principal bundle is defined as follows.

Definition 5.4.32. *Suppose that P is a right G-object equipped with a G-map $q : P \to M$ where G acts trivially on M. It is a* ***topological principal G-bundle*** *over M if q satisfies the following local triviality condition. B has a covering by open sets $(U_i)_{i \in I}$ such that there exist G-equivariant homeomorphisms $(\varphi_i : q^{-1}(U_i) \to U_i \times G)_{i \in I}$ such that the diagram*

$$\begin{tikzcd} q^{-1}(U_i) \arrow[r, "\varphi_i"] \arrow[d, "q"'] & U_i \times G \arrow[dl, "\pi_0"] \\ U_i & \end{tikzcd}$$

commutes. Here $U \times G$ has the right G-action $(u, g)h = (u, gh)$.

A morphism $\sigma : P \to Q$ of principal bundles over M is an equivariant morphism $\sigma : P \to Q$ over the identity of M, i.e. a morphism $\sigma : P \to Q$ such that

$$\begin{tikzcd} P \times G \arrow[r, "\sigma \times 1_G"] \arrow[d, "G\text{-action}"'] & Q \times G \arrow[d, "G\text{-action}"] \\ P \arrow[r, "\sigma"'] & Q \end{tikzcd} \qquad \begin{tikzcd} P \arrow[r, "\sigma"] \arrow[dr, "q_P"'] & Q \arrow[d, "q_Q"] \\ & M \end{tikzcd}$$

commute.

We will now show this definition actually is the same as a surjective, totally fibred principal bundle in the join-restriction category of topological spaces. For this, we first define this join restriction category.

Definition 5.4.33. *Let* $\mathbb{P}\mathrm{arTop}$ *be the following join restriction category:*

(i) Objects are topological spaces

(ii) Morphisms $A \to B$ *are continuous maps from an open subset* $U \in A$ *to* B*.*

(iii) Composition, identity and restrictions are like for partial maps of sets. In particular the restriction of a map $f : A \to B$ *that is defined on* $U \subset A$ *is the inclusion* $U \to A$*.*

Lemma 5.4.34. *Given a topological principal bundle* $(P, M, q, (U_i)_{i\in I}, (\varphi_i)_{i\in I})$ *like in Definition 5.4.32 the change of charts is given by the left-multiplication by a group element: There exists a map* $\tau_{ij} : U_i \cap U_j \to G$ *such that*

$$\varphi_i^{-1}\varphi_j = (\langle 1_{U_i\cap U_j}, \tau_{ij}\rangle \times 1_G)(1_{U_i\cap U_j} \times m) : (U_i \cap U_j) \times G \to (U_i \cap U_j) \times G.$$

Proof. We will define τ_{ij} as the composition of

$$U_i \cap U_j \xrightarrow{\langle 1,u\rangle} (U_i \cap U_j) \times G \xrightarrow{\varphi_i^{-1}\varphi_j} (U_i \cap U_j) \times G \xrightarrow{\pi_1} G.$$

In other words, τ_{ij} is the G-component of $\varphi_i^{-1}\varphi_j$ evaluated on the unit of the group. In order to show that $\varphi_i^{-1}\varphi_j = (\langle 1_{U_i\cap U_j}, \tau_{ij}\rangle \times 1_G)(1_{U_i\cap U_j} \times m)$, we will evaluate it on an element (x, g) of $(U_i \cap U_j) \times G$:

$$\varphi_i^{-1}\varphi_j(x, g) = (1_{U_i\cap U_j} \times m)\varphi_i^{-1}\varphi_j(x, u, g) = m(\varphi_i^{-1}\varphi_j(x, u), g) = m(x, \tau_{ij}(x), g) = (x, m(\tau_{ij}, g)),$$

where the second equality holds as both φ_i^{-1} and φ_j are G-equivariant. □

Theorem 5.4.35. *Let* G *be a total group object in the join restriction category* $\mathbb{P}\mathrm{arTop}$*, i.e. a topological group. The category of classical principal* G*-bundles over* M *(defined in Definition 5.4.32) is isomorphic to the subcategory of categorical principal bundles (defined in Definition 5.4.14) where*

- *the base space of every object is* M*,*
- *the underlying fibre bundle is totally fibred,*
- *the projection map* $q : P \to M$ *is surjective, and*
- *every morphism* $(f, g) : (P, M, q) \to (P', M, q')$ *has the identity as its second component:* $g = 1_M$*.*

Proof. Given a categorical principal bundle $P, M, q, (\alpha_i)_{i\in I}$ we obtain a classical principal bundle by taking U_i to be the subset of M where $\alpha_i^* : M \times G \to P$ is defined and $\varphi_i := \alpha_i : P \to M \times G$, defined on $q^{-1}(U_i)$ because the underlying fibre bundle is totally fibred. The space P is a right-G-object by Theorem 5.4.15 and q is a G-map to the trivial G-object M as $rq = \pi_0 q : P \times G \to M$.

Given a morphism $(f, 1_M) : (P \xrightarrow{q} M) \to (P' \xrightarrow{q'} M)$, the first component $f : P \to P'$ forms a morphism of classical principal bundles as, by Definition 5.4.17 it commutes with the G-action. It is total because $\bar{f} = \overline{fq'} = \overline{q1_M} = 1_P$.

Conversely, given a classical principal bundle $E \xrightarrow{q} M$, then it is also a categorical principal bundle with the α_i constructed from the φ_i as the partial map given by the span $P \hookleftarrow q^{-1}(U_i) \xrightarrow{\varphi_i} M \times G$. As the $(U_i)_{i\in I}$ form a covering, $\bigvee_{i\in\iota} \bar{a}_i = 1$. The partial inverse α_i^* is given by the span $M \times G \hookleftarrow U_i \times G \xrightarrow{\varphi_i^{-1}} E$. The change of charts is given by the left-multiplication with a group element according to Lemma 5.4.3. $\square$

Principal bundles in algebraic geometry

In [1], Definition 6.1.6, a notion of principal bundle over an algebraic variety is defined through algebraic properties instead of the local triviality condition used in Definition 5.4.1 and the differential geometry literature. These principal bundles are not necessarily locally trivial in the Zariski-topology. However they are locally trivial in the étale topology. Whether or not there is a join-restriction category where the categorical principal bundles are the principal bundles defined in [1] remains an open problem.

The difficulty here is that for a general Grothendieck topology and, in particular, for the étale topology, open subsets are described by morphisms that are not necessarily embeddings. Thus, if one follows the span construction of [26], the restriction idempotents are not idempotent, i.e. $\bar{f}\bar{f} \neq \bar{f}$. The resulting partial map category is not a restriction category.

We still think it is possible to describe principal bundles in algebraic geometry in a similar way to Definition 5.4.14, because it should be possible to describe Grothendieck topologies in the language of restriction categories using density relations. This is a new concept which is work in progress.

5.4.4 The torsor property

Let P be a principal bundle in a restriction category $\mathbb{X}$. In classical differential geometry, the group action $r : P \times G \to P$ coming from Theorem 5.4.15 is known to be free and proper and act transitively on fibres, as mentioned for example in the proof of Theorem 21.18(3) in [71]. In algebraic geometry, free and transitive group actions play such an important role that they were given a special name, torsors, in [72, Proposition III.4.1]. In [77] principal bundles are even introduced as torsors in the slice category. Often torsors are also thought of as groups without a specified unit, which is encoded as a ternary operation. In [10] these definitions are shown to be equivalent.

While there are multiple different definitions of torsors, we will use a generalization of the definition in [10, Section 2]. A torsor is an object with a free and transitive group action, which is described by the morphism in the definition below being an isomorphism.

Definition 5.4.36. *Given a group object G in a category, a **right G-torsor** is an object T with a right G-action $a : T \times G \to G$ such that the morphism*

$$T \times G \xrightarrow{\langle \pi_0, a \rangle} T \times T$$

is an isomorphism.

The Definition of [10] in addition requires the map from T to the terminal object to be a regular epimorphism in order to exclude the empty set from being a torsor. However, this would require us to make additional conditions on colimits which would be artificial in our setup. Thus we do not require this condition.

We will adapt Definition 5.4.36 slightly in the restriction category setting and distinguish partial and total torsors. Depending on the situation, we will either obtain partial or total torsors from principal bundles.

Definition 5.4.37. *Let G be a group object in a restriction category and let T an object with a right G-action $a : T \times G \to G$.*

*(i) Then (T, a) is a **partial right G-torsor** if the morphism*

$$T \times G \xrightarrow{\langle \pi_0, a \rangle} T \times T$$

is a partial isomorphism, i.e. it has a partial inverse.

*(ii) Then (T, a) is a **total right G-torsor** if the morphism*

$$T \times G \xrightarrow{\langle \pi_0, a \rangle} T \times T$$

is a total isomorphism, i.e. it is total and has a total inverse.

Since $\langle \pi_0, a \rangle$ is total, a partial right G-torsor is total if and only if the inverse $T \times T \to T \times G$ is total. We now want to investigate if a principal bundle in a join restriction category is a torsor in some kind of slice category. For this we define a restriction slice category. This definition stems from current work in progress by Robin Cockett, Geoff Cruttwell, Jonathan Gallagher and Dorette Pronk.

Definition 5.4.38. *Given an object M in a restriction category $\mathbb{X}$, we define the* ***restriction slice category*** $\mathbb{X} \wr M$ *as follows:*

(i) objects are pairs (A, f) consisting of an object A and a morphism $f : A \to M$,

(ii) morphisms from (A, f) to (B, g) are morphisms $\varphi : A \to B$ of $\mathbb{X}$ such that $f \geq \varphi g$, and

(iii) the restriction idempotent $\bar{\varphi}$ in the restriction slice category $X \wr M$ is the same as the restriction idempotent $\bar{\varphi}$ in the underlying category $\mathbb{X}$.

Composition and identity are given by the composition and identity operations of the underlying category.

A restriction slice category is, in particular, a restriction category. In order to describe G-torsors in the slice category the next lemma constructs a group object in the slice category $\mathbb{X} \wr M$ out of a group object in the underlying category $\mathbb{X}$.

Lemma 5.4.39. *Given a group object (G, m, u, ι) in a Cartesian restriction category $\mathbb{X}$, the pair $(M \times G, \pi_0)$ is a group object in the restriction slice category $\mathbb{X} \wr M$. The group multiplication, unit and inverse are*

$$\begin{aligned} 1_M \times m &: M \times G \times G \cong (M \times G) \times_M (M \times G) \to M \times G, \\ 1_M \times u &: M \times 1 \cong M \to M \times G, \text{ and} \\ 1_M \times \iota &: M \times G \to M \times G. \end{aligned}$$

Proof. The morphisms $1_M \times m, 1_M \times u$ and $1_M \times \iota$ are morphisms in the slice category since they preserve the projection to the group object,

$$(1_M \times m)\pi_0 = \pi_0 \ , \quad (1_M \times u)\pi_0 = \pi_0 \ , \text{and} \quad (1_M \times \iota)\pi_0 = \pi_0.$$

The fact that $1_M \times m, 1_M \times u$ and $1_M \times \iota$ fulfill the identities required in Definition 5.3.1 follows from the fact that m, u and ι satisfy the same identities. □

In light of Lemma 5.4.39 we will call a $(G \times M, \pi_1)$-torsor in the slice category $\mathbb{X} \wr M$ a G-torsor in $\mathbb{X} \wr M$ for brevity.

With these definitions, we can now formulate the main results of this section, that principal bundles are in fact torsors in the slice category $\mathbb{X} \wr M$. However, we need to distinguish between the general case and the totally fibred case. In the general case, we obtain only a partial right torsor, but in the totally fibred case, we will obtain a total right torsor.

In the category of sets, every group action is transitive on a certain subset, namely the orbit of a point. Therefore, in our opinion, total torsors are more useful than partial torsors, since they can be interpreted as group actions that are transitive, not just transitive on a subset.

Proposition 5.4.40. *Let $\mathbb{X}$ be a Cartesian join restriction category, let G be a group object in $\mathbb{X}$, let $(P, M, q, (\alpha_i)_{i\in I}, (\tau_{ij})_{i,j\in I})$ be a principal bundle in $\mathbb{X}$ such that $P \times_M P$ exists and let $r : P \times G \to P$ be the group action from Theorem 5.4.15. Then (P, q) is a partial right G-torsor in the restriction slice category $\mathbb{X} \wr M$. The partial inverse d^* of*

$$\langle \pi_0, r\rangle : P \times G \cong P \times_M (M \times G) \to P \times_M P$$

consists of components

$$P \times_M P \xrightarrow{\alpha_i \times_M \alpha_i} M \times G \times G \xrightarrow{\langle \pi_0, \pi_1, \langle \pi_1 \iota, \pi_2\rangle\rangle} M \times G \times G \xrightarrow{\alpha_i^* \times 1_G} P \times G \cong P \times_M (M \times G).$$

In order to prove this we will use the following technical observations about join restriction categories:

Lemma 5.4.41.

(i) In a join restriction category, let $(f_i)_{i\in I}$ and $(g_i)_{i\in I}$ be families of compatible functions indexed by the same set I, then

$$\bigvee_{i\in I} f_i \bigvee_{j\in I} g_j \geq \bigvee_{i\in I} f_i g_i.$$

(ii) In a restriction category, let $fg \geq \bar{f}$. Then $fg = \bar{f}$.

(iii) In a restriction category, let $\bar{f}g \leq f$. Then $\bar{f}g = \bar{g}f$ and therefore $f \smile g$.

Proof of Lemma 5.4.41.

(i) This follows since the join is associative and $\bigvee_{i\in I} f_i \bigvee_{j\in I} g_j = \bigvee_{i\in I} \bigvee_{j\in I} f_i g_j$ is a join over $I \times I$ and $\bigvee_{i\in I} f_i g_i$ is the join over the diagonal subset $\{(i,i)|i \in I\} \subset I \times I$.

(ii) This is true since $fg \geq \bar{f}$ means by definition of $\leq$ that $\bar{f} = \bar{\bar{f}}fg$ and therefore

$$fg = \bar{f}fg = \bar{\bar{f}}fg = \bar{f}.$$

(iii) This is true since $\bar{f}g \leq f$ means by definition of $\leq$ that

$$\bar{f}g = \overline{\bar{f}g}f = \bar{f}\bar{g}f = \bar{g}f.$$

$\square$

Proof of Proposition 5.4.40. In order to show that (P, q) is a partial right G-torsor in $\mathbb{X} \wr M$ we need to show that

$$\langle \pi_0, r\rangle : P \times G \to P \times_M P$$

is a partial isomorphism in $\mathbb{X}$. We do this by explicitly constructing its inverse as

$$d^* = \bigvee_{i \in I} (\alpha_i \times_M \alpha_i)\langle \pi_0, \pi_1, \langle \pi_1 \iota, \pi_2\rangle m\rangle(\alpha_i^* \times 1_G)$$

Before we show that this join is well-defined, we first show that, if it exists, it is a partial inverse for $\langle \pi_0, r\rangle$. We expand the definition of r from Theorem 5.4.15 and rewrite it in a form more useful for this proof.

$$\begin{aligned}\langle \pi_0, r\rangle &= \left\langle \pi_0 \bigvee_{i \in I} \alpha_i \alpha_i^*, \bigvee_{i \in I} (\alpha_i \times 1_G)(1_M \times m)\alpha_i^* \right\rangle \\ &= \bigvee_{i \in I} (\alpha_i \times 1_G)\langle \langle \pi_0, \pi_1\rangle \alpha_i^*, \langle \pi_0, \langle \pi_1, \pi_2\rangle m\rangle \alpha_i^*\rangle\end{aligned}$$

We show that $\langle \pi_0, r\rangle d^* = 1_{P \times G} = \overline{\langle \pi_0, r\rangle}$. We expand

$$\begin{aligned}&\langle \pi_0, r\rangle d^* = \\ &\bigvee_{i \in I} (\alpha_i \times 1_G)\langle \langle \pi_0, \pi_1\rangle \alpha_i^*, \langle \pi_0, \langle \pi_1, \pi_2\rangle m\rangle \alpha_i^*\rangle \bigvee_{j \in I} (\alpha_j \times_M \alpha_j)\langle \pi_0, \pi_1, \langle \pi_1 \iota, \pi_2\rangle m\rangle(\alpha_j^* \times 1_G).\end{aligned}$$

Applying Lemma 5.4.41(i) to the previous expansion we obtain the inequality

$$\begin{aligned}&\langle \pi_0, r\rangle d^* \geq \\ &\bigvee_{i \in I} (\alpha_i \times 1_G)\langle \langle \pi_0, \pi_1\rangle \alpha_i^*, \langle \pi_0, \langle \pi_1, \pi_2\rangle m\rangle \alpha_i^*\rangle(\alpha_i \times_M \alpha_i)\langle \pi_0, \pi_1, \langle \pi_1 \iota, \pi_2\rangle m\rangle(\alpha_i^* \times 1_G) \\ =&\bigvee_{i \in I} (\alpha_i \times 1_G)\left\langle \langle \pi_0, \pi_1\rangle \alpha_i^* \alpha_i, \left\langle \langle \pi_0, \pi_1\rangle \alpha_i^* \alpha_i \pi_1 \iota, \langle \pi_0, \langle \pi_1, \pi_2\rangle m\rangle \alpha_i^* \alpha_i \pi_1 \right\rangle m \right\rangle(\alpha_i^* \times 1_G).\end{aligned}$$

Since $\alpha_i^* \alpha_i = \bar{\alpha_i^*} = e_i \times 1_G$, this equals

$$\begin{aligned}&\bigvee_{i \in I} (\alpha_i \times 1_G)\left\langle \pi_0 e_i, \pi_1, \left\langle \langle \pi_0 e_i, \pi_1\rangle \pi_1 \iota, \langle \pi_0 e_i, \langle \pi_1, \pi_2\rangle m\rangle \pi_1 \right\rangle m \right\rangle(\alpha_i^* \times 1_G) \\ =&\bigvee_{i \in I} (\alpha_i \times 1_G)\overline{\pi_0 e_i}\left\langle \pi_0, \pi_1, \left\langle \pi_1 \iota, \langle \pi_1, \pi_2\rangle m \right\rangle m \right\rangle(\alpha_i^* \times 1_G).\end{aligned}$$

By associativity of m this equals

$$\bigvee_{i\in I}(\alpha_i \times 1_G)\overline{\pi_0 e_i}\langle\pi_0,\pi_1,\pi_2\rangle(\alpha_i^* \times 1_G)$$

and using $\bar{\alpha}_i^* = e_i \times 1_F = \overline{\pi_0 e_i}$ we obtain

$$\begin{aligned}
&\bigvee_{i\in I}(\alpha_i \times 1_G)(\bar{\alpha_i^*} \times 1_G)(\alpha_i^* \times 1_G)\\
=&\bigvee_{i\in I}(\bar{\alpha}_i \times 1_G)\\
=&\bigvee_{i\in I}\overline{\pi_0\alpha_i} = 1_{P\times G}.
\end{aligned}$$

This equations shows that $\langle\pi_0,r\rangle d^* \geq 1_{P\times G} = \overline{\langle\pi_0,r\rangle}$ which implies by Lemma 5.4.41(i) that $\langle\pi_0,r\rangle d^* = 1_{P\times G} = \overline{\langle\pi_0,r\rangle}$. Next we will show that $d^*\langle\pi_0,r\rangle = \bar{d}^*$. First we simplify the left hand side into

$$\begin{aligned}
&d^*\langle\pi_0,r\rangle =\\
&\bigvee_{j\in I}(\alpha_j \times_M \alpha_j)\langle\pi_0,\pi_1,\langle\pi_1\iota,\pi_2\rangle m\rangle(\alpha_j^* \times 1_G)\bigvee_{i\in I}(\alpha_i \times 1_G)\langle\langle\pi_0,\pi_1\rangle\alpha_i^*,\langle\pi_0,\langle\pi_1,\pi_2\rangle m\rangle\alpha_i^*\rangle\\
\geq&\bigvee_{i\in I}(\alpha_i \times_M \alpha_i)\left\langle\langle\pi_0,\pi_1\rangle\alpha_i^*,\left\langle\pi_0,\langle\pi_1,\langle\pi_1\iota,\pi_2\rangle m\rangle m\right\rangle\alpha_i^*\right\rangle\\
=&\bigvee_{i\in I}(\alpha_i \times_M \alpha_i)\left\langle\langle\pi_0,\pi_1\rangle\alpha_i^*,\langle\pi_0,\pi_2\rangle\alpha_i^*\right\rangle\\
=&\bigvee_{i\in I}\langle\pi_0\alpha_i\alpha_i^*,\pi_1\alpha_i\alpha_i^*\rangle\\
=&\bigvee_{i\in I}\overline{\pi_0\alpha_i}\,\overline{\pi_1\alpha_i}.
\end{aligned}$$

The right hand side is simplified to

$$\begin{aligned}
\bar{d}^* &= \overline{\bigvee_{i\in I}(\alpha_i \times_M \alpha_i)\langle\pi_0,\pi_1,\langle\pi_1\iota,\pi_2\rangle m\rangle(\alpha_i^* \times 1_G)}\\
&= \bigvee_{i\in I}\overline{\langle\pi_0\alpha_i\alpha_i^*,(\alpha_i \times_M \alpha_i)\langle\pi_1\iota,\pi_2\rangle m\rangle}\\
&= \bigvee_{i\in I}\overline{\pi_0\alpha_i}\,\overline{\langle\pi_0\alpha_i,\pi_1\alpha_i\rangle}\\
&= \bigvee_{i\in I}\overline{\pi_0\alpha_i}\,\overline{\pi_1\alpha_i}.
\end{aligned}$$

Therefore we showed that, assuming d^* is defined, it is a partial inverse to $\langle \pi_0, r\rangle$. It remains to show that the join

$$\bigvee_{i\in I} (\alpha_i \times_M \alpha_i)\langle \pi_0, \pi_1, \langle \pi_1\iota, \pi_2\rangle m\rangle(\alpha_i^* \times 1_G)$$

is defined, i.e. that for

$$d_i^* = (\alpha_i \times_M \alpha_i)\langle \pi_0, \pi_1, \langle \pi_1\iota, \pi_2\rangle m\rangle(\alpha_i^* \times 1_G)$$

the compatibility $d_i^* \smile d_j^*$ holds for all $i, j \in I$. By Lemma 5.4.41(iii) this amounts to showing the inequality

$$\bar{d}_i^* d_j^* \leq d_i^*.$$

For this we use that $\bar{d}_i^* = (\bar{\alpha}_i \times_M \bar{\alpha}_i)$. We see that

$$\begin{aligned}\bar{d}_i^* d_j^* =&(\bar{\alpha}_i \times_M \bar{\alpha}_i)(\alpha_j \times_M \alpha_j)\langle \pi_0, \pi_1, \langle \pi_1\iota, \pi_2\rangle m\rangle(\alpha_j^* \times 1_G)\\ =&\Big\langle \pi_0\bar{\alpha}_i\bar{\alpha}_j, (\alpha_i\alpha_i^*\alpha_j \times_M \alpha_i\alpha_i^*\alpha_j)\langle \pi_1\iota, \pi_2\rangle m\Big\rangle\end{aligned}$$

Since $(P, M, q, (\alpha_i)_{i\in I})$ is a principal bundle, we know that $\alpha_i^*\alpha_j\pi_1 = \langle \pi_0\tau_{ij}, \pi_1\rangle m$. Therefore the previous expression equals

$$=\Big\langle \pi_0\bar{\alpha}_i\bar{\alpha}_j, (\alpha_i \times_M \alpha_i)\Big\langle \langle \pi_0\tau_{ij}, \pi_1\rangle m\iota, \langle \pi_0\tau_{ij}, \pi_2\rangle m\Big\rangle m\Big\rangle$$

where the indices of the projections stem from the identification of $(M \times G) \times_M (M \times G))$ with $M \times G \times G$. The inverse of a product $\langle f, g\rangle m\iota$ is the same as the reversed product of the inverses $\langle g\iota, f\iota\rangle m$ and therefore this equals

$$\begin{aligned}&\Big\langle \pi_0\bar{\alpha}_i\bar{\alpha}_j, (\alpha_i \times_M \alpha_i)\Big\langle \langle \pi_1\iota, \pi_0\tau_{ij}\iota\rangle m, \langle \pi_0\tau_{ij}, \pi_2\rangle m\Big\rangle m\Big\rangle\\ =&\Big\langle \pi_0\bar{\alpha}_i\bar{\alpha}_j, (\alpha_i \times_M \alpha_i)\overline{\pi_0\tau_{ij}}\langle \pi_1\iota, \pi_2\rangle m\Big\rangle\\ \leq&\Big\langle \pi_0\alpha_i\alpha_i^*, (\alpha_i \times_M \alpha_i)\langle \pi_1\iota, \pi_2\rangle m\Big\rangle\\ =&(\alpha_i \times_M \alpha_i)\langle \pi_0, \pi_1, \langle \pi_1\iota, \pi_2\rangle m\rangle(\alpha_i^* \times 1_G) = d_i.\end{aligned}$$

The preceding calculation shows that $\bar{d}_i^* d_j^* \leq d_i^*$. Now it follows from Lemma 5.4.41(iii) that $d_i^* \smile d_j^*$ which concludes the proof of Theorem 5.4.40. □

While Proposition 5.4.40 gives a partial isomorphism, this isomorphism is, in general, not total. In order to obtain a total right G-torsor, as in Definition 5.4.37(iii), we need to require $(P, M, q, (\alpha_i)_{i\in I})$ to be totally fibred, as in Definition 5.4.1(iii).

