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An adjunction between finiteness spaces and algebras in Lefschetz spaces

by

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Abstract

In finite-dimensional categorical quantum mechanics, a crucial result is that orthonormal bases correspond precisely to the copyable elements of commutative dagger special Frobenius algebras. This thesis examines an extension of this correspondence into the infinite-dimensional setting of Lefschetz spaces.

Richard Blute conjectured that, by analogy, the copyable elements of a coalgebra on a reflexive Lefschetz space would similarly span the space. This would imply an equivalence between the category of finiteness spaces with functions and the category of dagger coalgebras in (reflexive) Lefschetz spaces with homomorphisms. Ehrhard has given a construction of a reflexive Lefschetz space from a finiteness space. To these reflexive spaces, we gave a coalgebra structure. However, we notice that if the coalgebra has a counit, then the underlying Lefschetz space must be discrete. However, spaces arising out of Ehrhard's construction were, in general, not discrete. Thus, to recapture the correspondence, necessitated dropping the counit and working with the broader category of dagger linear cosemigroups.

Finally, this thesis shows that one can construct a coreflection between the category of finiteness spaces with functions and dagger cosemigroups with homomorphisms – in reflexive Lefschetz spaces. However, a class of examples of cosemigroups is given whose copyables do not span the underlying space. Thus, contrary to the conjecture, this coreflection fails to be an equivalence.

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To maa,

for teaching me my first numbers

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Chapter 1

Introduction

Quantum mechanics is a resource sensitive theory. Unlike classical mechanics, where one can freely copy a resource, quantum mechanics does not allow one to clone (or delete) an arbitrary quantum state [32, 42].

The no-cloning theorem (and the no-deletion theorem) underpins the interpretation of quantum mechanics as a resource sensitive logic. Linear logic [20] is a refinement of classical logic, which emphasises the role of propositions as resources. Thus, one does not have, for any arbitrary formula A, rules of the form,

$$\frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta}\ \text{Contraction} \qquad\qquad \frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta}\ \text{Weakening}$$

That is, one can not duplicate or delete assumptions.

Linear logic also has an involutive negation $(\_)^{\perp}$, which allows one to move a resource from the succedent to the antecedent, or vice versa.

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, A^{\perp} \vdash \Delta}\ (\_)^{\perp}(L) \qquad\qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \Delta, A^{\perp}}\ (\_)^{\perp}(R)$$

Linear logic also has two connectives, called the multiplicatives, given by tensor $\otimes$ and par $\oplus$. The tensor models two resources being available at the same time, whereas the par is the De-Morgan dual to the tensor, $A \oplus B = (A^\perp \otimes B^\perp)^\perp$. The units for $\otimes$ and $\oplus$ are denoted by $\top$, $\perp$ respectively.

A category is said to provide a semantics for a formal system if the propositions and proofs can be modelled as objects and morphisms within that category. The program of categorical quantum mechanics [1] was initiated to capture the resource sensitive nature of quantum mechanics. The categorical semantics for finite-dimensional quantum mechanics is provided by dagger compact closed categories [37], which are a proof theory for compact linear logic (where $\otimes = \oplus$ and $\top = \perp$) with negation. The dagger is an abstraction of the notion of adjoint, given by the inner product in Hilbert spaces, which allows us to talk about observables (i.e. self-adjoint operators) and unitary operators (operators whose inverse is given by their adjoint).

However, compact closed categories can provide a model only for finite-dimensional quantum mechanics, as compactness breaks beyond the finite dimension [21]. Compact closed categories are the proof theory for the compact fragment of linear logic with negation. A natural generalisation of a compact closed category is, thus, a star ($*$)- autonomous category, where the tensor and par are not isomorphic. A further generalisation of $*$-autonomous category is a dagger ($\dagger$-) linearly distributive category (LDC), where the dagger need not necessarily come from the dualising functor. A semantics for arbitrary-dimensional quantum mechanics was proposed in [40]; given by mixed unitary categories (MUC). The main appeal of MUCs is that they still preserve compactness in the unitary core, which provides the finite-dimensional fragment of the theory. The core extends to a full $\dagger$- LDC, giving the structure of a MUC.

While an arbitrary quantum state can not be cloned, one can always clone the vectors of an orthonormal basis. Thus, a choice of orthonormal basis can be thought of as the classical structure sitting within the quantum realm. A central result in categorical quantum mechanics is that the orthonormal bases correspond precisely to the copyable elements of commutative dagger special Frobenius algebras [16]. A Frobenius algebra in a monoidal category is an algebra and a coalgebra that interact via the Frobenius axiom (See Chapter 6).

Richard Blute conjectured that a similar correspondence between bases and copyable elements of coalgebras holds in the mixed unitary category of reflexive Lefschetz spaces. This conjecture, if true, would mean that one can extend the correspondence between bases and algebras beyond the finite dimensions.