Theorem 5.4.42. *Let $(P, M, q, (\alpha_i)_{i\in I})$ be a totally fibred principal G-bundle in a join-restriction category $\mathbb{X}$. Then P is a total right G-torsor in the slice category $\mathbb{X} \wr M$ via the isomorphism in Proposition 5.4.40.*

Proof. Since the map $\langle \pi_0, r\rangle$ is total, we only need to show that

$$d^* = \bigvee_{i\in I} (\alpha_i \times_M \alpha_i)\langle \pi_0, \pi_1, \langle \pi_1 \iota, \pi_2\rangle m\rangle(\alpha_i^* \times 1_G)$$

is total too. As we calculated in the proof of Proposition 5.4.40.

$$\bar{d^*} = \bigvee_{i\in I} \bar{d_i^*} = \bigvee_{i\in I} (\bar{\alpha}_i \times_M \bar{\alpha}_i).$$

Since we assumed that $(P, M, q, (\alpha_i)_{i\in I})$ is totally fibred, $\bar{\alpha}_i = \overline{qe_i}$ implies

$$\bar{d^*} = \bigvee_{i\in I} (\overline{qe_i} \times_M \overline{qe_i}).$$

Since $\pi_0 q = \pi_1 q : P \times_M P \to M$,

$$\bar{d^*} = \bigvee_{i\in I} \overline{\pi_0 q e_i} = \bigvee_{i\in I} \overline{\pi_0 \bar{\alpha}_i} = \overline{\bigvee_{i\in I} \pi_0 \bar{\alpha}_i} = \overline{\pi_0 \bigvee_{i\in I} \bar{\alpha}_i} = \bar{\pi}_0 = 1_{P\times_M P}.$$

This shows that d^* is a partial inverse to $\langle \pi_0, r\rangle$ that is total and therefore a total inverse. $\square$

5.4.5 The vertical bundle of a principal bundle

Given a group object G and a principal G-bundle $P \xrightarrow{q} M$ in a Cartesian tangent join restriction category $\mathbb{X}$, the **vertical bundle** T_0P is defined to be the pullback

$$\begin{tikzcd}
T_0(P) \arrow[r, "Tq^*(0)"] \arrow[d, "pq"'] \arrow[dr, phantom, "\lrcorner", very near start] & T(P) \arrow[d, "T(q)"] \\
M \arrow[r, "0"'] & T(M).
\end{tikzcd}$$

It is known, for example from Section 21.18 of [71], that in classical differential geometry the vertical bundle is isomorphic to $P \times T(G)_u$, where $T(G)_u$ is the tangent space of G at the unit given by the (restriction) pullback

$$\begin{tikzcd}
T(G)_u \arrow[r, "p_u^*"] \arrow[d, "!"] \arrow[dr, phantom, "\lrcorner", very near start] & T(G) \arrow[d, "p"] \\
1 \arrow[r, "u"'] & G
\end{tikzcd}$$

as defined in Section 5.3.1. We now prove the same in the more general setting of a tangent restriction category.

Theorem 5.4.43. *Given a group object G and a principal G-bundle $P \xrightarrow{q} M$ in a Cartesian tangent join restriction category, if $T(G)_u$ exists, the vertical bundle exists and is isomorphic to $P \times T(G)_u$.*

In local charts (i.e. when considering partial isomorphisms) $P \cong M \times G$ and thus $T(P) \cong T(M) \times T(G) \cong T(M) \times G \times T(G)_u$ due to Theorem 5.3.3. Thus in local charts the diagram is

$$\begin{tikzcd}
M \times G \times T(G)_u \arrow[r, "0 \times 1_G \times 1_{T(G)_u}"] \arrow[d, "\pi_0"'] & T(M) \times G \times T(G)_u \arrow[d, "\pi_0"] \\
M \arrow[r, "0"'] & T(M)
\end{tikzcd}$$

which is a pullback. This shows that the statement of Theorem 5.4.43 makes sense in local coordinates. The result of Theorem 5.4.43 is stronger though. It states that there is a total isomorphism going globally between $T_0(P)$ and $P \times T(G)_u$.

Proof. We will show that the following is a pullback square:

$$\begin{tikzcd}
P \times T(G)_u \arrow[r, "(0 \times p_u^*)T(r)"] \arrow[d, "\pi_0 q"'] & T(P) \arrow[d, "T(q)"] \\
M \arrow[r, "0_M"'] & T(M)
\end{tikzcd}$$

The diagram commutes because the right action r preserves the basepoint (i.e. $rq = \pi_0 q$) and therefore

$$(0_P \times p^*(u))T(r)T(q) = (0_P \times p^*(u))\pi_0 T(q) = \pi_0 0_P T(q) = \pi_0 q 0_M.$$

Since all morphisms in the square are total we can use Lemma 5.2.8 to verify that the diagram is a pullback. We assume there is an object Z with (not necessarily total) morphisms $f : Z \to T(P)$ and $g : Z \to M$ such that $g0 = fT(q)$. We need to show that there is an unique morphism $\phi : Z \to P \times T(G)_u$ such that the diagram

$$
\begin{tikzcd}
Z \arrow[dr, dashed, "\phi"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "g"'] & & \\
& P \times T(G)_u \arrow[r, "(0\times p_u^*)T(r)"] \arrow[d, "\pi_0 q"'] & T(P) \arrow[d, "T(q)"] \\
& M \arrow[r, "0"'] & T(M)
\end{tikzcd}
\tag{5.8}
$$

commutes. We construct the morphism ϕ using maps

$$\xi_i = \langle p, \langle !, T(\alpha_i)\pi_1\langle p\iota 0, 1\rangle T(m)\rangle\rangle : T(P) \to P \times T(G)_u.$$

Define the morphism ϕ to be

$$\phi = \bigvee_{i\in I} f\xi_i.$$

It may be surprising that we do not need to use g. However, since $fT(q)p = g$, f contains all the information we need. The remainder of this proof consists of the following three steps.

(1) In order for ϕ to be well-defined we need to check that the morphisms $f\xi_i$ are compatible with each other, i.e. that $\overline{f\xi_i}f\xi_j = \overline{f\xi_j}f\xi_i$.

(2) The morphism ϕ must make the outer triangles of Diagram 5.8 commute, i.e. $\phi(0 \times p_u^*)T(r) = f$ and that $\phi\pi_0 q = g$.

(3) The morphism ϕ must be the unique such morphism.

For step (1) we will show that $\overline{f\xi_i}f\xi_j \le f\xi_i$ which implies compatiblity by Lemma 5.4.41. We simplify and expand $\overline{f\xi_i}f\xi_j$. By R.4 of Definition 5.2.1,

$$
\begin{aligned}
\overline{f\xi_i}f\xi_j =& f\bar{\xi}_i\xi_j \\
=& f\overline{T(\alpha_i)}\langle p, \langle !, T(\alpha_j)\pi_1\langle p\iota 0, 1\rangle T(m)\rangle\rangle.
\end{aligned}
$$

Here we used $\bar{\xi}_i = \overline{T(\alpha_i)}$ and the definition of ξ_i in order to simplify and expand the expression. In order to use the interaction between α_i and α_j we now compose

$$
\begin{aligned}
&\langle fp, \langle !, f\overline{T(\alpha_i)}T(\alpha_j)\pi_1\langle p\iota 0, 1\rangle T(m)\rangle\rangle \\
=&\langle fp, \langle !, fT(\alpha_i)T(\alpha_i^*)T(\alpha_j)\pi_1\langle p\iota 0, 1\rangle T(m)\rangle\rangle.
\end{aligned}
$$

Since P is a principal bundle we can rewrite $\overline{T(\alpha_i)}T(\alpha_j)$ into an expression of $T(\alpha_i)$ using Definition 5.4.14. Since P is a principal bundle, this equals

$$\begin{aligned}
&\langle fp, \langle !, fT(\alpha_i)\langle \pi_0 T(\tau_{ij}), \pi_1\rangle T(m)\pi_1\langle p\iota 0, 1\rangle T(m)\rangle\rangle \\
=&\langle fp, \langle !, \langle fT(\alpha_i)\pi_1 p\iota 0, fT(\alpha_i)\pi_0 T(\tau_{ij})p\iota 0, fT(\alpha_i)\pi_0 T(\tau_{ij}), fT(\alpha_i)\pi_1\rangle T(m_4)\rangle\rangle.
\end{aligned}$$

Now we have an expression containing only $T(\alpha_i)$ and we will continue by simplifying this expression. Since $\alpha_i\pi_0 \leq q$ the previous expression produces the inequality

$$\overline{f\xi_i}f\xi_j \leq \langle fp, \langle !, \langle fT(\alpha_i)\pi_1 p\iota 0, fT(q)p0T(\tau_{ij})T(\iota), fT(q)T(\tau_{ij}), fT(\alpha_i)\pi_1\rangle T(m_4)\rangle\rangle.$$

Since $fT(q)p0 = g0 = fT(q)$ the previous expression produces the inequality

$$\begin{aligned}
\overline{f\xi_i}f\xi_j \leq&\langle fp, \langle !, \langle fT(\alpha_i)\pi_1 p\iota 0, fT(\alpha_i)\pi_1\rangle T(m)\rangle\rangle \\
=&f\langle p, \langle !, T(\alpha_i)\pi_1\langle p\iota 0, 1\rangle T(m)\rangle\rangle = f\xi_i.
\end{aligned}$$

Using Lemma 5.4.41(iii) this implies that the morphisms $(f\xi_i)_{i\in I}$ are compatible with each other.

For step (2) we directly show the first equation $\phi(0\times p_u^*)T(r) = f$ by calulating

$$\begin{aligned}
&\phi(0\times p_u^*)T(r) \\
=&\bigvee_{i\in I} f\xi_i(0\times p_u^*)T(r) \\
=&\bigvee_{i\in I} f\langle p, \langle !, T(\alpha_i)\pi_1\langle p\iota 0, 1_{T(G)}\rangle T(m)\rangle\rangle(0\times p_u^*)T(r).
\end{aligned}$$

For every morphism h into $T(G)$, satisfying $hp = !u$, the universal property of the pullback guarantees that the equation $\langle !, h\rangle p_u^* = h$ holds. Using $h = T(\alpha_i)\pi_1\langle p\iota 0, 1_{T(G)}\rangle T(m)$, the previous line equals

$$\bigvee_{i\in I} f\langle p0, T(\alpha_i)\pi_1\langle p\iota 0, 1_{T(G)}\rangle T(m)\rangle T(r).$$

Now, expanding the definition of r from Theorem 5.4.15 the previous line equals

$$\begin{aligned}
&\bigvee_{i\in I}\bigvee_{j\in I} f\langle p0, T(\alpha_i)\pi_1\langle p\iota 0, 1_{T(G)}\rangle T(m)\rangle (T(\alpha_j)\times 1_{T(G)})(1_{T(M)}\times T(m))T(\alpha_j^*)\\
\geq &\bigvee_{i\in I} f\langle p0, T(\alpha_i)\pi_1\langle p\iota 0, 1_{T(G)}\rangle T(m)\rangle (T(\alpha_i)\times 1_{T(G)})(1_{T(M)}\times T(m))T(\alpha_i^*)\\
= &\bigvee_{i\in I} f\langle T(\alpha_i)p0, T(\alpha_i)\pi_1\langle p\iota 0, 1_{T(G)}\rangle T(m)\rangle (1_{T(M)}\times T(m))T(\alpha_i^*)\\
= &\bigvee_{i\in I} f\langle T(\alpha_i)\pi_0 p0, T(\alpha_i)\pi_1 p0, \langle T(\alpha_i)\pi_1 p\iota 0, T(\alpha_i)\pi_1\rangle T(m)\rangle (1\times T(m))T(\alpha_i^*).
\end{aligned}$$

Now due to the associativity, the inverse and the neutral property of the multiplication $T(\alpha_i)\pi_1 p0$ and $T(\alpha_i)\pi_1 p\iota 0$ cancel each other out. In addition $\alpha_i\pi_0 = \bar{\alpha}_i q$ and thus the previous expression simplifies to

$$\begin{aligned}
&\bigvee_{i\in I} f\langle T(q)p0, T(\alpha_i)\pi_1\rangle T(\alpha_i^*)\\
= &\bigvee_{i\in I}\langle g0p0, fT(\alpha_i)\pi_1\rangle T(\alpha_i^*)\\
= &\bigvee_{i\in I}\langle g0, fT(\alpha_i)\pi_1\rangle T(\alpha_i^*)\\
= &f\bigvee_{i\in I}\langle T(\alpha_i)\pi_0, T(\alpha_i)\pi_1\rangle T(\alpha_i^*) = f\bigvee_{i\in I} T(\alpha_i)T(\alpha_i^*) = f
\end{aligned}$$

where the last equality is due to Lemma 5.2.16. Since $\bar{\phi} = \bigvee_{i\in I}\overline{f\xi_i} \leq \bar{f}$, we also have that $\overline{\phi(0\times p_u^*)T(r)} \leq \overline{f}$. Together

$$\phi(0\times p_u^*)T(r) \geq f \quad\text{and}\quad \overline{\phi(0\times p_u^*)T(r)} \leq \overline{f}$$

imply that $\phi(0\times p_u^*)T(r) = f$ as required. Now we use $g0 = fT(q)$ and $\pi_0 q0 = (0\times p_u^*)T(r)T(q)$ to show the second equation

$$\phi\pi_0 q = \phi\pi_0 q0p = \phi(0\times p_u^*)T(r)T(q)p = fT(q)p = g0p = g.$$

Lastly, step (3), the uniqueness of ϕ, needs to be shown. Suppose there is another morphism $\phi' : Z \to P\times T(G)_u$ making these diagrams commute, then $\phi'(0\times p_u^*)T(r)\xi_i \smile \phi'(0\times p_u^*)T(r)\xi_j$ for all $i,j\in I$ because $\phi'(0\times p_u^*)T(r)\xi_i = f\xi_i$ and $f\xi_i \smile f\xi_j$. Then we have

$$\phi' = \bigvee_{i\in I}\phi'(0\times p_u^*)T(r)\xi_i = \bigvee_{i\in I} f\xi_i = \phi. \tag{5.9}$$

The first equality in 5.9 holds because

$$
\begin{aligned}
&\bigvee_{i\in I}\phi'(0\times p_u^*)T(r)\xi_i\\
=&\bigvee_{i\in I}\phi'(0\times p_u^*)(T(\alpha_i)\times 1_{T(G)})(1_{T(M)}\times T(m))T(\alpha_i^*)\langle p,\langle !,T(\alpha_i)\pi_1,\langle p\iota 0,1_{T(G)}\rangle T(m)\rangle\rangle\\
=&\bigvee_{i\in I}\phi'\langle T\pi_0 0T(\alpha_i)T(\pi_0),\langle\pi_0 0T(\alpha_i)\pi_1,\pi_1 p_u^*\rangle T(m)\rangle T(\alpha_i^*)\langle p,\langle !,T(\alpha_i)\pi_1\langle p\iota 0,1_{T(G)}\rangle T(m)\rangle\rangle.
\end{aligned}
$$

We will now calculate the two components of this morphism separately and show that $\bigvee_{i\in I}\phi'(0\times p_u^*)T(r)\xi_i=\phi'$ by showing that the first component is $\phi'\pi_0$ and the second component is $\phi'\pi_1$. The first component is

$$\bigvee_{i\in I}\phi'\Big\langle\pi_0 0T(\alpha_i)T(\pi_0),\langle\pi_0 0T(\alpha_i)T(\pi_1),\pi_1 p_u^*\rangle T(m)\Big\rangle T(\alpha_i^*)p.$$

Due to naturality of p, $0p=1$ and $p_u^*p=!u$, the previous line simplifies to

$$\bigvee_{i\in I}\phi'\Big\langle\pi_0\alpha_i\pi_0,\langle\pi_0\alpha_i\pi_1,!u\rangle m\Big\rangle\alpha_i^*.$$

Since u is the unit for m this equals

$$\bigvee_{i\in I}\phi'\langle\pi_0\alpha_i\pi_0,\pi_0\alpha_i\pi_1\rangle\alpha_i^*=\bigvee_{i\in I}\phi'\pi_0\alpha_i\alpha_i^*=\phi'\pi_0.$$

The second component of $\bigvee_{i\in I}\phi'(0\times p_u^*)T(r)\xi_i$ is

$$\bigvee_{i\in I}\Big\langle\pi_0 0T(\alpha_i)T(\pi_0),\langle\pi_0 0T(\alpha_i)T(\pi_1),\pi_1 p_u^*\rangle T(m)\Big\rangle T(\alpha_i^*)\Big\langle !,T(\alpha_i)\pi_1\langle p\iota 0,1\rangle T(m)\Big\rangle$$

which equals

$$\bigvee_{i\in I}\Bigg\langle !,\phi'\Big\langle\pi_0 0T(\alpha_i)T(\pi_0),\langle\pi_0 0T(\alpha_i)T(\pi_1),\pi_1 p_u^*\rangle T(m)\Big\rangle T(\alpha_i^*)T(\alpha_i)\pi_1\langle p\iota 0,1\rangle T(m)\Bigg\rangle.$$

Since $\alpha_i^*\alpha_i=\bar{\alpha}_i^*=e_i\times 1_G$ and for all morphisms a and b $\langle a,b\rangle\pi_1=\bar{a}b$, this equals

$$\bigvee_{i\in I}\Big\langle !,\phi'\overline{\pi_0 0T(\alpha_i)T(\pi_0)T(e_i)}\langle\pi_0 0T(\alpha_i)T(\pi_1),\pi_1 p_u^*\rangle T(m)\langle p\iota 0,1\rangle T(m)\Big\rangle.$$

Using naturality and totality of 0, $\pi_0 e_i = \bar{\alpha}_i^* \pi_0$ and $\alpha_i \bar{\alpha}_i^* = \alpha_i$ this equals

$$\bigvee_{i \in I} \left\langle !, \phi' \overline{\pi_0 \alpha_i} \Big\langle \langle \pi_0 0 T(\alpha_i) T(\pi_1), \pi_1 p_u^* \rangle T(m) p \iota 0, \langle \pi_0 0 T(\alpha_i) T(\pi_1), \pi_1 p_u^* \rangle T(m) \Big\rangle T(m) \right\rangle.$$

Using naturality of 0, that 0 is a section of p, and the fomula $\langle a, b \rangle m \iota = \langle b \iota, a \iota \rangle m$ for the inverse of a product, this becomes

$$\bigvee_{i \in I} \left\langle !, \phi' \overline{\pi_0 \alpha_i} \Big\langle \langle \pi_1 p_u^* p 0 T(\iota), \pi_0 0 T(\alpha_i \pi_1 \iota) \rangle T(m), \langle \pi_0 0 T(\alpha_i) T(\pi_1), \pi_1 p_u^* \rangle T(m) \Big\rangle T(m) \right\rangle.$$

Due to the associativity and inverse property of m and the defining property of p_u^*, $p_u^* p 0 = !u 0 = !T(u)$, this equals

$$\bigvee_{i \in I} \Big\langle !, \phi' \overline{\pi_0 \alpha_i} \langle !T(u) T(\iota), \pi_1 p_u^* \rangle T(m) \Big\rangle.$$

Since $\bigvee_{i \in I} \bar{\alpha}_i = 1_P$ and $u\iota = u$ is the unit for m this equals

$$\langle !, \phi' \pi_1 p_u^* \rangle = \phi' \pi_1 \langle !, p_u^* \rangle = \phi' \pi_1$$

where the last equality holds since ! and p_u^* are the projections out of the pullback $T(G)_u$ and thus $\langle !, p_u^* \rangle = 1_{T(G)_u}$. The two components together imply that $\phi' = \bigvee_{i \in I} \phi'(0 \times p_u^*) T(r) \xi_i$ which was missing to prove equation 5.9 which concludes the proof by proving uniqueness of ϕ.

□

This proof heavily uses the map $\langle !, \langle p\iota 0, 1 \rangle T(m) \rangle : T(G) \to T(G)$ in order to construct the ξ_i, thereby using similar methods to the ones used in Section 5.3.1.

5.5 Concluding remarks

The goal of this chapter was to understand group objects and principal bundles in the general categorical settings of tangent categories and join restriction categories. These structures have been introduced and we have shown that many results from classical differential geometry can be recovered. In particular our construction is a generalization of principal bundles in differential geometry and topology.

Aintablian and Blohmann have independently developed a categorical approach to Lie groupoids and

Lie algebroids in [2] which is similar to some of our results in Section 5.3. In particular, Theorem 7.5 of [2] is (up to small differences) equivalent to Theorem 5.3.21 in this chapter. The main focus of [2] is Lie groupoids and their behaviour, while the main focus of our work is group objects and principal bundles in restriction categories. In particular, in Section 4, we were primarily interested in examining the tangent bundle of a group object. These results are not contained in [2] since it works in the more general setup of groupoid objects, where these results may not hold. We relate the result about vector fields to the tangent space in Theorem 5.3.21. A technical difference between the constructions of [2] and our construction is that we do not assume that the tangent bundles have negatives (the fibres are monoids, not groups) nor scalar multiplication, though Theorem 5.3.15 shows that the relevant negatives always exist.

In future work, we hope, in particular, to explore how the notion of group objects and principal bundles behaves in algebraic geometry. Since there is a pre-existing notion of principal bundles in Algebraic Geometry, for example in [1]. As we explained in more detail in the end of Section 5.4.3, the goal of this will be to find a restriction structure for which the principal bundles of Definition 5.4.14 correspond to this pre-existing notion.

In addition we are interested in seeing in which cases we can recover a group object from its Lie algebra. Classically, one can recover a Lie Group from a Lie Algebra using the exponential function, a way to solve a differential equation on a manifold. This leads to the classical result that a Lie Group is determined by its Lie Algebra up to local isomorphism. In order to generalize this result, we need a notion of exponential functions and differential equations, like the ones provided in [63] and [23]. We hope to adapt those methods to our situation.

Another important continuation is the connection with differential bundles. In particular we want to generalize the classical result that principal GL_n-bundles correspond to vector bundles by showing that G-bundles, whose fibre is a differential object, are differential bundles. While this continuation is not part of the paper this chapter is based on, this is the main focus of Chapter 6 of this thesis.

Chapter 6

Relating principal and differential bundles

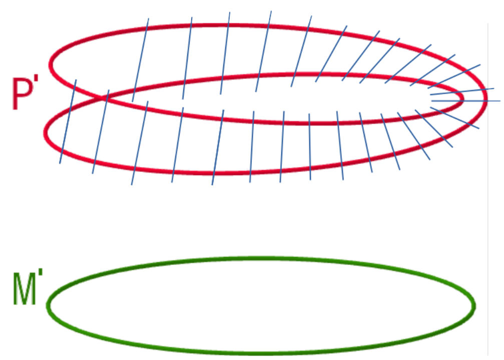

Figure 6.1: Given a principal bundle, one can form the associated bundle. The associated bundle is the differential bundle which is locally a product but globally twisted according to the principal bundle as pictured above.[2]

In Chapters 3 and 4 we focused on differential bundles in tangent categories. In Chapter 5 our focus was on G-bundles and principal G-bundles in Cartesian join restriction categories (often with gluings). In this chapter, we consider the confluence of these settings and connect principal and differential bundles in Cartesian tangent join restriction categories with gluings. In classical differential geometry, there is a close correspondence between differential bundles (i.e. vector bundles) and principal GL_n-bundles. In this chapter we show that every principal G-bundle together with a differential object with linear G-action has an associated differential bundle, but the reverse correspondence does not hold.

Since this chapter heavily relies on the constructions in Chapter 5, we will keep using diagrammatic composition order, i,e, we denote the composition of $f : A \to B$ and $g : B \to C$ as $fg : A \to C$.

[2]Figure created using GeoGebra 3D

In classical differential geometry, vector bundles with fibre $\mathbb{R}^n$ correspond to principal GL_n-bundles. This correspondence sends a principal GL_n-bundle to a vector bundle utilizing the functor

$$\mathtt{Build}^{\mathrm{tot}_G} : \mathbb{P}\mathrm{Bun}_G^{\mathrm{tot}} \times G - \mathbb{O}\mathrm{bj} \to \mathbb{B}\mathrm{un}_G^{\mathrm{tot}_G}$$

which forms part of the equivalence in Theorem 5.4.24, in the case when $G = \mathrm{GL}_n$. Given a principal GL_n-bundle and the standard GL_n action on the vector space $\mathbb{R}^n$, the $\mathtt{Build}$ functor produces a GL_n-bundle with fibre $\mathbb{R}^n$, called the "associated bundle". By the classical definition, a vector bundle with fiber $\mathbb{R}^n$ is the same as a GL_n-bundle with fiber $\mathbb{R}^n$ and the standard action of GL_n. The converse direction of the correspondence is formed by a construction called the "frame bundle". In the context of tangent categories, the notion of vector bundles is replaced by differential bundles. The generalization of vector bundles are the differential bundles which are the main focus of Chapter 4. Principal bundles and G-bundles are a major focus of Chapter 5, in particular in Section 5.4. The goal of this chapter is to show that (sometimes) a G-bundle is a differential bundle.

The main setup this chapter studies is given by Cartesian tangent join restriction categories with gluings, as defined in Definition 5.2.14 and Definition 5.2.12.

Since differential bundles $E \xrightarrow{q} M$ do not necessarily have a partial isomorphism to a Cartesian product $M \times F$, we do not expect the "frame bundle" direction to hold in this context. We give an explicit counterexample in Example 6.3.13.

This chapter generalizes the "associated bundle" direction of the correspondence. Theorem 5.4.24 already shows that the category of G-bundles is equivalent to the Cartesian product of the category of principal G-bundles and the category of G-spaces. All that remains is to determine under which circumstances G-bundles are differential bundles. That is the main subject of this chapter.

Section 6.1 establishes some language that we need to state and prove that certain G-bundles are differential bundles. Section 6.2 establishes some useful properties of the gluing which are required in the proofs of Section 6.3. Section 6.2.1 focuses on how a pullback in every chart leads to a pullback of the gluings. In Section 6.2.2, Proposition 6.2.5 proves that join restriction functors preserve gluings. The main results of this chapter will be in Section 6.3. Proposition 6.3.7 proves that, given a differential object F and a linear G-action on F, a G-bundle built out of the F and the action a is also a differential bundle. This resembles the classical case that considers the group GL_n and its (linear) action on $\mathbb{R}^n$. In Proposition 6.3.8 we will show that fibre bundle morphisms are also morphisms of differential bundles and that certain G-bundle homomorphisms are linear morphisms of differential bundles.

Theorem 6.3.10 will show that the assignments of objects and morphisms defined in Propositions 6.3.7

and 6.3.8 form a functor.

6.1 Epic fibre bundles

As we have seen in Theorem 5.4.27, classical fibre bundles over smooth manifolds correspond to totally fibred categorical fibre bundles in $\mathbb{P}$arMfld for which the projection q is an epimorphism. For any fibre bundle in any join restriction category, if the projection q is an isomorphism, the restriction idempotents $e_i : M \to M$ fulfill $\bigvee_{i\in I} e_i = 1_M$. Since the restriction idempotents e_i can be seen as the charts in which $E \cong M \times F$, this can be understood as saying that these charts cover all of M. If there is a point $x : 1 \to F$ in the fibre, F, then $\bigvee_{i\in I} e_i = 1_M$ is equivalent to the condition that q is an epimorphism.

Definition 6.1.1. *Let $(E, M, F, q, (\alpha_i)_{i\in I})$ be a fibre bundle (Definition 5.4.1) in a join restriction category $\mathbb{X}$. It is **epic** if $q : E \to M$ is an epimorphism.*

Intuitively this should mean that the image of the projection $q : E \to M$ covers all of M and not just a subset. Due to our partial isomorphisms α_i, we can express this intuitive description as a formal consequence using the restriction idempotents e_i fulfilling $\bar{\alpha}_i^* = e_i \times 1_F : M \times F \to M \times F$.

Proposition 6.1.2. *Let $(E, M, F, q, (\alpha_i)_{i\in I})$ be an epic fibre bundle in a join restriction category $\mathbb{X}$. Let $(e_i)_{i\in I}$ denote the restriction idempotents from Definition 5.4.1 that fulfill $\bar{\alpha}_i^* = e_i \times 1_F$. Then*

$$\bigvee_{i\in I} e_i = 1_M$$

Proof. In order to prove this we will show that $q \bigvee_{i\in I} e_i = q$. Since q is an epimorphism, this will imply $\bigvee_{i\in I} e_i = 1_M$. We calculate

$$q \bigvee_{i\in I} e_i = \bigvee_{i\in I} qe_i.$$

Since $\bigvee_{i\in I} \bar{\alpha}_i = 1_E$ this is equal to

$$\bigvee_{i,j\in I} \bar{\alpha}_i qe_j = \bigvee_{i,j\in I} \alpha_i \pi_0 e_j.$$

because $\bar{\alpha}_i q = \alpha_i \pi_0$ due to Definition 5.4.1 Since the join is the supremum with respect to the partial ordering, taking the join over fewer terms will produce something that is less than or equal to the original join. Therefore,

$$\bigvee_{i,j\in I} \alpha_i \pi_0 e_j \geq \bigvee_{i\in I} \alpha_i \pi_0 e_i = \bigvee_{i\in I} \alpha_i (e_i \times 1_F)\pi_0 = \bigvee_{i\in I} \alpha_i \bar{\alpha}_i^* \pi_0 = \bigvee_{i\in I} \alpha_i \pi_0 = \bigvee_{i\in I} \bar{\alpha}_i q = q.$$

□

Under very common circumstances, the implication in Proposition 6.1.2 becomes an equivalence: certain bundles $(E, M, F, q, (\alpha_i)_{i\in I})$ are epic if and only if $\bigvee_{i\in I} e_i = 1_M$. We will expand on this in the remainder of this section.

In order for the equation $\bigvee_{i\in I} e_i = 1_M$ to imply that the bundle is epic, we need one more condition. The fibre F needs to contain some point. In classical differential geometry this excludes the case when the fibre is the empty set.

Proposition 6.1.3. *Let $(E, M, F, q, (\alpha_i)_{i\in I})$ be a fibre bundle in a join restriction category $\mathbb{X}$. Let $(e_i)_{i\in I}$ denote the restriction idempotents from Definition 5.4.1 that fulfill $\bar{\alpha}_i^* = e_i \times 1_F$. Let $\mathbb{X}$ have a restriction terminal object 1 and let x be a point in F, i.e. a total morphism $x : 1 \to F$. Then $\bigvee_{i\in I} e_i = 1_M$ implies that q is an epimorphism.*

Proof. Suppose $\bigvee_{i\in I} e_i = 1_M$ and suppose X is an object and $f, g : M \to X$ are morphisms such that $qf = qg$. Then $\langle 1_M, x\rangle\pi_0 = 1_M$ and therefore

$$e_i f = \langle 1_M, x\rangle(e_i \times 1)\pi_0 = \langle 1_M, x\rangle\alpha_i^*\alpha_i\pi_0 f = \langle 1_M, x\rangle\alpha_i^*\bar{\alpha}_i qf.$$

Analogously

$$e_i g = \langle 1_M, x\rangle\alpha_i^*\bar{\alpha}_i qf$$

and since $qf = qg$ this implies that $e_i f = e_i g$. By assumption $\bigvee_{i\in I} e_i = 1_M$ and therefore

$$f = \bigvee_{i\in I} e_i f = \bigvee_{i\in I} e_i g = g.$$

Since f and g were arbitrary morphisms such that $qf = qg$ this proves that q is an epimorphism. □

The two Propositions 6.1.2 and 6.1.3 provide an equivalence in cases where F has a point. We summarize this in the following corollary.

Corollary 6.1.4. *Let $(E, M, F, q, (\alpha_i)_{i\in I})$ be a fibre bundle in a join restriction category $\mathbb{X}$. Let $(e_i)_{i\in I}$ denote the restriction idempotents from Definition 5.4.1 that fulfill $\bar{\alpha}_i^* = e_i \times 1_F$. Let $\mathbb{X}$ have a restriction terminal object 1 and let $x : 1 \to F$ be a point in F. Then q is an epimorphism if and only if $\bigvee_{i\in I} e_i = 1_M$.*

In particular, if the fibre is a group object (like for principal G-bundles), Corollary 6.1.4 applies, since every group object has the point $u : 1 \to G$

6.2 Gluing pullbacks and functors

This section provides some technical results that are needed in the proof of Proposition 6.3.7. Section 6.2.1 shows that, under certain conditions, the gluing of pullbacks is the pullback of gluings. Section 6.2.1 shows that join restriction functors preserve gluings.

6.2.1 Gluing pullbacks

We will first set up some context. Let I be a set and let

$$((U_i)_{i\in I},(u_{ij})_{i,j\in I}),\quad ((U'_i)_{i\in I},(u'_{ij})_{i,j\in I}),\quad ((U''_i)_{i\in I},(u''_{ij})_{i,j\in I}),\quad ((U'''_i)_{i\in I},(u'''_{ij})_{i,j\in I})$$

be atlases (Definition 5.2.10) in a join restriction category $\mathbb{X}$. Let

$$(A,(b_i)_{i\in I}:U_i\to A),\quad (A',(b'_i)_{i\in I}:U'_i\to A'),\quad (A'',(b''_i)_{i\in I}:U''_i\to A),\quad (A''',(b'''_i)_{i\in I}:U'''_i\to A''')$$

be their gluings (Definition 5.2.12). Let

$$\begin{tikzcd} U_i \arrow[r, "h_i"] \arrow[d, "k_i"'] \arrow[dr, phantom, "\lrcorner", very near start] & U'_i \arrow[d, "f_i"] \\ U''_i \arrow[r, "g_i"'] & U'''_i \end{tikzcd}$$

be a restriction pullback with total maps f_i, g_i, h_i, k_i for all $i\in I$ such that for $i,j\in I$ all faces of

$$\begin{tikzcd}[row sep=small, column sep=small] & U_j \arrow[rr, "h_j"] \arrow[dd, "k_j", near start] & & U'_j \arrow[dd, "f_j"] \\ U_i \arrow[ur, dashed, "u_{ij}"] \arrow[rr, "h_i", near end] \arrow[dd, "k_i"'] & & U'_i \arrow[ur, dashed, "u'_{ij}"'] \arrow[dd, "f_i", near start] & \\ & U''_j \arrow[rr, "g_j", near start] & & U'''_j \\ U''_i \arrow[ur, dashed, "u''_{ij}"] \arrow[rr, "g_i"'] & & U'''_i \arrow[ur, dashed, "u'''_{ij}"'] & \end{tikzcd}$$

commute.

Lemma 6.2.1. *There are unique morphisms $f : A' \to A'''$, $g : A'' \to A'''$, $h : A \to A'$ and $k : A \to A'$ such that*

$$\begin{tikzcd} U'_i \arrow[r, "f_i"] \arrow[d, "b'_i"'] & U'''_i \arrow[d, "b'''_i"] \\ A' \arrow[r, dashed, "f"'] & A''' \end{tikzcd} \qquad \begin{tikzcd} U''_i \arrow[r, "g_i"] \arrow[d, "b''_i"'] & U'''_i \arrow[d, "b'''_i"] \\ A'' \arrow[r, dashed, "g"'] & A''' \end{tikzcd} \qquad \begin{tikzcd} U_i \arrow[r, "h_i"] \arrow[d, "b_i"'] & U'_i \arrow[d, "b'_i"] \\ A \arrow[r, dashed, "h"'] & A' \end{tikzcd} \qquad \begin{tikzcd} U_i \arrow[r, "k_i"] \arrow[d, "b_i"'] & U''_i \arrow[d, "b''_i"] \\ A \arrow[r, dashed, "k"'] & A'' \end{tikzcd}$$

commute. The morphisms $f : A' \to A'''$, $g : A'' \to A'''$, $h : A \to A'$ and $k : A \to A'$ are total.

Proof. We will show the results for f, the other ones are analogous.

The morphisms $A_{ij} := f_i u'_{ij} = u_{ij} f_j$ for all $i, j \in I$ form an atlas morphism from $((U'_i)_{i\in I}, (u'_{ij})_{i,j\in I})$ to $((U'''_i)_{i\in I}, (u'''_{ij})_{i,j\in I})$.

Thus existence and uniqueness follow from the universal property of the gluingA'. Totality follows from

$$\begin{aligned}\bar{f} &= \overline{\bigvee_{i\in I} b_i'^* b_i' f} = \overline{\bigvee_{i\in I} b_i'^* f_i b_i'''} = \bigvee_{i\in I} \overline{b_i'^* f_i b_i'''} \geq \bigvee_{i\in I} \overline{b_i'^* f_i b_i''' b_i'''^*} = \bigvee_{i\in I} \overline{b_i'^* f_i u_{ij}'''^*} \\ &= \bigvee_{i\in I} \overline{b_i'^* u_{ij}'^* f_i} = \bigvee_{i\in I} \overline{b_i'^* b_i' b_i'^* f_i} = \bigvee_{i\in I} \overline{\overline{b_i'^*} b_i'^*} = \bigvee_{i\in I} \overline{b_i'^*} = 1_{A'}\end{aligned}$$

which concludes the proof. ☐

Proposition 6.2.2. *Under the circumstances described above,*

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "k"'] \arrow[dr, phantom, "\lrcorner", very near start] & A' \arrow[d, "f"] \\ A'' \arrow[r, "g"'] & A''' \end{tikzcd}$$

is a restriction pullback.

Proof. The square commutes because the induced map out of the gluing A is unique. In order to show that it is a restriction pullback, we will use the characterization of restriction pullbacks in Lemma 5.2.8. Let X be an object. Let $\varphi : X \to A'$ and $\psi : X \to A''$ be total morphisms such that $\varphi f = \psi g$, i.e.

$$\begin{tikzcd} X \arrow[drr, bend left, "\varphi"] \arrow[ddr, bend right, "\psi"'] & & \\ & A \arrow[r, "h"] \arrow[d, "k"'] & A' \arrow[d, "f"] \\ & A'' \arrow[r, "g"'] & A''' \end{tikzcd}$$

commutes. Then $\varphi b_i'^* : X \to U'_i$ and $\psi b_i''^* : X \to U''_i$ fulfill that

$$\varphi b_i'^* f_i = \varphi f b_i''^* = \psi g b_i'''^* = \psi b_i''^* g_i.$$

Thus there is an induced map

$$\langle \varphi b_i'^*, \psi b_i''^* \rangle : X \to U_i$$

into the pullback

$$\begin{tikzcd}
U_i \arrow[r, "h_i"] \arrow[d, "k_i"'] \arrow[dr, phantom, "\lrcorner", very near start] & U_i' \arrow[d, "f_i"] \\
U_i'' \arrow[r, "g_i"'] & U_i'''.
\end{tikzcd}$$

In order to glue the morphisms $X \to U_i$ together for all $i \in I$ and obtain a morphism $X \to A$, we now show that they form an atlas morphism from $\mathrm{At}(1_X)$ to $((U_i)_{i\in I}, (u_{ij})_{i,j\in I})$, The map $\langle \varphi b_i'^*, \psi b_i''^* \rangle u_{ij}$ into the pullback U_j is determined by its components and

$$\begin{aligned}
\langle \varphi b_i'^*, \psi b_i''^* \rangle u_{ij} h_j &= \langle \varphi b_i'^*, \psi b_i''^* \rangle h_i u_{ij}' = \overline{\psi b_i''^*} \varphi b_i'^* u_{ij}' = \overline{\psi b_i''^*} \varphi b_i'^* b_i' b_j'^* \\
&= \overline{\psi b_i''^*} \varphi \bar{b}_i'^* b_j'^* = \overline{\psi b_i''^*}\, \overline{\varphi b_i'^*} \varphi b_j'^* = \overline{\langle \varphi b_i'^*, \psi b_i''^* \rangle} \varphi b_j'^*
\end{aligned}$$

and analogously $\langle \varphi b_i'^*, \psi b_i''^* \rangle u_{ij} k_j = \overline{\langle \varphi b_i'^*, \psi b_i''^* \rangle} \psi b_j'' *$. Therefore

$$\langle \varphi b_i'^*, \psi b_i''^* \rangle u_{ij} = \overline{\langle \varphi b_i'^*, \psi b_i''^* \rangle} \langle \varphi b_j'^*, \psi b_j''^* \rangle$$

proving condition 5 of Definition 5.2.11 of an atlas morphism. Conditions 2 and 4 follow from condition 5. Conditions 1 and 3 are trivial since the domain is the atlas $\mathrm{At}(1_X)$. Thus $\langle \varphi b_i'^*, \psi b_i''^* \rangle$ is an atlas morphism from $\mathrm{At}(1_X)$ to $((U_i)_{i\in I}, (u_{ij})_{i,j\in I})$. By composition of atlas morphisms,

$$\theta_{\varphi,\psi} = \bigvee_{i\in I} \langle \varphi b_i'^*, \psi b_i''^* \rangle b_i : X \to A$$

is defined. We will now show that it fulfills the required equations to be the induced map into a pullback. Since $\bar{\varphi} = \bar{\psi}$, we calculate

$$\begin{aligned}
\theta_{\varphi,\psi} h &= \bigvee_{i\in I} \langle \varphi b_i'^*, \psi b_i''^* \rangle b_i h = \bigvee_{i\in I} \langle \varphi b_i'^*, \psi b_i''^* \rangle h_i b_i' = \bigvee_{i\in I} \overline{\psi b_i''^*} \varphi b_i'^* b_i' \\
&= \overline{\bigvee_{i\in I} \psi b_i''^*} \bigvee_{i\in I} \varphi \bar{b}_i'^* = \overline{\psi \bigvee_{i\in I} \bar{b}_i''^*} \varphi \bigvee_{i\in I} \bar{b}_i'^* = \bar{\psi}\varphi = \varphi
\end{aligned}$$

and analogously also $\theta_{\varphi,\psi} k = \psi$.