A reflexive Lefschetz space is a linear topological vector space for which the canonical evaluation map is an isomorphism. The term linear here means that open neighbourhood bases around the origin are linear subspaces. The category of reflexive Lefschetz spaces with continuous linear maps forms a $*$- autonomous category [4], and the unitary core of these spaces is exactly finite-dimensional vector spaces.

Just as one can construct a finite-dimensional Hilbert space from a finite set of basis elements, one can similarly construct a reflexive Lefschetz space on what is called a finiteness space. Ehrhard provided a construction of a reflexive Lefschetz space on a finiteness space, whose *basis* is given by the web of the finiteness space. This basis, however, is not a basis in the usual algebraic sense. That is to say, taking just finite linear combinations over this basis, does not span the Lefschetz space Ehrhard constructed. However, we have access to the Lefschetz topology and, therefore, we can take arbitrary convergent linear combinations – which makes the web, in Ehrhard's construction, a basis in this stronger sense. And the convergent nets over this basis are exactly those that have finitary support.

Conversely, the finitary sets are encoded in the basis of a Lefschetz space as the support of convergent nets of finite sums over the basis. We use this encoding to provide a functor from coalgebras in reflexive Lefschetz spaces to finiteness spaces (with functions) by studying the support of convergent nets over the copyable elements.

The basis vectors of a reflexive Lefschetz space arising out of Ehrhard's construction are copyable, and give a coalgebra structure on the space. We noticed that, if the coalgebra has a counit, then the underlying space must be discrete. However, the reflexive Lefschetz spaces arising out of Ehrhard's construction are not, in general, discrete. Thus, to capture the correspondence, necessitates dropping the counit and working with the broader category of dagger linear cosemigroups.

Finally, this thesis does establish a coreflection between the category of finiteness spaces with functions and the category of cosemigroups with homomorphisms – in reflexive Lefschetz spaces. However, a class of examples of cosemigroups is given whose copyables do not span the underlying space. Thus, contrary to the conjecture, this coreflection fails to be an equivalence.

1.1 Outline and contributions

Chapter 2: This chapter introduces the background from category theory and basic topology needed in this thesis. All definitions and results in this chapter are from the literature.

Chapter 3: This chapter develops the theory of mixed unitary categories. This chapter is almost entirely based on [40], except for Proposition 3.2.9, where we made a small observation that in a unitary category, the construction of a stationary dagger defined in [40] can be extended functorially.

Chapter 4: This chapter develops the theory of Lefschetz spaces, showing it is a model of a mixed unitary category. Here, we first show that a full subcategory of Lefschetz spaces forms a $*$- autonomous category. We follow the development in [4] up to Section 4.2. Then we use the equivalence between the notion of an LDC with duals and of $*$-autonomous category [14] to develop the theory of reflexive Lefschetz space within the framework of a linearly distributive category. The material from Section 4.3 to Section 4.5 is novel. Notably, this includes developing the notion of a basis for a Lefschetz space in Section 4.4. We conclude this chapter showing that the reflexive Lefschetz spaces form a mixed unitary category and give a complete characterisation of unitary isomorphisms in this category (Proposition 4.5.3).

Chapter 5: This chapter describes two constructions, one from a finiteness space to a reflexive Lefschetz space, due to Ehrhard [19], and a second construction from a reflexive Lefschetz space to a finiteness space. The construction in Section 5.3 is based on [19]. We prove many novel propositions about Lefschetz spaces arising out of Ehrhard's construction. Proposition 5.3.10 to Corollary 5.3.19 are novel, with the exception of Proposition 5.3.11 [5], of which we provide an alternative proof. An alternative proof of Lemma 9 from [19] is provided in Lemma 5.3.20. Notably, we observe scalar invariant convergence over basis in Lefschetz spaces arising out of Ehrhard's construction (Proposition 5.3.12). This is the key observation which allows us to construct a finiteness space on a reflexive Lefschetz space (with a given basis) in Section 5.4, which we prove to be functorial in Section 5.5. We also prove that Ehrhard's construction, on objects, is inverse to the construction of a finiteness space from a reflexive Lefschetz space given in Section 5.4. This allows us to show that any reflexive Lefschetz space's topology arises as a finiteness topology, allowing us to characterise topology on reflexive Lefschetz spaces (Corollary 5.4.10, 5.4.8) due to Ehrhard's classification [19].