Therefore it only remains to show that $\theta_{\varphi,\psi}$ is the only morphism such that $\theta_{\varphi,\psi} h = \varphi$ and $\theta_{\varphi,\psi} k = \psi$. Let $\eta_{\varphi,\psi} : X \to A$ be any morphism fulilling $\eta_{\varphi,\psi} h = \varphi$ and $\eta_{\varphi,\psi} k = \psi$. Then

$$\eta_{\varphi,\psi} b_i^* h_i = \eta_{\varphi,\psi} h b_i'^* = \varphi b_i'^*$$

and analogously $\eta_{\varphi,\psi} k_i = \psi b_i''^*$. Since morphisms into the pullback U_i are given by their components, this implies

$$\eta_{\varphi,\psi} b_i^* = \langle \varphi b_i'^*, \psi b_i''^* \rangle = \theta_{\varphi,\psi} b_i^*.$$

Thus

$$\eta_{\varphi,\psi} = \bigvee_{i \in I} \eta_{\varphi,\psi} b_i^* b_i = \bigvee_{i \in I} \theta_{\varphi,\psi} b_i^* b_i = \theta_{\varphi,\psi},$$

which concludes the proof. □

6.2.2 Gluing functors

In this section we will investigate how functors, in particular join restriction functors, preserve gluings. First we define join restriction functors. While all join restriction functors are restriction functors, Example 6.2.4 will showcase a restriction functor that is not a join restriction functor, proving that not all restriction functors are join restriction functors. Then we will continue proving that join restriction functors preserve gluings. While these results seem to be well known to experts in the theory of restriction categories, we were unable able to find any proof of them in the literature, so a proof is included here.

Definition 6.2.3. [47, Section 3.1.2] *For join restriction categories $\mathbb{X}$ and $\mathbb{Y}$, a restriction functor $F : \mathbb{X} \to \mathbb{Y}$ is called a* ***join restriction functor****, if for any $\smile$-compatible subset S, $F(\bigvee_{s \in S} s) = \bigvee_{s \in S} F(s)$.*

Many restriction functors between join restriction categories happen to be join restriction functors. In the following we will give a combinatorial example of a restriction functor that is not a join restriction functor.

Example 6.2.4. *Given a set X, let the join restriction category $\mathbb{X}_X$ be defined as follows.*

- *Its single object is the set X.*
- *The set of morphisms is the power set* $\mathrm{Hom}(X, X) = \{f \subseteq X\}$.
- *Composition is given by intersection of subsets.*
- *Identity is given by the entire set $X \subseteq X$.*
- *Any morphism is a restriction idempotent $\bar{f} = f$.*
- *The join is given by union of subsets.*

In fact, the category $\mathbb{X}_X$ is a subcategory of the category $\mathbb{P}\mathrm{arSet}$ *of sets and partial functions. For the sets* $[1] := \{0, 1\}$ *and* $[2] := \{0, 1, 2\}$, *define the functor $F : \mathbb{X}_{[1]} \to \mathbb{X}_{[2]}$ as follows*

- *It sends the unique object* $[1] := \{0, 1\}$ *to the unique object* $[2] := \{0, 1, 2\}$.

- *It sends a morphism $f \subseteq [1]$ to its embedding in $[2]$, unless it is the identity. It sends the identity to the identity, i.e.*

$$F(f) = \begin{cases} f & \textit{if } f \neq [1] \\ [2] & \textit{if } f = [1] \end{cases}$$

It preserves identity by definition. It preserves composition (intersection) of non-identities, since it is just the embedding with them. It preserves composition with the identity since it sends the identity to the identity. It preserves restrictions, since in both the source and target the restriction operation is the identity.

However, it does not preserve the join since

$$F(\{0\} \vee \{1\}) = F(\{0,1\}) = \{0,1,2\} \neq \{0,1\} = \{0\} \vee \{1\} = F(\{0\}) \vee F(\{1\}).$$

This example shows that the notions of restriction functors and join restriction functors are indeed distinct. There are other examples of restriction functors in the literature which are not join restriction functors, but this example is simpler than others which we are aware of.

Proposition 6.2.5. *Let $\mathbb{X}$ and $\mathbb{Y}$ be join restriction categories. Let $((U_i)_{i\in I}, (u_{ij})_{i,j\in I})$ be an atlas in $\mathbb{X}$ whose gluing is (E, g_i). Let $F : \mathbb{X} \to \mathbb{Y}$ be a join restriction functor. Then $((F(U_i))_{i\in I}, (F(u_{ij}))_{i,j\in I})$ is an atlas whose gluing is $(F(E), F(g_i))$.*

Proof. It follows immediately from the definition of an atlas and functoriality that $((F(U_i))_{i\in I}, (F(u_{ij}))_{i,j\in I})$ is an atlas.

In ordert to show that $(F(E), F(g_i))$ is its gluing, we use the characterization of gluings in Remark 5.2.13 as an object E with a family of partial isomorphisms $g_i : U_i \to E$ (with partial inverses g_i^*) such that

(a) $u_{ij} g_j \leq g_i$,

(b) $u_{ij} = g_i g_j^*$, and

(c) $1_E = \bigvee_i g_i^* g_i$.

Applying the restriction functor T preserves inequalities: For any parallel morphisms f, g of $\mathbb{X}$, $f \leq g$ implies $F(f) \leq F(g)$. Therefore

$$F(u_{ij})F(g_j) \leq F(g_i).$$

Since $F(g_i)F(g_i^*) = F(g_i g_i^*) = F(\bar{g}_i) = \overline{F(g_i)}$ and analogously $F(g_i^*)F(g_i) = \overline{F(g_i^*)}$, the the partial inverse of $F(g_i)$ is $F(g_i^*) = F(g_i)^*$. Thus the second equation

$$F(u_{ij}) = F(g_i)F(g_j^*)$$

follows from functoriality. Since F preserves joins

$$1_{F(E)} = F(1_E) = F\left(\bigvee_i g_i^* g_i\right) = \bigvee_i F(g_i^*)F(g_i),$$

which was the remaining property (c) to show that $F(G_U)$ is the gluing. $\square$

The most important application for the next section is that the tangent bundle endofunctor preserves gluings.

Corollary 6.2.6. *Let $(\mathbb{X}, T, p, 0, +, \ell, c)$ be a tangent join restriction category. Let $((U_i)_{i\in I}, (u_{ij})_{i,j\in I})$ be an atlas in $\mathbb{X}$ whose gluing is (E, g_i). Then $((T(U_i))_{i\in I}, (T(u_{ij}))_{i,j\in I})$ is an atlas in $\mathbb{X}$ whose gluing is $(T(E), T(g_i))$.*

Proof. In [20, Proposition 6.15] Cockett and Cruttwell show that the tangent bundle endofunctor T preserves joins. Thus we can apply Proposition 6.2.5 to obtain the result. $\square$

6.3 Constructing differential bundles from (principal) G-bundles

In classical differential geometry, there is an equivalence between principal GL_n-bundles and vector bundles with n-dimensional fibre. In this section we will generalize the construction that turns (principal) G-bundles (Definition 5.4.14 and the main topic of Section 5.4) into differential bundles (Definition 2.3.1 and the main topic of Chapter 4) in the more abstract setting of tangent join restriction categories.

Differential bundles and differential objects in split restriction categories were defined by Lanfranchi in [58, Definition 4.26]. We will use the same definition for all restriction categories here. It closely resembles Definition 2.3.1 of differential bundles in tangent categories, except that the structure morphisms are required to be total and that strict limits are replaced by restriction limits.

Definition 6.3.1. [58, Definition 4.26] *Let $(\mathbb{X}, T, p, 0, +, \ell, c)$ be a tangent restriction category. Then a **restriction differential bundle** consists of two objects M and E and four total morphisms*

$$q : E \to M \ , \ \zeta : M \to E \ , \ \sigma : E_2 \to E \ , \ \lambda : E \to T(E)$$

where E_n is the restriction n-fold pullback of q along itself. The total morphisms q, ζ, σ and λ satisfy the *equational axioms (4) and (5) in Definition 2.3.1. In addition, the diagram*

$$
\begin{tikzcd}
E_2 \arrow[r, "\langle \pi_0\lambda, \pi_1 0\rangle T(\sigma)"] \arrow[d, "\pi_0 q"'] & T(E) \arrow[d, "T(q)"] \\
M \arrow[r, "0"'] & T(M)
\end{tikzcd}
\tag{6.1}
$$

is required to be a restriction pullback diagram whose universal property is preserved by the tangent bundle functor.

*If M is a restriction terminal object, the differential bundle is called a **restriction differential object**.*

Note that Diagram 6.1 replaces the universality of the vertical lift diagram in Definition 2.3.1. Instead of requiring Diagram 6.1 to be a strict pullback as in Definition 2.3.1, we require it to be a restriction pullback here. For the following definition recall the definition of Cartesian tangent restriction categories in Definition 5.2.14.

Definition 6.3.2. *Let $(E, M, q, \zeta, \sigma, \lambda)$ and $(E', M', q', \zeta', \sigma', \lambda')$ be restriction differential bundles in a Cartesian tangent restriction category $(\mathbb{X}, T, p, 0, +, \ell, c)$. A pair (Φ, ϕ) of total morphisms $\Phi : E \to E'$, $\phi : M \to M'$ is a **morphism of restriction differential bundles**, if*

$$\Phi q' = q\phi.$$

*It is called **linear** if*

$$\Phi\lambda' = \lambda T(\Phi)$$

These definitions of differential bundles and their morphisms are completely analogous to Definitions 2.3.1 and 2.3.5. A choice that we made when defining them is that we only consider total morphisms, both for the structure morphisms $q, \zeta, \sigma, \lambda$ of a differential bundle and the components of a morphism of differential bundles.

Like in the tangent category case, a linear morphism of restriction differential bundles also preserves the addition and the zero section.

Proposition 6.3.3. *Let $(E, M, q, \zeta, \sigma, \lambda)$ and $(E', M', q', \zeta', \sigma', \lambda')$ be restriction differential bundles in a Cartesian tangent restriction category $(\mathbb{X}, T, p, 0, +, \ell, c)$. For a linear morphism of differential bundles*

$$(\Phi, \phi) : (E, M, q, \zeta, \sigma, \lambda) \to (E', M', q', \zeta', \sigma', \lambda'),$$

the diagrams

$$\begin{tikzcd} M \arrow[r, "\phi"] \arrow[d, "\zeta"'] & M' \arrow[d, "\zeta'"] \\ E \arrow[r, "\Phi"'] & E' \end{tikzcd} \qquad \begin{tikzcd} E_2 \arrow[r, "\Phi_2"] \arrow[d, "\sigma"'] & E_2' \arrow[d, "\sigma'"] \\ E \arrow[r, "\Phi"'] & E \end{tikzcd}$$

commute, where $\Phi_2 : E_2 = E \times_M E \to E_2' = E' \times_{M'} E'$ *is the map into the pullback induced by* $\Phi : E \to E$ *and* $\phi : E \to E$.

Proof. The proof follows the exactly same strategy as in [29, Proposition 2.16], where the analogous statement for differential bundles in tangent categories is shown. The proof is summarized below.

Using the restriction pullback diagram 6.1 and the uniqueness of morphisms into the pullback from Lemma 5.2.8, we can show that λ and λ' are monomorphisms. Since q and q' are retractions, they are epimorphisms.

Now some elementary chains of equations which can be found in the proof of [29, Proposition 2.16] show that $\sigma\Phi\lambda' = \Phi_2\sigma'\lambda'$ and $q\zeta\Phi = q\phi\zeta'$. Since λ' was monic and q was epic, the required equations follow. □

Before turning G-bundles into differential bundles, we first consider the case of products. This case is relevant, since restricted to a chart, a G-bundle is isomorphic to a product.

Let $(F, \zeta, \sigma, \lambda)$ be a differential object and X be an object in a Cartesian tangent join restriction category $(\mathbb{X}, T, p, 0, +, \ell, c)$. Then $X \times F$ is a differential bundle over $X \cong X \times 1$ (where 1 is the restriction terminal object of $\mathbb{X}$) with structure maps

$$\begin{aligned} \pi_0 &: X \times F \to X \\ 1_X \times \zeta &: X \times 1 \to X \times F \\ 1_X \times \sigma &: X \times F \times F \cong (X \times F) \times_X (X \times F) \to X \times F \\ 0_X \times \lambda &: X \times F \to T(X) \times T(F) \cong T(X \times F) \end{aligned}$$

All the properties follow from the fact that $(F, \zeta, \sigma, \lambda)$ was a differential object.

Definition 6.3.4. *Let* (G, m, u, i) *be a group object (Definition 5.3.1) and* $(F, \sigma, \zeta, \lambda)$ *be a differential object in a Cartesian tangent join restriction category* $(\mathbb{X}, T, p, 0, +, \ell, c)$. *A* ***linear action*** *of* (G, m, u, i) *on* $(F, \sigma, \zeta, \lambda)$ *is a total morphism* $a : G \times F \to F$ *that is an action (in the sense of Definition 5.4.10) such that* $(a, !) : (G \times F, G) \to (F, 1)$ *is a linear morphism of differential bundles.*

Before we will use that definition, we will first give an overview over the consequences of this definition, in particular the following commutative diagrams. Diagram 6.2 commutes since this is the linearity demanded in Definition 6.3.4, expressed using Definition 6.3.2. Diagrams 6.3 and 6.4 commute because of Proposition 6.3.3.

$$\begin{tikzcd} G \times F \arrow[r, "a"] \arrow[d, "0\times\lambda"'] & F \arrow[d, "\lambda"] \\ T(G) \times T(F) \arrow[r, "T(a)"'] & T(F) \end{tikzcd} \tag{6.2}$$

$$\begin{tikzcd} G \times F \times F \arrow[r, "\langle\langle\pi_0,\pi_1\rangle a,\langle\pi_0,\pi_2\rangle a\rangle"] \arrow[d, "1_G\times\sigma"'] & F \times F \arrow[d, "\sigma"] \\ G \times F \arrow[r, "a"'] & F \end{tikzcd} \tag{6.3}$$

$$\begin{tikzcd} G \arrow[r, "!"] \arrow[d, "\langle 1_G,!\zeta\rangle"'] & 1 \arrow[d, "\zeta"] \\ G \times F \arrow[r, "a"'] & F \end{tikzcd} \tag{6.4}$$

These diagrams appear in equivalent ways of characterizing the action a as a linear action, as we point out in the following Lemma.

Lemma 6.3.5. *Let $(F, \zeta, \sigma, \lambda)$ be a differential object and (G, m, u, i) be a group object in a Cartesian tangent join restriction category $(\mathbb{X}, T, p, 0, +, \ell, c)$. Let $a : G \times F \to F$ be an action in the sense of Definition 5.4.10. Then the the following are equivalent.*

(a) The action a is linear.

(b) Diagram 6.2 commutes.

(c) Diagrams 6.2, 6.3 and 6.4 commute.

Proof. All of the equivalences follow from Definition 6.3.2 of additive and linear morphisms between differential bundles. The equivalence $(a) \Leftrightarrow (b)$ holds, because Diagram 6.2 is just the linearity condition for the morphism $(a, !)$. The implication $(c) \Rightarrow (b)$ is true as a tautology. The implication $(b) \Rightarrow (c)$ holds because Proposition 6.3.3 shows that every linear morphism of differential bundles is additive. □

Next we point out some technical lemma that shows how we can form the pullback power E_2 of a fibre bundle E.

Lemma 6.3.6. *Let $(E, M, F, q, (\alpha_i)_{i\in I})$ be a fibre bundle with fibre F in a Cartesian join restriction category with gluings. Then the restriction pullback*

$$\begin{tikzcd} E_2 \arrow[r, "\pi_0"] \arrow[d, "\pi_1"'] & E \arrow[d, "q"] \\ E \arrow[r, "q"'] & M \end{tikzcd}$$

is the gluing of $(M \times F \times F, (u_{ij} \times_M u_{ij})_{i,j \in I})$ *where the morphisms* $u_{ij} \times_M u_{ij}$ *are the composition*

$$M \times F \times F \xrightarrow{\cong} (M \times F) \times_M (M \times F) \xrightarrow{u_{ij} \times_{1_M} u_{ij}} (M \times F) \times_M (M \times F) \xrightarrow{\cong} M \times F \times F.$$

Proof. This is a direct consequence of Proposition 6.2.2, applied to Definition 5.4.1.

□

Proposition 6.3.7. *Let* $(F, \zeta, \sigma, \lambda)$ *be a differential object and* (G, m, u, i) *be a group object in a Cartesian tangent join restriction category* $(\mathbb{X}, T, p, 0, +, \ell, c)$ *with gluings. Let* $a : G \times F \to F$ *be a linear action of* (G, m, u, i) *on* $(F, \zeta, \sigma, \lambda)$*. Let* $(E, M, q, (\alpha_i)_{i \in I})$ *be a totally fibred epic G-bundle with fibre F with the action a. Then* $E \xrightarrow{q} M$ *is a differential bundle.*

The proof is lengthy because we need to construct the differential bundle structure for $E \xrightarrow{q} M$ from the differential object structure on F. This proceeds by constructing the total morphisms

$$q : E \to M,\ \zeta_E : M \to E, \sigma_E : E_2 \to E \text{ and } \lambda_E : E \to T(E)$$

out of ζ, σ, λ using gluings. In the end, we will show that Diagram 6.1 is a restriction pullback.

Proof. We first define all the structure we need. **The projection** $q : E \to M$ is already defined. For the remaining structure we will use the property from Theorem 5.4.9 that E is a gluing of $(M \times F, (u_{ij}))$ with $u_{ij} = \alpha_i^* \alpha_j$ and the inclusion morphisms $\alpha_i^* : M \times F \to E$.

The zero section $\zeta_E : M \to E$ is induced by the morphisms $1_M \times \zeta : M \times 1 \to M \times F$. Concretely it is given by

$$\zeta_E = \bigvee_{i \in I} \langle 1_M, !\zeta \rangle \alpha_i^*.$$

Intuitively, this is well-defined (i.e. $\langle 1_M, !\zeta \rangle \alpha_i^* \smile \langle 1_M, !\zeta \rangle \alpha_j^*$) because the partial isomorphisms $u_{ij} = \alpha_i^* \alpha_j$ (corresponding to the changes of charts in classical differential geometry) are given by group actions and Diagram 6.4 commutes. Concretely $\langle 1_M, !\zeta \rangle \alpha_i^* \smile \langle 1_M, !\zeta \rangle \alpha_j^*$ can be verified as follows.

Using standard properties of restriction categories in Definition 5.2.1, the definitions of the partial isomorphisms α_i in Definition 5.4.1 and the definition of the restriction idempotents e_i through $\bar{\alpha}_i^* = e_i \times 1_F$ in Definition 5.4.1, we derive

$$\bar{u}_{ji} = \overline{\alpha_j^* \alpha_i} = \overline{\alpha_j^* \bar{\alpha}_i} = \overline{\alpha_j^* \overline{q e_i}} = \overline{\alpha_j^* q e_i} = \overline{\alpha_j^* \bar{\alpha}_j q e_i} = \overline{\bar{\alpha}_j^* \pi_0 e_i} = \overline{\pi_0 e_j e_i}.$$

Using the equation above, we observe that

$$\begin{aligned}
\overline{\langle 1_M, !\zeta\rangle\alpha_i^*}\langle 1_M, !\zeta\rangle\alpha_j^* &= \overline{\langle 1_M, !\zeta\rangle(e_i \times 1_F)}\langle 1_M, !\zeta\rangle\alpha_j^* && \text{by Definition 5.4.1(i)}\\
&= \langle e_i, !\zeta\rangle\alpha_j^* \\
&= \langle 1_M, !\zeta\rangle\overline{\pi_0 e_i}\alpha_j^* && \text{by Definition 5.2.6}\\
&= \langle 1_M, !\zeta\rangle\bar{u}_{ji}\alpha_j^* && \text{since } \bar{u}_{ji} = \overline{\pi_0 e_j e_i}.
\end{aligned}$$

Since the $(\alpha_i^*)_{i\in I}$ are a gluing, $u_{ji}\alpha_i^* = \overline{u_{ji}\alpha_i^*}\alpha_j^* = \bar{u}_{ij}\alpha_j^*$ and therefore the previous line is equal to

$$\begin{aligned}
&\langle 1_M, !\zeta\rangle u_{ji}\alpha_i^* \\
&= \langle 1_M, !\zeta\rangle\langle \pi_0, \pi_0\tau_{ji}, \pi_1\rangle(1_M \times a)\alpha_i^* && \text{by Definition 5.4.14}\\
&= \langle 1_M, \tau_{ji}\langle 1_G, !\zeta\rangle a\rangle\alpha_i^* \\
&= \bar{\tau}_{ji}\langle 1_M, !\zeta\rangle\alpha_i^* && \text{by Diagram 6.4}\\
&\leq \langle 1_M, !\zeta\rangle\alpha_i^*.
\end{aligned}$$

and thus $\langle 1_M, !\zeta\rangle\alpha_i^* \smile \langle 1_M, !\zeta\rangle\alpha_j^*$. This shows that $\zeta_E : M \to E$ is well-defined. Next we will prove that it is total by calculating

$$\bar{\zeta}_E = \overline{\bigvee_{i\in I}\langle 1_M, !\zeta\rangle\alpha_i^*} = \overline{\langle 1_M, !\zeta\rangle\bigvee_{i\in I}\bar{\alpha}_i^*} = \overline{\langle 1_M, !\zeta\rangle\bigvee_{i\in I}(e_i \times 1_F)} = \overline{\langle 1_M, !\zeta\rangle\overline{\bigvee_{i\in I}(e_i \times 1_F)}}$$

using Property (iii) of Lemma 5.2.2. Since q was assumed to be an epimorphism, Proposition 6.1.2 shows that $\bigvee_{i\in I} e_i = 1$. This shows that the previous line is equal to

$$\overline{\langle 1_M, !\zeta\rangle} = 1_M$$

proving that ζ_E is total.

The addition $\sigma_E : E_2 \to E$ is induced by the addition $\sigma : F \times F \to F$ as

$$\sigma_E := \bigvee_{i\in I}(\alpha_i \times_M \alpha_i)(\sigma/1_M)\alpha_i^*.$$

where $\sigma/1_M : (M \times F) \times_M (M \times F) \to M \times F$ is given by the composition

$$(M \times F) \times_M (M \times F) \xrightarrow[\cong]{\langle \pi_0\pi_0, \pi_0\pi_1, \pi_1\pi_1\rangle} M \times F \times F \xrightarrow{1_M \times \sigma} M \times F.$$

Intuitively, this is well-defined (i.e. $(\alpha_i \times_M \alpha_i)(1_M \times \sigma)\alpha_i^* \smile (\alpha_j \times_M \alpha_j)(1_M \times \sigma)\alpha_j^*$) since partial isomorphisms $u_{ij} = \alpha_i^* \alpha_j$ (corresponding to the changes of charts in classical differential geometry) are given by the group action a (Definition 5.4.14) and Diagram 6.3 commutes. Concretely we calculate

$$
\begin{aligned}
&\overline{(\alpha_j \times_M \alpha_j)(\sigma/1_M)\alpha_j^*}(\alpha_i \times_M \alpha_i)(\sigma/1_M)\alpha_i^* && \\
&\leq \overline{(\alpha_j \times_M \alpha_j)}(\alpha_i \times_M \alpha_i)(\sigma/1_M)\alpha_i^* && \\
&= (\bar{\alpha}_j \alpha_i \times_M \bar{\alpha}_j \alpha_i)(\sigma/1_M)\alpha_i^* && \\
&= (\alpha_j \alpha_j^* \alpha_i \times_M \alpha_j \alpha_j^* \alpha_i)(\sigma/1_M)\alpha_i^* && \\
&= (\alpha_j \times_M \alpha_j)(u_{ji} \times_M u_{ji})(\sigma/1_M)\alpha_i^* && \text{since } u_{ji} = \alpha_j^* \alpha_i \\
&= (\alpha_j \times_M \alpha_j)(\langle \pi_0, \langle \pi_0 \tau_{ji}, \pi_1 \rangle a \rangle \times_M \langle \pi_0, \langle \pi_0 \tau_{ji}, \pi_1 \rangle a \rangle)(\sigma/1_M)\alpha_i^* && \text{by Definition 5.4.14} \\
&= (\alpha_j \times_M \alpha_j)\langle \pi_0\pi_0, \langle \langle \pi_0\pi_0\tau_{ji}, \pi_0\pi_1 \rangle a, \langle \pi_0\pi_0\tau_{ji}, \pi_1\pi_1 \rangle a \rangle \sigma \alpha_i^* && \text{by Definition of } \sigma/1_M \\
&= (\alpha_j \times_M \alpha_j)\big\langle \pi_0\pi_0, \langle \pi_0\pi_0\tau_{ji}, \pi_0\pi_1, \pi_1\pi_1 \rangle \langle \langle \pi_0, \pi_1 \rangle a, \langle \pi_0, \pi_2 \rangle a \rangle \sigma \big\rangle \alpha_i^* && \\
&= (\alpha_j \times_M \alpha_j)\big\langle \pi_0\pi_0, \langle \pi_0\pi_0\tau_{ji}, \pi_0\pi_1, \pi_1\pi_1 \rangle (1_G \times \sigma) a \big\rangle \alpha_i^* && \text{by Diagram 6.3} \\
&= (\alpha_j \times_M \alpha_j)(\sigma/1_M)\langle \pi_0, \langle \pi_0 \tau_{ji}, \pi_1 \rangle a \rangle \alpha_i^* && \text{by Definition of } \sigma/1_M \\
&= (\alpha_j \times_M \alpha_j)(\sigma/1_M) u_{ji} \alpha_i^* && \text{by Definition 5.4.14} \\
&= (\alpha_j \times_M \alpha_j)(\sigma/1_M)\alpha_j^* \alpha_i \alpha_i^* && \\
&= (\alpha_j \times_M \alpha_j)(\sigma/1_M)\alpha_j^* \bar{\alpha}_i && \\
&\leq (\alpha_j \times_M \alpha_j)(\sigma/1_M)\alpha_j^*. &&
\end{aligned}
$$

Thus, due to Lemma 5.2.2(v) and the Definition of compatibility in Definition 5.2.1(ii),

$$(\alpha_i \times_M \alpha_i)(1_M \times \sigma)\alpha_i^* \smile (\alpha_j \times_M \alpha_j)(1_M \times \sigma)\alpha_j^*,$$

proving that σ_E is well-defined.

In order to show that σ_E is total, we calculate

$$\begin{aligned}
\bar{\sigma}_E &= \overline{\bigvee_{i\in I}(\alpha_i \times_M \alpha_i)(\sigma/1_M)\alpha_i^*} \\
&= \overline{\bigvee_{i\in I}(\alpha_i \times_M \alpha_i)(\sigma/1_M)\bar{\alpha}_i^*} \\
&= \overline{\bigvee_{i\in I}(\alpha_i \times_M \alpha_i)\langle\pi_0\pi_0, \pi_0\pi_1, \pi_1\pi_1\rangle(1_M \times \sigma)(e_i \times 1_F)} \\
&= \overline{\bigvee_{i\in I}(\alpha_i \times_M \alpha_i)\langle\pi_0\pi_0, \pi_0\pi_1, \pi_1\pi_1\rangle\overline{(e_i \times \sigma)}}.
\end{aligned}$$

Since σ is total this equals

$$\begin{aligned}
&\overline{\bigvee_{i\in I}(\alpha_i \times_M \alpha_i)\pi_0\pi_0 e_i} \\
&= \overline{\bigvee_{i\in I}\overline{\pi_1\alpha_i}\pi_0\alpha_i\pi_0 e_i} \\
&= \overline{\bigvee_{i\in I}\overline{\pi_1\alpha_i}\pi_0\alpha_i\overline{\pi_0 e_i}}.
\end{aligned}$$

Since $\bar{\alpha}_i^* = e_i \times 1 = \overline{\pi_0 e_i}$ this can be simplified into

$$\begin{aligned}
&\overline{\bigvee_{i\in I}\overline{\pi_1\alpha_i}\pi_0\alpha_i\bar{\alpha}_i^*} \\
&= \overline{\bigvee_{i\in I}\overline{\pi_1\alpha_i}\pi_0\alpha_i}.
\end{aligned}$$

Since $(E, M, F, q, (\alpha_i)_{i\in I})$ is totally fibred, $\bar{\alpha}_i = \overline{qe_i}$ and therefore this equals

$$\overline{\bigvee_{i\in I}\overline{\pi_0 qe_i}\,\overline{\pi_0 qe_i}}.$$

Since the domain is the pullback $E_2 = E \times_M E$, $\pi_0 q = \pi_1 q$ and therefore we obtain

$$\overline{\bigvee_{i\in I}\pi_0 qe_i} = \overline{\pi_0\bigvee_{i\in I}\bar{\alpha}_i} = \bar{\pi}_0 = 1_{E_2}.$$

This shows that $\sigma_E : E_2 \to E$ is total.

The vertical lift $\lambda_E : E \to T(E)$ is induced by the lift $\lambda : F \to T(F)$ as

$$\bigvee_{i \in I} \alpha_i(0_M \times \lambda)T(\alpha_i^*).$$

It is well-defined (i.e. $\alpha_i(0_M \times \lambda)T(\alpha_i^*) \smile \alpha_j(0_M \times \lambda)T(\alpha_j^*)$) since Diagram 6.2 commutes. Concretely,

$$\begin{aligned}
\bar{\alpha}_j \alpha_i(0 \times \lambda)T(\alpha_i^*) &= \alpha_j u_{ji}(0 \times \lambda)T(\alpha_i^*) && \text{since } \bar{\alpha}_j = \alpha_j \alpha_j^* \\
&= \alpha_j \langle \pi_0, \langle \pi_0 \tau_{ji}, \pi_1 \rangle a \rangle (0 \times \lambda)T(\alpha_i^*) \\
&= \alpha_j \langle \pi_0 0, \langle \pi_0 \tau_{ji}, \pi_1 \rangle a\lambda \rangle T(\alpha_i^*) \\
&= \alpha_j \langle \pi_0 0, \langle \pi_0 \tau_{ji} 0, \pi_1 \lambda \rangle T(a) \rangle T(\alpha_i^*) && \text{by Diagram 6.2} \\
&= \alpha_j (0 \times \lambda) \langle \pi_0, \langle \pi_0 T(\tau_{ji}), \pi_1 \rangle T(a) \rangle T(\alpha_i^*) \\
&= \alpha_j (0 \times \lambda) T(u_{ij}) T(\alpha_i^*) && \text{by Definition 5.4.14} \\
&= \alpha_j (0 \times \lambda) T(\alpha_j) T(\bar{\alpha}_j) \\
&\leq \alpha_j (0 \times \lambda) T(\alpha_j)
\end{aligned}$$

and therefore $\alpha_i(0_M \times \lambda)T(\alpha_i^*) \smile \alpha_j(0_M \times \lambda)T(\alpha_j^*)$ showing that $\lambda_E : E \to T(E)$ is well-defined. In order to show that λ_E is total we use naturality of 0 and totality of 0 and λ. Concretely we calculate

$$\begin{aligned}
\bar{\lambda}_E &= \bigvee_{i \in I} \overline{\alpha_i(0_M \times \lambda)T(\alpha_i^*)} \\
&= \bigvee_{i \in I} \overline{\alpha_i(0_M \times \lambda)T(\bar{\alpha}_i^*)} \\
&= \bigvee_{i \in I} \overline{\alpha_i(0_M \times \lambda)T(e_i \times 1_F)} && \text{by Definition 5.4.1} \\
&= \bigvee_{i \in I} \overline{\alpha_i(e_i \times 1_F)(0_{TM} \times \lambda)} && \text{since 0 is natural} \\
&= \bigvee_{i \in I} \overline{\alpha_i \bar{\alpha}_i^* \overline{(0_{TM} \times \lambda)}} \\
&= \bigvee_{i \in I} \overline{\alpha_i} = 1_E && \text{since 0 and } \lambda \text{ are total.}
\end{aligned}$$

This proves that $\lambda_E : E \to T(E)$ is total. All equations for q, ζ_E, σ_E and λ_E in Definition 6.3.1 follow from the fact that $(M \times F, M, \pi_0, \langle 1_M, !\zeta \rangle, \sigma/1_M, 0_M \times \lambda)$ is a differential bundle.

It remains to show the **universality of the vertical lift**. In order to show the universality of the vertical lift, we will use Proposition 6.2.2. As usually we denote $\alpha_i^* \alpha_j$ by u_{ij}. Due to Lemma 6.3.6, $(E_2, \alpha_i^* \times \alpha_i^*)$

is the gluing of $((M \times F) \times_M (M \times_F), u_{ij} \times_M u_{ij})$. Due to Corollary 6.2.6, $(T(E), T(\alpha_i^*))$ is the gluing of $(T(M \times F), T(u_{ij}))$. Since $\bigvee_{i\in I} e_i = 1_M$, $(M, (e_i)_{i\in I})$ is the gluing of $(M, (e_ie_j)_{i,j\in I})$ and analogously $(T(M), (T(e_i))_{i\in I})$ is the gluing of $(T(M), (T(e_ie_j))_{i,j\in I})$.

In each chart, the diagram

$$\begin{tikzcd}
(M \times F) \times_M (M \times F) \arrow[r, "\langle \pi_0(0\times\lambda), \pi_1 0\rangle T(\sigma/1_M)"] \arrow[d, "\pi_0\pi_0"'] & T(M \times F) \arrow[d, "T(\pi_0)"] \\
M \arrow[r, "0"'] & T(M)
\end{tikzcd}$$

is a pullback because $(M \times F, M, \pi_0, \langle 1_M, !\zeta\rangle, \sigma/1_M, 0_M \times \lambda)$ is a differential bundle. In order to apply Proposition 6.2.2, we additionally need to show that

$$\begin{tikzcd}
& (M \times F) \times_M (M \times F) \arrow[rr, "\langle \pi_0(0\times\lambda), \pi_1 0\rangle T(\sigma/1_M)"] \arrow[dd, "\pi_0\pi_0"' near start] & & T(M \times F) \arrow[dd, "T(\pi_0)"] \\
(M \times F) \times_M (M \times F) \arrow[ur, "u_{ij} \times_M u_{ij}"] \arrow[rr, "\langle \pi_0(0\times\lambda), \pi_1 0\rangle T(\sigma/1_M)" near end] \arrow[dd, "\pi_0\pi_0"'] & & T(M \times F) \arrow[ur, "T(u_{ij})"] \arrow[dd, "T(\pi_0)" near start] & \\
& M \arrow[rr, "0"' near end] & & T(M) \\
M \arrow[ur, "e_ie_j"'] \arrow[rr, "0"'] & & T(M) \arrow[ur, "T(e_ie_j)"'] &
\end{tikzcd} \tag{6.5}$$

commutes. The front and back squares commute and is even a pullback. The bottom square commutes because of naturality of 0. The left square commutes because

$$(u_{ij} \times_M u_{ij})\pi_0\pi_0 = \overline{\pi_1 u_{ij}}\pi_0 u_{ij}\pi_0 = \overline{\pi_1 \alpha_i^* \alpha_j}\pi_0 \alpha_i^* \alpha_j \pi_0.$$

We observe that $\alpha_j\pi_0 = \bar{\alpha}_j q$. Since $(E, M, F, q, (\alpha_i)_{i\in I})$ is totally fibred, $\bar{\alpha}_j = \overline{qe_j}$ and thus the previous line is equal to

$$\overline{\pi_1 \alpha_i^* q e_j}\pi_0 \alpha_i^* \overline{qe_j} q = \overline{\pi_1 \alpha_i^* q e_j}\pi_0 \alpha_i^* q e_j.$$

Since $\alpha_i^* q = \bar{\alpha}_i^* \pi_0 = \pi_0 e_i$, this equals

$$\overline{\pi_1 \pi_0 e_i e_j}\pi_0\pi_0 e_i e_j.$$

Since the domain is the pullback $(M \times F) \times_M (M \times F)$, the equation $\pi_0\pi_0 = \pi_1\pi_0$ has to hold and therefore the previous line equals

$$\overline{\pi_0 \pi_0 e_i e_j}\pi_0\pi_0 e_i e_j = \pi_0\pi_0 e_i e_j$$

which proves that the left square in Diagram 6.5 commutes.

The right square in Diagram 6.5 commutes since

$$T(u_{ij})T(\pi_0) = T(\alpha_i^*\alpha_j\pi_0) = T(\alpha_i^*\bar{\alpha}_j q) = T(\alpha_i^*\overline{qe_j}q) = T(\alpha_i^* q e_j)$$
$$= T(\bar{\alpha}_i^*\pi_0 e_j) = T((e_i \times 1)\pi_0 e_j) = T(\pi_0)T(e_i e_j)$$

For showing that the top square commutes, we first expand

$$\begin{aligned}\langle \pi_0(0\times\lambda), \pi_1 0\rangle T(\sigma/1_M) &= \langle \pi_0(0\times\lambda), \pi_1 0\rangle T(\langle \pi_0\pi_0, \pi_0\pi_1, \pi_1\pi_1\rangle(1_M\times\sigma)) \\ &= \langle \pi_0\pi_0 0, \pi_0\pi_1\lambda, \pi_1\pi_1 0\rangle T(1_M\times\sigma) \\ &= \langle \pi_0\pi_0 0, \langle \pi_0\pi_1\lambda, \pi_1\pi_1 0\rangle T(\sigma)\rangle.\end{aligned}$$

Using this expansion we now calculate

$$(u_{ij}\times_M u_{ij})\langle \pi_0(0\times\lambda), \pi_1 0\rangle T(\sigma/1_M) = \langle \pi_0 u_{ij}\pi_0 0, \langle \pi_0 u_{ij}\pi_1\lambda, \pi_1 u_{ij}\pi_1 0\rangle T(\sigma)\rangle.$$

Since $u_{ij} = \langle \pi_0, \langle \pi_0\tau_{ij}, \pi_1\rangle a\rangle$ and $\pi_0\pi_0 = \pi_0\pi_1$, this is equal to

$$\Big\langle \pi_0\pi_0 0, \big\langle \langle \pi_0\pi_0\tau_{ij}, \pi_0\pi_1\rangle a\lambda, \langle \pi_0\pi_0\tau_{ij}, \pi_1\pi_1\rangle a0\big\rangle T(\sigma)\Big\rangle.$$

Using Diagram 6.2 of Definition 6.3.4 and the naturality of 0, we can rewrite this as

$$\Big\langle \pi_0\pi_0 0, \big\langle \langle \pi_0\pi_0 0 T(\tau_{ij}), \pi_0\pi_1\lambda\rangle T(a), \langle \pi_0\pi_0 0 T(\tau_{ij}), \pi_1\pi_1 0\rangle T(a)\big\rangle T(\sigma)\Big\rangle.$$

Next we use Diagram 6.3 of Definition 6.3.4 to rewrite the previous expression into

$$\Big\langle \pi_0\pi_0 0, \big\langle \pi_0\pi_0 0 T(\tau_{ij}), \langle \pi_0\pi_1\lambda, \pi_1\pi_1 0\rangle T(\sigma)\big\rangle T(a)\Big\rangle.$$

Since $u_{ij} = \langle \pi_0, \langle \pi_0\tau_{ij}, \pi_1\rangle a\rangle$, this is equal to

$$\big\langle \pi_0\pi_0 0, \langle \pi_0\pi_1\lambda, \pi_1\pi_1 0\rangle T(\sigma)\big\rangle T(u_{ij}) = \langle \pi_0(0\times\lambda), \pi_1 0\rangle T(\sigma/1_F)T(u_{ij}),$$

which shows that the top square of Diagram 6.5 commutes.

This proves that Diagram 6.5 commutes. Now Proposition 6.2.2 shows that

$$\begin{tikzcd} E_2 \arrow[r, "\langle \pi_0\lambda_E, \pi_1 0\rangle T(\sigma_E)"] \arrow[d, "\pi_0 q"'] \arrow[dr, phantom, "\lrcorner", very near start] & TE \arrow[d, "T(q)"] \\ M \arrow[r, "0"'] & TM \end{tikzcd}$$

is a restriction pullback and thereby the universality of the vertical lift holds. Since this was the last part of Definition 6.3.1, this concludes the proof of Proposition 6.3.7 $\square$

Next we show that that morpisms of G-bundles are morphisms of differential bundles.

Proposition 6.3.8. *Let (G, m, u, i) be a group object and $(F, \zeta, \sigma, \lambda), (F', \zeta', \sigma', \lambda')$ be differential objects in a Cartesian tangent join restriction category $(\mathbb{X}, T, p, 0, +, \ell, c)$. Let $a : G \times F \to F, a' : G \times F' \to F'$ be linear actions. Let $(E, M, q, (\alpha_i)_{i\in I}), (E', M', q', (\alpha'_i)_{i\in I})$ be totally fibred epic G-bundles with fibres F and F' and actions a and a' respectively. Let $(E, M, q, \zeta_E, \sigma_E, \lambda_E)$ and $(E', M', q', \zeta_{E'}, \sigma_{E'}, \lambda_{E'})$ be the corresponding differential bundles from Proposition 6.3.7 Let $\Phi : E \to E'$ and $\phi : M \to M'$ be total morphisms in $\mathbb{X}$.*

(a) Then the pair (Φ, ϕ) is a morphism of fibre bundles if and only if is a morphism of differential bundles.

(b) Given a total linear morphism $\psi : F \to F'$ of differential objects, if the triple (Φ, ϕ, ψ) is part of a morphism of G-bundles, then (Φ, ϕ) is a linear morphism of differential bundles.

Proof. Due to Definition 5.4.1, (Φ, ϕ) is a morphism of fibre bundles if and only if $\Phi q' = q\phi$. Due to Definition 6.3.2, it is a morphism of differential bundles if and only if $\Phi q' = q\phi$. Since these definitions coincide, (a) holds.

Due to Definition 5.4.19, if (Φ, ϕ) is a morphism of G-bundles, then there is a morphism $\psi : F \to F'$ such that $(\psi \times 1_G)a' = a\psi$ and a collection of morphisms $T_{ik} : M \to G$ such that $T_{ik} \geq (T_{jk}, \tau_{ij})m$, $T_{ik} \geq (\phi\tau'_{lk}, T_{il})m$, and $\bar{T}_{ik}T_{ih} = (\phi\tau_{kh}, T_{ik})m$ and such that

$$\begin{tikzcd} M \times F \arrow[r, "\langle \pi_0\phi, (T_{ik} \times \psi)a'\rangle"] \arrow[d, "\alpha_i^*"'] & M' \times F' \\ E \arrow[r, "\Phi"'] & E' \arrow[u, "\alpha'_k"'] \end{tikzcd}$$

commutes. This means in particular that

$$\Phi = \bigvee_{i\in I} \bigvee_{k\in I'} \alpha_i \langle \pi_0\phi, (T_{ik} \times \psi)a'\rangle {\alpha'_k}^*.$$

The equation $q\phi = \Phi q'$ holds since $q\phi \geq \Phi q'$ and both are total. Due to the construction of λ and λ',

$$\lambda_E = \bigvee_{i \in I} \alpha_i (0_M \times \lambda) T(\alpha_i^*) \;\text{ and }\; \lambda_{E'} = \bigvee_{k \in I'} \alpha'_k (0_{M'} \times \lambda') T({\alpha'_k}^*).$$

Due to Definition 6.3.2, the pair (Φ, ϕ) is a linear morphism of differential bundles if and only if $\Phi\lambda_{E'} = \lambda_E T(\Phi)$.

In order to check this equation, we substitute the formulas we have for Φ, λ_E and $\lambda_{E'}$. Due to naturality of 0 and the fact that $\bar{\alpha'_k}^* = e'_k \times 1_{F'}$, the inequality

$$\begin{aligned}
\Phi\lambda_E &= \bigvee_{i \in I} \bigvee_{k \in I'} \bigvee_{l \in I'} \alpha_i \langle \pi_0 \phi, (T_{ik} \times \psi) a' \rangle \alpha_k^* \alpha'_l (0 \times \lambda') T({\alpha'_k}^*) \\
&\geq \bigvee_{i \in I} \bigvee_{k \in I'} \alpha_i \langle \pi_0 \phi, (T_{ik} \times \psi) a' \rangle (0 \times \lambda') T({\alpha'_k}^*) \\
&= \bigvee_{i \in I} \bigvee_{k \in I'} \alpha_i \langle \pi_0 \phi 0, (T_{ik} \times \psi) a' \lambda' \rangle T({\alpha'_k}^*)
\end{aligned}$$

holds. In fact, since both sides of the inequality are total, equality

$$\Phi\lambda_E = \bigvee_{i \in I} \bigvee_{k \in I'} \alpha_i \langle \pi_0 \phi 0, (T_{ik} \times \psi) a' \lambda' \rangle T({\alpha'_k}^*)$$

holds. Due to Diagram 6.2 and naturality of zero this is equal to

$$\Phi\lambda_E = \bigvee_{i \in I} \bigvee_{k \in I'} \alpha_i \langle \pi_0 0 T(\phi), (T_{ik} \times \psi)(0 \times \lambda') T(a') \rangle T({\alpha'_k}^*).$$

Since ψ is linear and 0 is natural this is equal to

$$\begin{aligned}
\Phi\lambda_E &= \bigvee_{i \in I} \bigvee_{k \in I'} \alpha_i \langle \pi_0 0 T(\phi), (0 \times \lambda)(T(T_{ik}) \times T(\psi)) T(a') \rangle T({\alpha'_k}^*) \\
&= \bigvee_{i \in I} \bigvee_{k \in I'} \alpha_i (0 \times \lambda) T(\langle \pi_0 \phi, (T_{ik} \times \psi) a' \rangle) T({\alpha'_k}^*).
\end{aligned}$$

Since for $\bar{\alpha}_i^* = e_i \times 1_F$ and due to naturality of 0, this is equal to

$$\Phi\lambda_E = \bigvee_{i \in I} \bigvee_{k \in I'} \alpha_i (0 \times \lambda) T(\alpha_i^*) T(\alpha_i) T(\langle \pi_0 \phi, (T_{ik} \times \psi) a' \rangle) T({\alpha'_k}^*)$$

which implies the inequality

$$\Phi\lambda_E \geq \bigvee_{i\in I}\bigvee_{j\in I}\bigvee_{k\in I'} \alpha_i(0\times\lambda)T(\alpha_i^*)T(\alpha_j)T(\langle\pi_0\phi,(T_{ik}\times\psi)a'\rangle)T({\alpha'_k}^*) = \lambda_E T(\Phi).$$

Since both sides of the inequality are total, equality $\Phi\lambda_E = \lambda_E T(\Phi)$ holds, proving (b). $\square$

While Proposition 6.3.7 assigns differential bundles to G-bundles, Proposition 6.3.8 sends morphisms of G-bundles to morphisms of differential bundles. In order to express these results of Propositions 6.3.7 and 6.3.8 as functors, we first define certain subcategories of fibre bundles and G-bundles.

Recall from Definition 5.4.14 that a G-bundle $(E, M, F, (\alpha_i)_{i\in I}, \tau_{ij})$ contains a designated action $a : G\times F \to F$ of G on the fibre F. Further recall from Definition 5.4.19, that a morphism of G-bundles comes equipped with a morphism $\psi : F \to F'$ of between the fibres. Together G-bundles and G-bundle morphisms form the category $\mathbb{B}\mathrm{un}_G$.

Definition 6.3.9. *Let G be a group object in a Cartesian join restriction category.*

(a) Let $\mathbb{F}\mathrm{ibBun}_G^{\mathrm{diff}}$ denote the full subcategory of $\mathbb{F}\mathrm{ibBun}$ consisting of G-bundles where the fibre F is a differential object and where the action $a : G\times F \to F$ is linear morphism of differential objects.

(b) Let $G - \mathbb{O}\mathrm{bj}^{\mathrm{lin}}$ denote the subcategory of $G - \mathbb{O}\mathrm{bj}$ consisting of

- *(objects) G-objects where the underlying object F is a differential object and where the action $a : G\times F \to F$ is linear morphism of differential objects, and*
- *(morphisms) G-object morphisms where the underlying morphism $\psi : F \to F'$ is a linear morphism of differential objects.*

(c) Analogously let $\mathbb{B}\mathrm{un}_G^{\mathrm{lin}}$ denote the subcategory of $\mathbb{B}\mathrm{un}_G$ consisting of

- *(objects) G-bundles where the fibre F is a differential object and where the action $a : G\times F \to F$ is linear morphism of differential objects, and*
- *(morphisms) G-bundle morphisms where the morphism $\psi : F \to F'$ is a linear morphism of differential objects.*

(d) Let $\mathbb{P}\mathrm{Bun}_G^{\mathrm{ep,tot}}$, $\mathbb{B}\mathrm{un}_G^{\mathrm{lin,ep,tot}}$ and $\mathbb{F}\mathrm{ibBun}_G^{\mathrm{diff,ep,tot}}$ denote the full subcategories of $\mathbb{P}\mathrm{Bun}_G$, $\mathbb{B}\mathrm{un}_G^{\mathrm{lin}}$ and $\mathbb{F}\mathrm{ibBun}_G^{\mathrm{diff}}$ respectively, consisting of all totally fibred epic bundles.

Since differential objects always contain a point (zero), every object of $\mathbb{B}\mathrm{un}_G^{\mathrm{lin,ep,tot}}$ is totally G-fibred due to Lemma 5.4.22(iv).

The category $\mathbb{P}\mathrm{Bun}_G^{\mathrm{ep,tot}}$ is the full subcategory of classical principal bundles. As shown in Proposition 5.4.31, they are exactly the principal bundles defined in differential geometry.

Propositions 6.3.7 and 6.3.8 define assignments of objects and morphisms

$$\texttt{IsDifferential} : \mathrm{Tot}(\mathbb{F}\mathrm{ibBun}_G^{\mathrm{diff,ep,tot}}) \to \mathbb{D}\mathrm{Bun}$$
$$\texttt{IsDifferential}^{\mathrm{lin}} : \mathrm{Tot}(\mathbb{B}\mathrm{un}_G^{\mathrm{lin,ep,tot}}) \to \mathbb{D}\mathrm{Bun}^{\mathrm{lin}},$$

which send a G-bundle $E \xrightarrow{q} M$ to the differential bundle $E \xrightarrow{q} M$. The functor `IsDifferential` sends a total morphism (Φ, ϕ) of fibre bundles to the morphism (Φ, ϕ) of differential bundles. The functor `IsDifferential`$^{\mathrm{lin}}$ sends as total morphism $(\phi, (T_{ik})_{i\in I, k\in J}, \psi, \Phi)$ of G-bundles to the linear morphism (Φ, ϕ) of differential bundles.

Theorem 6.3.10. *Let $(\mathbb{X}, T, p, 0, +, \ell, c)$ be a Cartesian tangent join restriction category. The assignments*

$$\texttt{IsDifferential} : \mathrm{Tot}(\mathbb{F}\mathrm{ibBun}_G^{\mathrm{diff,ep,tot}}) \to \mathbb{D}\mathrm{Bun}$$
$$\texttt{IsDifferential}^{\mathrm{lin}} : \mathrm{Tot}(\mathbb{B}\mathrm{un}_G^{\mathrm{lin,ep,tot}}) \to \mathbb{D}\mathrm{Bun}^{\mathrm{lin}},$$

described above, are functors. The functor `IsDifferential` $: \mathrm{Tot}(\mathbb{F}\mathrm{ibBun}_G^{\mathrm{diff}}) \to \mathbb{D}\mathrm{Bun}$ *is fully faithful.*

Proof. The assignments are well-defined due to Propositions 6.3.7 and 6.3.8. Since they send morphisms of fibre bundles to the morphism of differential bundles given by the same data, functoriality is immediate. For the same reason `IsDifferential` is faithful.

The functor `IsDifferential` is full because in Proposition 6.3.8, a pair (Φ, ϕ) is a differential bundle morphism if and only if it is a fibre bundle morphism. □

This shows that $\mathrm{Tot}(\mathbb{F}\mathrm{ibBun}_G^{\mathrm{diff}})$ is a full subcategory of $\mathbb{D}\mathrm{Bun}$. In Theorem 5.4.24, we proved the equivalence

$$\texttt{Build}^{\mathrm{tot}_G} : \mathbb{P}\mathrm{Bun}_G^{\mathrm{tot}} \times G - \mathbb{O}\mathrm{bj} \xrightarrow{\cong} \mathbb{B}\mathrm{un}_G^{\mathrm{tot}_G}$$

We will now restrict this equivalence to the subcategory of linear G-objects and the subcategory of epic and totally fibred principal bundles. As shown in Theorem 5.4.27, the totally fibred epic bundles in $\mathbb{P}\mathrm{arMfld}$ are exactly the bundles that are considered principal bundles in classical differential geometry. Classically, the linearity condition for $G-\mathbb{O}\mathrm{bj}^{\mathrm{lin}}$ means that its objects are vector spaces and its morphisms are linear maps. The linearity condition for $\mathbb{B}\mathrm{un}_G^{\mathrm{lin,ep,tot}}$ ensures that each object in $\mathbb{B}\mathrm{un}_G^{\mathrm{lin,ep,tot}}$ is a vector bundle.

Lemma 6.3.11. *Let $\mathbb{X}$ be a Cartesian tangent join restriction category with gluings and G be a group object* *in $\mathbb{X}$. Let $(P, M, q, (\alpha_i)_{i\in I}, (\tau_{ij})_{i,j\in I})$ be a totally fibred principal G-bundle in $\mathbb{X}$. Let (F, a) be a G-object in $\mathbb{X}$.*

(a) The G-bundle $\mathtt{Build}^{\mathrm{tot}_G}((P, M, q, (\alpha_i)_{i\in I}, (\tau_{ij})_{i,j\in I}), (F, a))$ is epic if and only if the principal G-bundle $(P, M, q, (\alpha_i)_{i\in I}, (\tau_{ij})_{i,j\in I})$ is epic.

(b) There is an equivalence of categories

$$\mathtt{Build}^{\mathrm{lin,ep,tot}} : \mathbb{P}\mathrm{Bun}_G^{\mathrm{ep,tot}} \times G - \mathbb{O}\mathrm{bj}^{\mathrm{lin}} \xrightarrow{\cong} \mathbb{B}\mathrm{un}_G^{\mathrm{lin,ep,tot}}.$$

(c) There is an equivalence of categories

$$\mathrm{Tot}(\mathtt{Build}^{\mathrm{lin,ep,tot}}) : \mathrm{Tot}(\mathbb{P}\mathrm{Bun}_G^{\mathrm{ep,tot}}) \times \mathrm{Tot}(G - \mathbb{O}\mathrm{bj}^{\mathrm{lin}}) \xrightarrow{\cong} \mathrm{Tot}(\mathbb{B}\mathrm{un}_G^{\mathrm{lin,ep,tot}}).$$

Proof. (a) Recall from Definition 5.4.23(iii), that the total space E of the G-bundle

$$\mathtt{Build}^{\mathrm{tot}_G}((P, M, q, (\alpha_i)_{i\in I}, (\tau_{ij})_{i,j\in I}), (F, a))$$

is constructed as the gluing of

$$u_{ij} := ((1_M, \tau_{ij}) \times 1_F)(1_M \times a) : M \times F \to M \times F.$$

The local trivializations $\alpha'_i : E \to M \times F$ are the partial inverses of the partial isomorphisms into the gluing, as characterized in Remark 5.2.13. Since it follows from Definition 5.4.14, that $\bar{\tau}_{ii} = e_i$, the equation

$$\bar{\alpha}_i'^{*} = \alpha_i'^{*}\alpha'_i = \bar{u}_{ii} = \overline{\pi_0\tau_{ii}} = \overline{\pi_0 e_i},$$

holds. Since $\bar{\alpha}_i'^{*}$ is of the form $1 \times e'_i$ (Definition 5.4.1), the restriction idempotent $e'_i : M \to M$ of the G-bundle $\mathtt{Build}((P, M, q, (\alpha_i)_{i\in I}, (\tau_{ij})_{i,j\in I}), (F, a))$ equals the restriction idempotent e_i of the input principal bundle$(P, M, q, (\alpha_i)_{i\in I}, (\tau_{ij})_{i,j\in I})$.

Now we can apply Corollary 6.1.4 to show that q' is epic if and only if $\bigvee_{i\in I} e'_i = 1$ which happens if and only if $\bigvee_{i\in I} e_i = 1$. In turn, this happens if and only if q is epic.

(b) We will prove that the object / morphism the functor

$$\mathtt{Build}^{\mathrm{tot}_G} : \mathbb{P}\mathrm{Bun}_G^{\mathrm{tot}} \times G - \mathbb{O}\mathrm{bj} \xrightarrow{\cong} \mathbb{B}\mathrm{un}_G^{\mathrm{tot}_G}$$

produces lies in $\mathbb{B}\mathrm{un}_G^{\mathrm{lin,ep,tot}}$ if and only if $\texttt{Build}^{\mathrm{tot}_G}$ is evaluated on an object / morphism of $\mathbb{P}\mathrm{Bun}_G^{\mathrm{ep,tot}} \times G - \mathbb{O}\mathrm{bj}^{\mathrm{lin}}$. We will first focus on objects and then on morphisms.

Let $((P, M, q, (\alpha_i)_{i\in I}, (\tau_{ij})_{i,j\in I}), (F, a))$ be an object of $\mathbb{P}\mathrm{Bun}_G^{\mathrm{tot}} \times G - \mathbb{O}\mathrm{bj}$. Denote the G-bundle from Definition 5.4.23,

$$\texttt{Build}^{\mathrm{tot}_G}((P, M, q, (\alpha_i)_{i\in I}, (\tau_{ij})_{i,j\in I}), (F, a)) = (E, M, F, q', (\alpha'_i)_{i\in I}).$$

Inspecting Definition 5.4.23, we see that the fibre of $\texttt{Build}^{\mathrm{tot}_G}((P, M, q, (\alpha_i)_{i\in I}, (\tau_{ij})_{i,j\in I}), (F, a))$ is exactly the G-object (F, a) which the functor $\texttt{Build}^{\mathrm{tot}_G}$ was applied to.

Thus the fibre F of the G-bundle $\texttt{Build}^{\mathrm{tot}_G}((P, M, q, (\alpha_i)_{i\in I}, (\tau_{ij})_{i,j\in I}), (F, a))$ is a differential object with a linear G-action a if and only if (F, a) was a differential object with linear G-action to begin with, i.e. an object of $G - \mathbb{O}\mathrm{bj}^{\mathrm{lin}}$.

Together with part (a) this shows that the object $\texttt{Build}^{\mathrm{tot}_G}((P, M, q, (\alpha_i)_{i\in I}, (\tau_{ij})_{i,j\in I}), (F, a)$ is in $\mathbb{B}\mathrm{un}_G^{\mathrm{lin,ep,tot}}$ if and only if $(P, M, q, (\alpha_i)_{i\in I}, (\tau_{ij})_{i,j\in I})$ is an object of $\mathbb{P}\mathrm{Bun}_G^{\mathrm{ep,tot}}$ and (F, a) is an object of $G - \mathbb{O}\mathrm{bj}^{\mathrm{lin}}$.

Let

$$((\phi, \Phi, T_{ik}), \psi) : ((P, M, q, (\alpha_i)_{i\in I}, (\tau_{ij})_{i,j\in I}), (F, a)) \to ((P', M', q', (\alpha'_k)_{k\in J}, (\tau_{kl})_{k,l\in J}), (F', a'))$$

be a morphism of $\mathbb{P}\mathrm{Bun}_G^{\mathrm{tot}} \times G - \mathbb{O}\mathrm{bj}$ where (F, a) and (F', a') are objects of $G - \mathbb{O}\mathrm{bj}^{\mathrm{lin}}$. Recall that, as explained in Definition 5.4.23,

$$\texttt{Build}^{\mathrm{tot}_G}((\phi, T_{ik}), \psi) = (\phi, \psi, T_{ik}, \Phi),$$

where Φ is constructed from T_{ik}. The ψ component is exactly the same as the ψ which the functor $\texttt{Build}^{\mathrm{tot}_G}$ was applied to. Thus $\texttt{Build}^{\mathrm{tot}_G}((\phi, T_{ik}), \psi)$ is linear if and only if ψ was linear to start with, i.e. $((\phi, T_{ik}), \psi)$ was a morphism of $\mathbb{P}\mathrm{Bun}_G^{\mathrm{ep,tot}} \times G - \mathbb{O}\mathrm{bj}^{\mathrm{lin}}$.

(c) The subcategory of total morphisms contains all isomorphisms. Thus it suffices to show that the functors $\texttt{Build}^{\mathrm{tot}_G}$ and $\langle\texttt{Principal}^{\mathrm{tot}_G}, \texttt{Fibre}\rangle$ produce total morphisms when evaluated on total morphisms. Let $((\phi, T_{ik}), \psi)$ be a total morphism in $\mathbb{P}\mathrm{Bun}_G^{\mathrm{ep,tot}} \times G - \mathbb{O}\mathrm{bj}^{\mathrm{lin}}$. Due to Proposition 5.4.18, this means that $\bar{\phi} = 1_M$ and $\bar{\psi} = 1_F$. So in order to show that $\texttt{Build}^{\mathrm{tot}_G}((\phi, T_{ik}), \psi) = (\phi, \psi, T_{ik}, \Phi)$ is total, it only remains to show that Φ is total. We use the commutative diagram in Definition 5.4.19

to calculate

$$\begin{aligned}
\bar{\Phi} &\geq \overline{\bigvee_{i,k} \alpha_i \langle \pi_0\phi, (T_{ik} \times \psi)a' \rangle \alpha_k^*} \\
&= \overline{\bigvee_{i,k} \alpha_i \langle \pi_0\phi, (T_{ik} \times \psi)a' \rangle (e_k \times 1_F)} \\
&= \overline{\bigvee_{i,k} \alpha_i \langle \pi_0\phi e_k, (T_{ik} \times \psi)a' \rangle} \\
&= \overline{\bigvee_{i,k} \alpha_i \overline{\pi_0\phi e_k}\, \overline{\pi_0 T_{ik}}} \\
&= \overline{\bigvee_{i,k} \alpha_i \overline{\pi_0\phi e_k}\, \overline{\pi_0 e_i \phi e_k}} && \text{since } \bar{T}_{ik} = \overline{e_i\phi e_k} \\
&= \overline{\bigvee_{i,k} \alpha_i \pi_0 e_i \phi e_k} \\
&= \overline{\bigvee_{i,k} \bar{\alpha}_i q e_i \phi e_k} \\
&= \overline{\bigvee_{i,k} \overline{qe_i} q e_i \phi e_k} && \text{totally fibred} \\
&= \overline{q \bigvee_i e_i \phi \bigvee_k e_k} \\
&= \overline{q\phi} = 1_E && \text{since the bundles are epic.}
\end{aligned}$$

Conversely, let $(\phi, \psi, T_{ik}, \Phi)$ be a total morphism in $\mathbb{B}\mathrm{un}_G^{\mathrm{lin,ep,tot}}$. Then ϕ, ψ and Φ are total and therefore $\langle \texttt{Principal}^{\mathrm{tot}_G}, \texttt{Fibre} \rangle(\phi, \psi, T_{ik}, \Phi) = ((\phi, T_{ik}), \psi)$ is total.

□

Composing the component $\mathrm{Tot}(\texttt{Build}^{\mathrm{lin,ep,tot}}) : \mathrm{Tot}(\mathbb{P}\mathrm{Bun}_G^{\mathrm{ep,tot}}) \times \mathrm{Tot}(G - \mathbb{O}\mathrm{bj}^{\mathrm{lin}}) \to \mathrm{Tot}(\mathbb{B}\mathrm{un}_G^{\mathrm{lin,ep,tot}})$ of this equivalence with the functor $\texttt{IsDifferential}^{\mathrm{lin}} : \mathbb{B}\mathrm{un}_G^{\mathrm{lin,ep,tot}} \to \mathbb{D}\mathrm{Bun}^{\mathrm{lin}}$ from Theorem 6.3.10, we obtain a functor of bundle categories.

Theorem 6.3.12. *There is a functor*

$$\mathrm{Tot}(\texttt{Build}^{\mathrm{lin,ep,tot}}); \texttt{IsDifferential}^{\mathrm{lin}} : \mathbb{P}\mathrm{Bun}_G^{\mathrm{ep,tot}} \times G - \mathbb{O}\mathrm{bj}^{\mathrm{lin}} \to \mathbb{D}\mathrm{Bun}^{\mathrm{lin}}.$$

This shows how we can construct a differential bundle out of a principal bundle.

In the join restriction category of smooth manifolds and partial maps, we can choose the group object G to be the Lie group GL_n and the differential object with a linear G-action to be $\mathbb{R}^n$ with the usual action of GL_n on $\mathbb{R}^n$. Classically, every principal bundle can be constructed this way. However, in a general restriction category, not every differential bundle is in the image of $\mathrm{Tot}(\texttt{Build}^{\mathrm{lin,ep,tot}});\texttt{IsDifferential}^{\mathrm{lin}}$, as we show in the example below.

Example 6.3.13. *Let* $\mathbb{T}\mathrm{otMfld}$ *be the restriction category of smooth manifolds and smooth total maps where the restriction idempotents are all identity, i.e. for* $f : A \to B$, $\bar{f} = 1_A$. *The restriction category* $\mathbb{T}\mathrm{otMfld}$ *is a join restriction category, since* $f \smile g$ *is equivalent to* $f = g$ *and thus for* $\bigvee_{i\in I} f_i$ *to be defined, all* f_i *are the same and thus equal to their join. The join restriction category* $\mathbb{T}\mathrm{otMfld}$ *has all gluings, since any partial isomorphism* $u_{ij} : U_i \to U_j$ *is a total isomorphism and therefore any atlas is isomorphic to a trivial one.*

The standard Cartesian tangent structure on smooth manifolds makes $\mathbb{T}\mathrm{otMfld}$ *into a Cartesian tangent join restriction category with gluings.*

Let now $E \xrightarrow{q} M$ *be any vector bundle of smooth manifolds that is not globally trivial* $E \ncong M \times F$ *(for example the tangent bundle of the sphere* S^2*, see the "hairy ball theorem" in* [71, Section III.11.10]*). Since all partial isomorphisms in* $\mathbb{T}\mathrm{otMfld}$ *are total isomorphisms, there can be no partial isomoprhism* $\alpha_i : E \to M \times F$*, thus it can not be a* G*-bundle and in particular not be in the image of* $\texttt{Build}^{\mathrm{lin,ep,tot}};\texttt{IsDifferential}^{\mathrm{lin}}$.

Thus $\mathbb{T}\mathrm{otMfld}$ *is an example of a Cartesian tangent join restriction category with gluings where not all differential bundles come from a principal bundle.*

The functor of Theorem 6.3.12 is only defined on the category of total morphisms. The main reason is that morphisms of differential bundles were defined to be total in Definition 6.3.2. A possible generalization in the future could be to define partial morphisms of restriction differential bundles and construct a functor similar to the one in Theorem 6.3.12.

Chapter 7

Conclusions and future work

This thesis focused on further developing our understanding of tangent bundles, differential bundles and principal bundles in category-theoretical contexts.

7.1 Summary

We developed a categorical notion of dimension that generalizes the dimension of vector spaces and smooth manifolds and which givens an interpretation for the universality of the vertical lift which tangent bundles and differential bundles need to fulfill. We characterized differential bundles as functors, thereby connecting the axiomatic definition of differential bundles in [29] with the functorial characterization of tangent categories in [64] and [44]. Furthermore, we showcased that group objects have many desirable properties in tangent categories, in particular their tangent bundle is a product of the group object with a tangent space. We defined fibre bundles, G-bundles and principal G-bundles and used the properties of group objects to show that these bundles are well-behaved.

For the categorical dimension, tangent structures and differential bundles need to fulfill the dimension equations of Theorems 3.2.7 and 3.2.8. Therefore we can use dimensions to show that an endofunctor is not a tangent structure, which can lead to results classifying all tangent structures on an underlying category. In addition, one can use Theorems 3.2.13 and 3.2.14 to show that something is not a differential bundle.

Since we characterized differential bundles as functors, this provides a new way to show that something is differential bundle. We envision this can be particularly useful in order to combine differential bundles with other categorical constructions like Kleisli categories and opposites. In addition the functorial perspective is easier to generalize to higher categories, for example the tangent infinity categories of [6] and some bicategory related to [7].

The properties of group objects can be used for many purposes, since group objects frequently appear. One example of such a usage is the definition and properties of principal bundles. Principal bundles now allow us to unify approaches from different perspectives on geometry. Additionally, the principal bundles in restriction categories can be used to encode symmetries in categories that are not traditionally considered parts of geometry.

In Chapter 6, we showed a relation between these concepts, generalizing a well-established equivalence from differential geometry. We now know how to construct a differential bundle from a principal bundle and that this construction is a functor, though in general not an equivalence.

7.2 Future work

Due to the structure of this thesis, many approaches to continue this work have been already proposed in Sections 3.8, 4.6 and 5.5. Each of them discusses the future developments following each of the papers.

In addition, we would like to highlight that the collection of these projects will enables us to generalize methods from classical physical systems based on smooth manifolds to other systems that are not based on smooth manifolds.

Two major characteristics of physical systems are their dynamics and the symmetries the system admits. Classically, the dynamics is governed by vector fields, i.e. sections of vector bundles. Classical dynamics often is described by vector fields on the phase space, the space consisting of positions and momenta. These vector fields determine the differential equations governing classical dynamics. Symmetries are often described by group actions on the phase space. Gauge symmetries (redundancies in the descriptions of the physical system) are encoded by principal bundles. Group actions and principal bundles have been central themes in this thesis.

We can apply our bundle constructions in order to describe symmetries and dynamics of systems that do not necessarily admit a description in classical geometry, like networks. Networks are particularly relevant, since many systems that are important for real-life applications are given by networks, for example power grids and neural networks. Category theory has already been used to describe the dynamics of many important real-life systems [5],[86] and combining these approaches with categorical geometry promises to be a useful extension.

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Appendix A

When the pushout is not the double mapping cylinder

The purpose of this appendix is to prove the statements made in Example 3.7.7. Concretely, this appendix is a proof of Proposition A.0.2. The goal of this is to show that, while they form a dimension on the category of CW complexes and cellular maps, our proof does not work for the Betti numbers as a dimension on the category of Hausdorff spaces and continuous maps.

First we will recall some definitions used in the statements.

Definition A.0.1. *Let X, Y be topological spaces and $f, f' : X \to Y$ be continuous maps. Let $I = [0,1] \subset \mathbb{R}$ denote the closed unit interval.*

*(a) The maps f and f' are **homotopic**, if there is a continous map $h : I \times X \to Y$ such that $h(0,x) = f(x)$ and $h(1,x) = f'(x)$ for all $x \in X$. Such a map h is called a **homotopy**.*

*(b) A **homotopy equivalence** between X and Y consists of continuous maps $g : X \to Y$ and $g' : Y \to X$ and homotopies h between $g \circ g'$ and the identity 1_Y and h' between $g' \circ g$ and the identity 1_X.*

*(c) Two points $x, y \in X$ are **path-connected** if there is a path between them, i.e. a continuous map $\gamma : I \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$. A **path-component** of X is a maximal path-connected subset.*

(d) Given a span $X \xleftarrow{f} Y \xrightarrow{g} Z$ of topological spaces, the double mapping cylinder ${}_XM_Z$ is defined as

$$ {}_XM_Z = (X \sqcup (Y \times I) \sqcup Z)/\sim, \text{ where } (y,0) \sim f(y) \text{ and } (y,1) \sim g(y), \ \forall y \in Y. $$

The following proposition is the central result of this appendix.

Proposition A.0.2. *For the Hausdorff spaces*

$$\begin{aligned}
A' =& \left\{ x \in \mathbb{R} | x = \frac{1}{n} \text{ for some } n \in \mathbb{N}_{>0} \right\}, \\
B' =& \left\{ (x,y,z) \in \mathbb{R}^3 | (x,y,z) = \left(\frac{1}{n}, \frac{1}{n}\sin(\varphi), \frac{1}{n}\cos(\varphi) \right) \text{ for some } n \in \mathbb{N}_{>0} \ , \ \varphi \in [0, 2\pi] \right\}, \\
A =& \{0\} \cup A', \text{ and} \\
B =& \{(0,0,0)\} \cup B'
\end{aligned}$$

and the pushout diagram

$$\begin{tikzcd}
* \arrow[r, "i_0"] \arrow[d, "i_{(0,0,0)}"'] & A \arrow[d] \\
B \arrow[r] & P \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

there is no homotopy equivalence between P and the double mapping cylinder ${}_AM_B$

Before we prove this result, we will state the intuition that homotopy-equivalences preserve path components.

Lemma A.0.3. *Let A, B be topological spaces and let there be a homotopy equivalence between A and B consisting of $f : A \to B$, $g : B \to A$, $h : A \times I \to A$ and $k : B \times I \to B$.*

(i) If $a, a' \in A$ can be connected by a path $I \to A$, $f(a)$ and $f(a')$ can be connected by a path $I \to B$.

(ii) If a, a' can not be connected by a path $I \to A$, $f(a)$ and $f(a')$ can not be connected by a path $I \to B$.

(iii) Let $X \subset A$ be a path component of A and $Y \subset B$ a path component of B and let for some $x \in X$ the image $f(x)$ be in Y. Then the restrictions $(f|_X, g|_Y, h|_{X\times I}, k|_{Y\times I})$ form a homotopy equivalence between X and Y.

Proof. Part (i) is immediate using the the post-composition of the path with f.

For part (ii), suppose that $f(a)$ and $f(a')$ are connected by a path $\gamma : I \to B$. Then $g \circ \gamma : I \to A$ is a path in A connecting $g(f(a))$ with $g(f(a'))$. The homotopy h between $g \circ f$ and 1_A now provides paths $h(a, -) : I \to A$ and $h(a', -)$ connecting a with $g(f(a))$ and a' with $g(f(a'))$. Composing these paths gives a path from a to a' proving part (ii).

For part (iii), we need to show that the image of the restrictions $f|_X$ and $g|_Y$ is within Y and X respectively.

Given a x' within the same path component X as x, part (i) shows that $f(x')$ will be within the same path component as $f(x)$ which is Y. For any $t \in I$, $h(x', t)$ has a path connecting it to $h(x', 1) = x'$, so it is within the same path component X as x'.

Since $h(x, -) : I \to A$ is a path between $g(f(x))$ and x, the element $f(x) \in Y$ is sent to the connected component of x. Thus we can use the same argument as in the previous paragraph to obtain that the images of $g|_Y$ and $k|_{Y \times I}$ land in X and Y respectively.

Now since we showed that the restrictions have the types $f|_X : X \to Y$, $g|_Y : Y \to X$, $h|_{X \times I} : X \times I \to X$ and $k|_{Y \times I} : Y \times I \to Y$, they form a homotopy equivalence between X and Y, since (f, g, h, k) formed a homotopy equivalence between A and B. $\square$

Equipped with this Lemma we now can prove Proposition A.0.2.

Proof of Proposition A.0.2. We use the explicit descriptions

$$P = \left\{(-\frac{1}{n}, 0, 0) | n \in \mathbb{N}_{>0}\right\} \cup \{(0,0,0)\} \cup \left\{(\frac{1}{n}, \frac{1}{n}\sin(\varphi), \frac{1}{n}\cos(\varphi)) | n \in \mathbb{N}_{>0}, \varphi \in [0, 2\pi]\right\} \subset \mathbb{R}^3 \text{ and}$$

$${}_AM_B \cong \left\{(-\frac{1}{n}, 0, 0) | n \in \mathbb{N}_{>0}\right\} \cup [0,1] \times \{(0,0)\} \cup \left\{(1 + \frac{1}{n}, \frac{1}{n}\sin(\varphi), \frac{1}{n}\cos(\varphi)) | n \in \mathbb{N}_{>0}, \varphi \in [0, 2\pi]\right\} \subset \mathbb{R}^3.$$

These explicit descriptions are visualized in Figure 3.3. This proof proceeds by contradiction. Suppose there is a homotopy equivalence (f, g, h, k) between P and ${}_AM_B$. Then for $\vec{0} = (0,0,0)$ we consider the point $p_0 = f(\vec{0}) \in {}_AM_B$ and a small open neighbourhood U_0 around it. Since f is continuous $f^{-1}(U_0)$ is an open neighbourhood around $\vec{0}$. Thus for any open neighbourhood U_0 of p_0, $f^{-1}(U_0)$ contains infinitely many points of the form $(-1/n, 0, 0)$ and infinitely many circles of the form $\{(1/n, 1/n\sin(\varphi), 1/n\cos(\varphi)) | \varphi \in [0, 2\pi]\}$. We will now distinguish different cases where p_0 could be, showing that each of them leads to a contradiction

- If $p_0 = (-1/n, 0, 0) \in {}_AM_B$ for some $n \in \mathbb{N}_{>0}$, U_0 can be chosen to be just the set $U_0 = \{p_0\}$ which is open in subset topology as an open ball of radius $1/n - 1/(n+1)$. Since $f^{-1}(U_0)$ contains infinity many path components, f sends infinity many path components to one path component. By Lemma A.0.3(ii) this contradicts the assumption that (f, g, h, k) was a homotopy-equivalence.

- If $p_0 = (1 + 1/n, 1/n\sin(\varphi), 1/n\cos(\varphi)) \in {}_AM_B$ for some $n \in \mathbb{N}_{>0}$ and some $\varphi \in [0, 2\pi]$, U_0 can be chosen to be the open set $\{(1 + 1/n, 1/n\sin(\varphi), 1/n\cos(\varphi)) | \varphi \in [0, 2\pi]\}$ (open in subspace topology). Since $f^{-1}(U_0)$ contains infinity many path components, f sends infinity many path components to one path component. By Lemma A.0.3(ii) this contradicts the assumption that (f, g, h, k) was a homotopy-equivalence.

- If $p_0 = (x, 0, 0) \in {}_AM_B$ for some $x \in (0, 1)$, U_0 can be chosen to be $(0, 1) \times \{(0, 0)\}$ (open in subspace topology). Since $f^{-1}(U_0)$ contains infinity many path components, f sends infinity many path components to one path component. By Lemma A.0.3(ii) this contradicts the assumption that (f, g, h, k) was a homotopy-equivalence.

- If $p_0 = (0, 0, 0) \in {}_AM_B$, U_0 can be chosen to have a first coordinate below $1/2$. Since $f^{-1}(U_0)$ contains a circle of the form $\{(1/n, 1/n \sin(\varphi), 1/n \cos(\varphi)) | \varphi \in [0, 2\pi]\}$ and these circles are in a different path component than $\vec{0}$, Lemma A.0.3(ii) implies that this circle is sent to a point of the form $(-1/m, 0, 0)$. Due to Lemma A.0.3(i) and the fact that all these point are path-disconnected, the entire circle is sent to one point. By Lemma A.0.3(iii) this implies that (f, g, h, k) restricts to a homotopy equivalence between a circle and a point. It is a classical result that such a homotopy equivalence does not exist (proven for example by observing that their fundamental groups differ) leading to the required contradiction.

- If $p_0 = (1, 0, 0) \in {}_AM_B$, U_0 can be chosen to have a first coordinate above $1/2$. Since $f^{-1}(U_0)$ contains a isolated point of the form $(-1/n, 0, 0)$, Lemma A.0.3(ii) implies that this point is sent to one of the circles of the form $\{(1 + 1/m, 1/m \sin(\varphi), 1/m \cos(\varphi)) | \varphi \in [0, 2\pi]\}$. By Lemma A.0.3(iii) this implies that (f, g, h, k) restricts to a homotopy equivalence between a circle and a point. Again, this leads to the required contradiction.

This shows that our proof strategy of Proposition 3.7.4 does not work in the more general setup of all Hausdorff spaces. It still remains open, if the generalized version for all Hausdorff spaces holds true, though we doubt it. $\square$

Appendix B

Naturality diagrams for $\texttt{Ind}$(E)

In this appendix we will check that the functor $\texttt{Ind}(E) : \mathbb{N}^\bullet$ to $\mathbb{X}$ defined in Proposition 4.4.9 with components $(F_E, \hat{\alpha})$, the $\hat{\alpha} : F_E \circ D_\bullet \to T_\bullet \circ (1 \times F_E)$ is natural. The functors $F_E \circ D_\bullet$ and $T_\bullet \circ (1 \times F_E)$ both go from $\mathbb{W}\mathrm{eil} \times \mathbb{N}^\bullet$ to $\mathbb{X}$.

The following Lemma is a summary of some of Leung's observations about generators of $\mathbb{W}\mathrm{eil}$ in [64].

Lemma B.0.1. [64, Proposition 9.5] *In the category* $\mathbb{W}\mathrm{eil}$ *every map is inductively generated by* p, 0, $+$, ℓ, c *and the projections* π_i *using compositions, coproducts and foundational pullbacks, i.e. pullbacks of the form*

$$\begin{tikzcd} A \otimes (B \times C) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \otimes B \arrow[d] \\ A \otimes C \arrow[r] & A. \end{tikzcd}$$

Recall from Definition 4.4.8 that the transformation $\hat{\alpha} : F_E \circ D_\bullet \to T_\bullet \circ F_E$ is defined inductively from the $W, \mathbb{N}^1$ component:

$$\hat{\alpha}_{W,\mathbb{N}^1} := T(\sigma) \circ \langle 0 \circ \pi_0, \lambda \circ \pi_1 \rangle : E_2 \to TE \qquad \hat{\alpha}_{W,\mathbb{N}^k} := (\hat{\alpha}_{W,\mathbb{N}})_k : \left(\frac{E_2}{M}\right)^k \to \left(\frac{T(E)}{T(M)}\right)^k = T(E_k).$$

This is the map in the diagram for the universality of the vertical lift in Definition 2.3.1. In particular, for $k = 0$, we define $\hat{\alpha}_{W,\mathbb{N}^0} = 0_M : M \to T(M)$. For a product W^n, we define

$$\hat{\alpha}_{W^n,\mathbb{N}^1} := (\hat{\alpha}_{W,\mathbb{N}^1})_n : (E_2)_n \to T_n(E) \qquad \hat{\alpha}_{W^n,\mathbb{N}^k} := (\hat{\alpha}_{W^n,\mathbb{N}^1})_k : (E_{2n})_k \to (T_n(E))_k$$

In particular for the case $n = 0$, we define $\hat{\alpha}_{\mathbb{N},\mathbb{N}^k} := 1_{E_k}$. Now, for coproducts $A \otimes B \in \mathrm{Obj}(\mathbb{W}\mathrm{eil})$, we define $\hat{\alpha}_{A \otimes B,-}$ using the commuting diagrams in Definition 2.2.14. Concretely, for $A, B \in \mathrm{Obj}(\mathbb{W}\mathrm{eil})$ and

$X \in \mathrm{Obj}(\mathbb{N}^\bullet)$, define

$$\hat{\alpha}_{A\otimes B,X} := T_A\hat{\alpha}_{B,X} \circ \hat{\alpha}_{A,T_B X} : F \circ T_A \circ T_B(X) \to T_A \circ T_B \circ F(X)$$

and

$$\alpha_{\mathbb{N},X} = 1_{F_E(X)} : F_E(D_{\mathbb{N}}X) = F_E(X) \to F_E(X) = T_{\mathbb{N}}(F_E(X)).$$

Proposition B.0.2. *This transformation $\hat{\alpha} : F_E \circ D_\bullet \to T_\bullet \circ (1 \times F_E)$ is a natural transformation between functors* $\mathbb{W}\mathrm{eil} \times \mathbb{N}^\bullet \to \mathbb{X}$

Recall from Theorem 4.2.11 that for any tangent actegory $(\mathbb{X}, T_\bullet)$, the morphisms of $\mathbb{W}\mathrm{eil}$ produce the tangent structures in $\mathbb{X}$ via the functor $T_\bullet$ and the equality $(\mathbb{X}, T, p, 0, +, \ell, c) = (\mathbb{X}, T_W, T_p, T_0, T_+, T_\ell, T_c)$. Thus the symbols $(p, 0, +, \ell, c)$ are used both for the morphisms in $\mathbb{W}\mathrm{eil}$ from Remark 4.2.9 and the morphisms in $\mathbb{X}$ that come from the tangent structure.

Analogously, F_E sends the morphisms $(\zeta : \mathbb{N}^0 \to \mathbb{N}^1, \sigma : \mathbb{N}^2 \to \mathbb{N}^1)$ of $\mathbb{N}^\bullet$ to the additive bundle structure $(\zeta : M \to E, \sigma : E_2 \to E)$ underlying the differential bundle $(E, M, q, \zeta, \sigma, \lambda)$. Thus the symbols σ and ζ are used both for the maps in $\mathbb{N}^\bullet$ and their image in $\mathbb{X}$.

Proof. We need to show that $\hat{\alpha}$ is natural in $\mathbb{N}^\bullet$ and $\mathbb{W}\mathrm{eil}$. We will accomplish this by proving that it is natural in the morphisms generating $\mathbb{N}^\bullet$ and $\mathbb{W}\mathrm{eil}$.

First we show it is natural in $\mathbb{N}^\bullet$.

Since $\alpha_{A,-}$ is built as a composition of pullbacks of $\alpha_{W,-}$ it suffices to check naturality of $\alpha_{W,-}$ in $\mathbb{N}^\bullet$. Let $f : \mathbb{N}^k \to \mathbb{N}^n$ be a morphism in $\mathbb{N}^\bullet$. As shown in Lemma 4.2.22, its components $(f_i : \mathbb{N}^k \to \mathbb{N})_{i=1,\ldots,n}$ are given by

$$f_n = \sigma_{f_n^1+\ldots+f_n^k} \circ (\Delta_{f_n^1} \times_M \ldots \times_M \Delta_{f_n^k}).$$

The diagonal is the map induced by the identity. Therefore this can be rewritten into

$$f_n = \sigma_{N_n} \circ \langle \pi_1, \ldots, \pi_1, \ldots, \pi_k, \ldots, \pi_k \rangle$$

where $N_n = f_n^1 + \ldots + f_n^k$ and the projection π_i appears f_n^i times. For the case $f_n^i = 0$, the zero-fold sum is the zero map $\sigma_0 = \zeta : \mathbb{N}^0 \to \mathbb{N}^1$ and the zero-fold diagonal is the unique map to the terminal object $\Delta_0 =! : \mathbb{N}^1 \to \mathbb{N}^0$.

The k-fold summation σ_k can be written as a composition of pullbacks over the terminal object of the binary summation $\sigma : \mathbb{N}^2 \to \mathbb{N}$ and $T_\bullet \circ F_E$ preserve these pullbacks. As we saw above, f can be expressed in terms of σ_k and projections. Thus we only need to show that α is natural with respect to the addition

$\sigma : \mathbb{N}^2 \to \mathbb{N}^1$, the zero $\zeta : \mathbb{N}^0 \to \mathbb{N}^1$ (zero-fold addition), the pullback-projections $\pi_i : \mathbb{N}^k \to \mathbb{N}^1$ and the unique map $! : \mathbb{N}^1 \to \mathbb{N}^0$ into the terminal object (zero-fold diagonal). In what follows, we check naturality for each of them in turn. Here we extensively use that the morphisms

$$\sigma : E_2 \to E \ , \ \zeta : M \to E \ , \ \pi_i : E_k \to E \ , \ q : E \to M$$

are given by the functor $F_E : \mathbb{N}^\bullet \to \mathbb{X}$ evaluated on the corresponding morphisms

$$\sigma : \mathbb{N}^2 \to \mathbb{N}^1 \ , \ \zeta : \mathbb{N}^0 \to \mathbb{N}^1 \ , \ \pi_i : \mathbb{N}^k \to \mathbb{N}^1 \ , \ ! : \mathbb{N}^1 \to \mathbb{N}^0$$

of $\mathbb{N}^\bullet$:

$$\sigma = F_E(\sigma) \ , \ \zeta = F_E(\zeta) \ , \ \pi_i = F_E(\pi_i) \ , \ q = F_E(!).$$

Additionally we use the correspondence between the tangent bundle endofunctors $T : \mathbb{X} \to \mathbb{X}$ and $D : \mathbb{N}^\bullet \to \mathbb{N}^\bullet$ with the Weil-actions $T_\bullet$ and $D_\bullet$ by $T = T_W$ and $D = D_W$.

Naturality for the addition $\sigma : \mathbb{N}^2 \to \mathbb{N}$ is given by the following square:

$$\begin{array}{ccc}
E_4 = F_E \circ D_\bullet(W, \mathbb{N}^2) & \xrightarrow{F_E \circ D_W(\sigma)} & F_E \circ D_\bullet(W, \mathbb{N}) = E_2 \\
{\scriptstyle \hat{\alpha}_{W,\mathbb{N}^2}} \downarrow & & \downarrow {\scriptstyle \hat{\alpha}_{W,\mathbb{N}}} \\
T(E_2) = T_\bullet(W, F_E(\mathbb{N}^2)) & \xrightarrow[T_W(F_E(\sigma))]{} & T_\bullet(W, F_E(\mathbb{N})) = T(E)
\end{array}$$

Since $\hat{\alpha}_{W,\mathbb{N}^2} = \left(\frac{\alpha_{W,\mathbb{N}}}{\alpha_{W,\mathbb{N}^0}}\right)$, the composition along the left path through the diagram is

$$T(\sigma) \circ (T(\sigma) \circ \langle 0 \circ \pi_0, \lambda \circ \pi_1 \rangle \times_{TM} T(\sigma) \circ \langle 0 \circ \pi_0, \lambda \circ \pi_1 \rangle).$$

Since $F \circ D_W(\sigma)$ is given by the pullback $\sigma \times_E \sigma = \langle \sigma \circ \langle \pi_0, \pi_2 \rangle, \sigma \circ \langle \pi_1, \pi_2 \rangle \rangle$, the right path through the square amounts to

$$T(\sigma) \circ \langle 0 \circ \pi_0, \lambda \circ \pi_1 \rangle \circ \langle \sigma \circ \langle \pi_0, \pi_2 \rangle, \sigma \circ \langle \pi_1, \pi_3 \rangle \rangle.$$

Using the naturality of 0 and that $(\lambda, 0)$ is an additive bundle morphism the right path is equal to

$$= T(\sigma) \circ \langle T(\sigma) \circ \langle 0 \circ \pi_0, 0 \circ \pi_2 \rangle, T(\sigma) \circ \langle \lambda \circ \pi_1, \lambda \circ \pi_3 \rangle \rangle.$$

By commutativity and associativity of σ this is equal to the left path.

Naturality for the zero map $\zeta : \mathbb{N}^0 \to \mathbb{N}^1$ is given by the following square:

$$\begin{array}{ccc}
M = F_E \circ D_\bullet(W, \mathbb{N}^0) & \xrightarrow{F_E \circ D_\bullet(1_W, \zeta)} & F_E \circ D_\bullet(W, \mathbb{N}^1) = E_2 \\
{\scriptstyle \hat{\alpha}_{W,\mathbb{N}^0}}\downarrow & & \downarrow{\scriptstyle \hat{\alpha}_{W,\mathbb{N}^1}} \\
T(M) = T_\bullet(W, F_E(\mathbb{N}^0)) & \xrightarrow[T_\bullet(1_W, F_E(\zeta))]{} & T_\bullet(W, F_E(\mathbb{N}^1)) = T(E)
\end{array}$$

Since $\alpha_{W,\mathbb{N}^0} = 0 : M \to T(M)$ and $F_E(\zeta) = \zeta$, the composition along the left path through the diagram amounts to $T(\zeta) \circ 0$. The composition along the right path is

$$T(\sigma) \circ \langle 0 \circ \pi_0, \lambda \circ \pi_1 \rangle \circ \langle \zeta \circ \pi_0, \zeta \circ \pi_1 \rangle.$$

Since, $0 \circ \zeta = \lambda \circ \zeta$, this equals

$$T(\sigma) \circ (0 \circ \zeta \times_{T(E)} 0 \circ \zeta)$$

and since $0 : 1_{\mathbb{X}} \to T$ is a natural transformation, this equals

$$T(\sigma) \circ (T(\zeta) \circ 0 \times_{T(E)} T(\zeta) \circ 0)$$

which equals $T(\zeta) \circ 0$ because ζ is the zero for the addition σ.

Naturality for the projections $\pi_i : \mathbb{N}^k \to \mathbb{N}$ is given by the following square:

$$\begin{array}{ccc}
E_{2k} = F_E \circ D_\bullet(W, \mathbb{N}^k) & \xrightarrow{F_E \circ D_W(\pi_i)} & F_E \circ D_\bullet(W, \mathbb{N}) = E_2 \\
{\scriptstyle \hat{\alpha}_{W,\mathbb{N}^k}}\downarrow & & \downarrow{\scriptstyle \hat{\alpha}_{W,\mathbb{N}}} \\
T(E_k) = T_\bullet(W, F_E(\mathbb{N}^k)) & \xrightarrow[T_W(F(\pi_i))]{} & T_\bullet(W, F_E(\mathbb{N})) = T(E)
\end{array}$$

The composition along the left path is $\pi_i \circ (T(\sigma) \circ \langle 0 \circ \pi_0, \lambda \circ \pi_1 \rangle)_k$, the composition along the right path is $T(\sigma) \circ (0 \times \lambda) \circ \langle \pi_{2i}, \pi_{2i+1} \rangle$ and they are equal as projecting to the sum of the i-th pair is the same as taking the sum of the pair in indices $2i$ and $2i+1$.

Naturality for the unique morphism to the terminal object, $! : \mathbb{N}^1 \to \mathbb{N}^0$ is given by the following square:

$$\begin{array}{ccc}
E_2 = F_E \circ D_\bullet(W, \mathbb{N}^1) & \xrightarrow{F_E \circ D_\bullet(1_W, !)} & F_E \circ D_\bullet(W, \mathbb{N}^0) = M \\
{\scriptstyle \hat{\alpha}_{W,\mathbb{N}^1}}\downarrow & & \downarrow{\scriptstyle \hat{\alpha}_{W,\mathbb{N}^0}} \\
T(E) = T_\bullet(W, F_E(\mathbb{N}^1)) & \xrightarrow[T_\bullet(1_W, F_E(!))]{} & T_\bullet(W, F_E(\mathbb{N}^0)) = T(M)
\end{array}$$

The composition along the left path in the diagram gives

$$T(q) \circ T(\sigma) \circ \langle 0 \circ \pi_0, \lambda \circ \pi_1 \rangle$$

which equals $q \circ \pi_0 = q \circ \pi_1$ by the commutativity of the diagram for the universality of the vertical lift. The composition along the right path in the diagram is

$$0 \circ \langle q, q \rangle$$

where $\langle q, q \rangle = q \circ \pi_0 = q \circ \pi_1$ is the morphism $E \times_M E \to M \times_M M = M$.

Since every morphism in $\mathbb{N}^\bullet$ is generated by σ, ζ, π_i and $!$, this concludes the proof that $\hat{\alpha}$ is natural in $\mathbb{N}^\bullet$.

Now we show $\hat{\alpha}$ is natural in $\mathbb{W}$eil. Since $\hat{\alpha}_{-,\mathbb{N}^k}$ is a pullback power of $\hat{\alpha}_{-,\mathbb{N}^1}$ we only need to check the naturality of $\hat{\alpha}_{-,\mathbb{N}^1}$.

By Lemma B.0.1 (Proposition 9.5 in [64]) every morphism in $\mathbb{W}$eil can be constructed from $0, p, \ell, +, c$ using composition, tensors and foundational pullbacks.

We first prove that $\hat{\alpha}$ is natural with respect to the generating morphisms $0, p, \ell, +, c$ and then prove that the naturality diagrams are preserved under tensors and foundational pullbacks.

Naturality with respect to $0 : \mathbb{N} \to \mathbb{W}$ amounts to the following square:

$$\begin{array}{ccc}
E = F_E \circ D_\bullet(\mathbb{N}, \mathbb{N}^1) & \xrightarrow{F_E \circ D_\bullet(0, 1_{\mathbb{N}^1})} & F_E \circ D_\bullet(W, \mathbb{N}^1) = E_2 \\
{\scriptstyle \hat{\alpha}_{\mathbb{N},\mathbb{N}^1}} \downarrow & & \downarrow {\scriptstyle \hat{\alpha}_{W,\mathbb{N}^1}} \\
E = T_\bullet(\mathbb{N}, F_E(\mathbb{N}^1)) & \xrightarrow[T_\bullet(0, 1_{F_E(\mathbb{N}^1)})]{} & T_\bullet(W, F_E(\mathbb{N}^1)) = T(E)
\end{array}$$

Since $\hat{\alpha}_{\mathbb{N},\mathbb{N}}$ is the identity map (the empty pullback), the left path is just the zero-map $0 : E \to T(E)$. Due to the explicit formula in Example 2.2.11, we have that $D_\bullet(0, 1_{\mathbb{N}^1}) = \langle 1, 0 \rangle : \mathbb{N}^1 \to \mathbb{N}^2$ where the zero in $\langle 1, 0 \rangle$ is the zero matrix and thus can be described by $\zeta \circ !$. Therefore, the composition along the right path of the diagram is

$$T(\sigma) \circ (0 \times_{T(M)} \lambda) \circ \langle 1, \zeta \circ q \rangle.$$

Since $(\lambda, 0)$ is an additive bundle morphism from (E, M) to $(T(E), T(M))$, this is equal to

$$T(\sigma) \circ \langle 0, T(\zeta) \circ 0 \circ q \rangle = 0$$

which is the zero-map because ζ is the neutral element with respect to the addition σ.

Naturality with respect to $p : W \to \mathbb{N}$ amounts to the following square:

$$\begin{tikzcd}
E_2 = F_E \circ D_\bullet(W, \mathbb{N}^1) \arrow[r, "F_E \circ D_\bullet(p, 1_{\mathbb{N}^1})"] \arrow[d, "\hat{\alpha}_{W,\mathbb{N}^1}"'] & F_E \circ D_\bullet(\mathbb{N}, \mathbb{N}^1) = E \arrow[d, "\hat{\alpha}_{\mathbb{N},\mathbb{N}^1} = 1_{F(\mathbb{N})}"] \\
T(E) = T_\bullet(W, F_E(\mathbb{N}^1)) \arrow[r, "T_\bullet(p, 1_{F_E(\mathbb{N}^1)})"'] & T_\bullet(\mathbb{N}, F_E(\mathbb{N}^1)) = E
\end{tikzcd}$$

It commutes since the composition of the left path is $p \circ T(\sigma) \circ \circ (0 \times \lambda)$ which by naturality of p euqals to $\sigma \circ (p \circ 0 \times p \circ \lambda)$. Since (λ, ζ) is an additive bundle morphism this is equal to $\sigma \circ (1 \times \zeta \circ q)$. This equals π_0 because ζ is zero with respect to the addition σ.

Naturality with respect to $+ : W^2 \to W$ amounts to the following square:

$$\begin{tikzcd}
E_3 = \left(\frac{E_2}{E}\right)^2 = F_E \circ D_\bullet(W^2, \mathbb{N}^1) \arrow[r, "F_E \circ D_\bullet(+, 1_{\mathbb{N}^1})"] \arrow[d, "(\hat{\alpha}_{W,\mathbb{N}^1})_2"'] & F_E \circ D_\bullet(W, \mathbb{N}^1) = E_2 \arrow[d, "\hat{\alpha}_{W,\mathbb{N}^1}"] \\
T_2(E) = T_\bullet(W^2, F_E(\mathbb{N}^1)) \arrow[r, "T_\bullet(+, 1_{F_E(\mathbb{N}^1)})"'] & T_\bullet(W, F_E(\mathbb{N}^1)) = T(E)
\end{tikzcd}$$

Because $\hat{\alpha}_{W,\mathbb{N}} = T(\sigma) \circ (0 \times_{TM} \lambda)$, the map $(\hat{\alpha}_{W,\mathbb{N}})_2 : E_3 \to T_2 E$ is given by

$$(\hat{\alpha}_{W,\mathbb{N}})_2 = \langle T(\sigma) \circ \langle 0 \circ \pi_0, \lambda \circ \pi_1 \rangle, T(\sigma) \circ \langle 0 \circ \pi_0, \lambda \circ \pi_2 \rangle \rangle.$$

The composition along the left path gives

$$+ \circ \langle T(\sigma) \circ \langle 0 \circ \pi_0, \lambda \circ \pi_1 \rangle, T(\sigma) \circ \langle 0 \circ \pi_0, \lambda \circ \pi_2 \rangle \rangle.$$

By naturality of $+$ this equals

$$\begin{aligned}
&T(\sigma) \circ +_{E_2} \circ \langle \langle 0 \circ \pi_0, \lambda \circ \pi_1 \rangle, \langle 0 \circ \pi_0, \lambda \circ \pi_2 \rangle \rangle \\
=&T(\sigma) \circ \langle + \circ \langle 0 \circ \pi_0, 0 \circ \pi_0 \rangle, + \circ \langle \lambda \circ \pi_1, \lambda \circ \pi_2 \rangle \rangle.
\end{aligned}$$

Since 0 is the zero with respect to the addition $+$ and (λ, ζ) is an additive bundle morphism, this equals

$$T(\sigma) \circ \langle 0 \circ \pi_0, \lambda \circ \sigma \circ \langle \pi_1, \pi_2 \rangle \rangle = T(\sigma) \circ (0 \times_{TM} \lambda) \circ (1_E \times_M \sigma)$$

which is the composition along the right path in the square.

Naturality with respect to $\ell : W \to W \otimes W$ amounts to the following square:

$$\begin{tikzcd}
E_2 = F_E \circ D_\bullet(W,\mathbb{N}) \arrow[r, "F_E\circ D_\bullet(\ell,1_\mathbb{N})"] \arrow[d, "\hat{\alpha}_{W,\mathbb{N}}"'] & F_E \circ D_\bullet(W\otimes W,\mathbb{N}) = E_4 \arrow[d, "T_\bullet(1_W,\hat{\alpha}_{W,\mathbb{N}})\circ\hat{\alpha}_{W,\mathbb{N}^2}"] \\
T(E) = T_\bullet(W, F_E(\mathbb{N})) \arrow[r, "T_\bullet(\ell,1_{F_E(\mathbb{N})})"'] & T_\bullet(W\otimes W, F_E(\mathbb{N})) = T^2(E)
\end{tikzcd}$$

The composition along the right path gives

$$T^2(\sigma)\circ\langle T(0)\circ\pi_0, T(\lambda)\circ\pi_1\rangle\circ(T(\sigma)\circ\langle 0\circ\pi_0,\lambda\circ\pi_1\rangle\times_{TM} T(\sigma)\circ\langle 0\circ\pi_0,\lambda\circ\pi_1\rangle)\circ\langle\pi_0,\zeta\circ q\circ\pi_0,\zeta\circ q\circ\pi_1,\pi_1\rangle$$

which equals

$$T^2(\sigma)\circ\langle T(0)\circ T(\sigma)\circ\langle 0\circ\pi_0,\lambda\circ\zeta\circ q\circ\pi_0\rangle, T(\lambda)\circ T(\sigma)\circ\langle 0\circ\zeta\circ q\circ\pi_1,\lambda\circ\pi_1\rangle\rangle.$$

In the last part of the composition we used the fact that

$$\zeta\circ q\circ\pi_1 = \zeta\circ q\circ\pi_0 : E_2\to E.$$

Due to the additive bundle morphism properties $\lambda\circ\zeta = 0\circ\zeta = T(\zeta)\circ 0$ and therefore the right path gives

$$T^2(\sigma)\circ\langle T(0)\circ T(\sigma)\circ\langle 0\circ\pi_0, T(\zeta)\circ 0\circ q\circ\pi_0\rangle, T(\lambda)\circ T(\sigma)\circ\langle T(\zeta)\circ 0\circ\pi_1,\lambda\circ\pi_1\rangle\rangle.$$

Since ζ is the zero of the addition σ this equals

$$T^2(\sigma)\circ\langle T(0)\circ 0\circ\pi_0, T(\lambda)\circ\lambda\circ\pi_1\rangle = \ell\circ T(\sigma)\circ\langle 0\circ\pi_0,\lambda\circ\pi_1\rangle$$

which is the composition along the left path.

Naturality with respect to $c : W\otimes W \to W\otimes W$ amounts to the following square:

$$\begin{tikzcd}
E_4 = F_E \circ D_\bullet(W\otimes W,\mathbb{N}) \arrow[r, "F_E\circ D_\bullet(c,1_\mathbb{N})"] \arrow[d, "T_\bullet(1_W,\hat{\alpha}_{W,\mathbb{N}})\circ\hat{\alpha}_{W,\mathbb{N}^2}"'] & F_E \circ D_\bullet(W\otimes W,\mathbb{N}) = E_4 \arrow[d, "T_\bullet(1_W,\hat{\alpha}_{W,\mathbb{N}})\circ\hat{\alpha}_{W,\mathbb{N}^2}"] \\
T^2(E) = T_\bullet(W\otimes W, F_E(\mathbb{N})) \arrow[r, "T_\bullet(c,1_{F_E(\mathbb{N})})"'] & T_\bullet(W\otimes W, F_E(\mathbb{N})) = T^2(E)
\end{tikzcd}$$

The composition along the left path gives

$$\begin{aligned}
&c \circ T^2(\sigma) \circ \langle T(0) \circ \pi_0, T(\lambda) \circ \pi_1 \rangle \circ (T(\sigma) \circ \langle 0 \circ \pi_0, \lambda \circ \pi_1 \rangle \times_{TM} T(\sigma) \circ \langle 0 \circ \pi_0, \lambda \circ \pi_1 \rangle) \\
=&c \circ T^2(\sigma) \circ \langle T(0) \circ T(\sigma) \circ \langle 0 \circ \pi_0, \lambda \circ \pi_1 \rangle, T(\lambda) \circ T(\sigma) \circ \langle 0 \circ \pi_2, \lambda \circ \pi_3 \rangle \rangle.
\end{aligned}$$

Due to naturality of the canonical flip of the tangent structure $c : T^2 \Rightarrow T^2$, since $c \circ T(0_E) = 0_{TE}$ and since (λ, ζ) is an additive bundle morphism this is equal to

$$T^2(\sigma) \circ \langle 0_{TE} \circ T(\sigma) \circ \langle 0 \circ \pi_0, \lambda \circ \pi_1 \rangle, c \circ T^2(\sigma) \circ \langle T(\lambda) \circ 0 \circ \pi_2, T(\lambda) \circ \lambda \circ \pi_3 \rangle \rangle.$$

By naturality of c and since $c \circ T(\lambda) \circ \lambda = c \circ \ell \circ \lambda = \ell \circ \lambda = T(\lambda) \circ \lambda$ this is equal to

$$T^2(\sigma) \circ \langle T^2(\sigma) \circ \langle 0_{TE} \circ 0 \circ \pi_0, 0_{TE} \circ \lambda \circ \pi_1 \rangle, T^2(\sigma) \circ \langle c \circ T(\lambda) \circ 0 \circ \pi_2, T(\lambda) \circ \lambda \circ \pi_3 \rangle \rangle.$$

By naturality of zero and since $c \circ 0_{TE} = T(0_E)$ this is equal to

$$T^2(\sigma) \circ \langle T^2(\sigma) \circ \langle T(0) \circ 0 \circ \pi_0, T(\lambda) \circ 0 \circ \pi_1 \rangle, T^2(\sigma) \circ \langle T(0) \circ \lambda \circ \pi_2, T(\lambda) \circ \lambda \circ \pi_3 \rangle \rangle.$$

Now using associativity and commutativity of σ we can rearrange the terms to obtain

$$T^2(\sigma) \circ \langle T^2(\sigma) \circ \langle T(0) \circ 0 \circ \pi_0, T(0) \circ \lambda \circ \pi_2 \rangle, T^2(\sigma) \circ \langle T(\lambda) \circ 0 \circ \pi_1, T(\lambda) \circ \lambda \circ \pi_3 \rangle \rangle.$$

By naturality of 0 and since $(\lambda, 0)$ is an additive bundle morphism, this is equal to

$$\begin{aligned}
&T^2(\sigma) \circ \langle T(0) \circ T(\sigma) \circ \langle 0 \circ \pi_0, \lambda \circ \pi_2 \rangle, T(\lambda) \circ T(\sigma) \circ \langle 0 \circ \pi_1, \lambda \circ \pi_3 \rangle \rangle \\
=&T^2(\sigma) \circ \langle T(0) \circ \pi_0, T(\lambda) \circ \pi_1 \rangle \circ (T(\sigma) \circ \langle 0 \circ \pi_0, \lambda \circ \pi_1 \rangle \times_{TM} T(\sigma) \circ \langle 0 \circ \pi_0, \lambda \circ \pi_1 \rangle) \circ \langle \pi_0, \pi_2, \pi_1, \pi_3 \rangle
\end{aligned}$$

which is the composition along the right path in the square.

This shows that $\hat{\alpha}$ is natural with respect to the building blocks $p, +, 0, \ell, c$ and projections out of products. In order to show that $\hat{\alpha}$ is natural with respect to all morphisms in $\mathbb{W}$eil, we show next that $\hat{\alpha}$ is compatible with coproducts (tensor products) and pullbacks of the form $A \otimes (B \times C)$ from Lemma B.0.1 (which include products).

For tensor products $f_1 \otimes f_2 : A \otimes B \to A' \otimes B'$ in $\mathbb{W}$eil, $\hat{\alpha}$ is natural since $\hat{\alpha}_{A \otimes B}$ is defined as $T_A(\hat{\alpha}_B) \circ \hat{\alpha}_A$.

For pullbacks of the form $A \otimes (B \times C)$, suppose $\hat{\alpha}_{-,\mathbb{N}^k}$ is natural with respect to the Weil-morphisms

$f_1 : K \to A \otimes B$ and $f_2 : K \to A \otimes C$.

From the tangent category axioms, the following diagram is a pullback in the tangent category $(\mathbb{X}, T_\bullet)$:

$$
\begin{tikzcd}
T_A \circ T_{B\times C} \circ F_E \arrow[r, "T_A\circ T_{\pi_1}\circ F_E"] \arrow[d, "T_A\circ T_{\pi_0}\circ F_E"'] \arrow[dr, phantom, "\lrcorner", very near start] & T_A \circ T_C \circ F_E \arrow[d, "T_A\circ T_!\circ F_E"] \\
T_A \circ T_B \circ F_E \arrow[r, "T_A\hat{\circ}T_!\circ F"'] & T_A \circ F_E
\end{tikzcd}
$$

Therefore, due to the universal property of the pullback, the naturality diagram

$$
\begin{tikzcd}
F_E \circ D_K \arrow[r, "F_E\circ D_{\langle f_1, f_2\rangle} = F_E\circ\langle D_{f_1}, D_{f_2}\rangle"] \arrow[d, "\hat{\alpha}_K"'] & F_E \circ D_{A\otimes(B\times C)} \arrow[d, "\hat{\alpha}_{A\otimes(B\times C)}"] \\
T_K \circ F_E \arrow[r, "T_{\langle f_1, f_2\rangle}\circ F_E = \langle T_{f_1}, T_{f_2}\rangle\circ F_E"'] & T_{A\otimes(B\times C)} \circ F_E
\end{tikzcd}
$$

commutes if and only if the naturality diagrams

$$
\begin{tikzcd}
F_E \circ D_K \arrow[r, "F_E\circ D_{f_1}"] \arrow[d, "\hat{\alpha}_K"'] & F_E \circ D_{A\otimes B} \arrow[d, "\hat{\alpha}_{A\otimes B}"] \\
T_K \circ F_E \arrow[r, "T_{f_1}\circ F_E"'] & T_{A\otimes B} \circ F_E
\end{tikzcd}
\qquad
\begin{tikzcd}
F_E \circ D_K \arrow[r, "F_E\circ D_{f_2}"'] \arrow[d, "\hat{\alpha}_K"'] & F_E \circ D_{A\otimes C} \arrow[d, "\hat{\alpha}_{A\otimes C}"] \\
T_K \circ F_E \arrow[r, "T_{f_2}\circ F_E"] & T_{A\otimes C} \circ F_E
\end{tikzcd}
$$

for the components f_1 and f_2 commute (which we assumed).

This concludes the proof that $\hat{\alpha}$ is natural in $\mathbb{W}\mathrm{eil}$. Thus $\hat{\alpha}$ is natural in $\mathbb{N}^\bullet$ and in $\mathbb{W}\mathrm{eil}$ and therefore natural in $\mathbb{N}^\bullet \times \mathbb{W}\mathrm{eil}$. $\square$