Chapter 6: This chapter first gives the correspondence between bases of finite- dimensional Hilbert spaces and (commutative dagger special) Frobenius algebras as in [16]. This is from Section 6.2 to Section 6.7. Then we introduce our categories of interest in Section 6.8, where we recall the category of finiteness spaces and functions FinF and define the category of commutative dagger linear (co) semigroups in reflexive Lefschetz spaces with (co)algebra morphisms $\mathsf{CDLS}(\mathsf{RLef})$. Then we define a functor $F : \mathsf{FinF} \longrightarrow \mathsf{CDLS}(\mathsf{RLef})$ in Section 6.9 using Ehrhard's construction, followed by copying the basis vectors. Then in Section 6.10 we first take the subspace generated by copyables and then apply the support functor to get a functor $G : \mathsf{CDLS}(\mathsf{RLef}) \longrightarrow \mathsf{FinF}$. We show in Section 6.11 that this pair of functors gives an adjunction $(\eta, \varepsilon) : F \dashv G : \mathsf{FinF} \longrightarrow \mathsf{CDLS}(\mathsf{RLef})$. We finish this chapter by giving a class of examples showing that this pair of functors does not lead to an equivalence, as there exist cosemigroup structures on reflexive Lefschetz spaces whose copyables do not span the underlying space.

Chapter 7 This chapter provides a conclusion for this thesis and discusses avenues for future work.

1.2 Notation

Composition is written in the diagrammatic order: $f\,g$ means that we apply f followed by g. Note that the opposite is the usual applicative order, where the composition of $f : X \to Y$ and $g : Y \to Z$ is written as $g \circ f$. String diagrams in this thesis are to be read from top to bottom.

Chapter 2

Background

In this chapter, we recall the definitions and theorems of category theory and basic topology needed in this thesis. All definitions and theorems in this chapter are from the literature. For further reading, see [11, 26, 30, 34].

2.1 Elementary category theory

Definition 2.1.1. A **category** $\mathbb{X}$ consists of

- a collection of objects $\mathbb{X}_0$, whose elements are denoted by $X, Y, Z, \ldots$
- a collection of morphisms $\mathbb{X}_1$, whose elements are denoted by $f, g, h, \ldots$

such that

- each morphism $f \in \mathbb{X}_1$ has a specified domain object $\mathrm{dom}(f)$, and a codomain object $\mathrm{cod}(f)$; a morphism f with domain X and codomain Y is denoted as $f : X \to Y$;
- Each object $X \in \mathbb{X}_0$ has a designated identity morphism $\mathrm{id}_X : X \to X$;

- For $f, g \in \mathbb{X}_1$ such that $\mathrm{cod}(f) = \mathrm{dom}(g)$, we have a specified composite morphism $fg \in \mathbb{X}_1$ such that $\mathrm{dom}(fg) = \mathrm{dom}(f)$ and $\mathrm{cod}(fg) = \mathrm{cod}(g)$. Note that the composition is written in the diagrammatic order.

This data is subject to the following conditions:

- **Unitality** : For any $f : X \to Y$, $\mathrm{id}_X f = f = f\,\mathrm{id}_Y$;
- **Associativity** : For any $f : X \to Y$, $g : Y \to Z$, $h : Z \to W$, $(fg)h = f(gh)$.

When the context is clear, we will simply say $X \in \mathbb{X}$ and $f : X \to Y \in \mathbb{X}$ to denote that X is an object of $\mathbb{X}$ and $f : X \to Y$ is a morphism in $\mathbb{X}$. The collection of all the maps between two objects X, Y in a category $\mathbb{X}$ is denoted by $\mathbb{X}(X, Y)$. When this is a set (and not a proper class), the category is said to be **locally small**. All the examples we consider in this thesis are locally small.

Example 2.1.2.

- **Set**

  Objects: Sets;

  Morphisms: functions;

  Composition: usual composition of functions;

  Identity: identity function $\mathrm{id}_X : X \to X$, $\mathrm{id}(x) := x$ for any $x \in X$.

- **Rel**

  Objects: Sets;

Morphisms: a relation $R : X \to Y$ is a subset $R \subseteq X \times Y$;

Composition: the composition of relations $R : X \to Y, S : Y \to Z$ is given by $xRSz$ if and only if there exists $y \in Y$ such that xRy and ySZ;

Identity: the identity relation $R : X \to X$ is given by the diagonal $\mathrm{id}_X := \{(x,x) : x \in X\}$.

- **FdHilb**

Objects: Finite-dimensional Hilbert spaces over a field $\mathbb{K}$. In this thesis, the base field in all our examples will be $\mathbb{C}$.

Morphisms: linear transformation, or equivalently, matrices, by choosing a basis;

Composition: composition of (underlying) functions or equivalently matrix multiplication, when the basis is fixed;

identity: identity linear transformation.

Definition 2.1.3. For any category $\mathbb{X}$, we can define a **dual category** $\mathbb{X}^{\mathsf{op}}$ which has the same object as $\mathbb{X}$ and $f^{\mathsf{op}} : Y \to X \in \mathbb{X}_1^{\mathsf{op}}$ if and only if $f : X \to Y \in \mathbb{X}_1$, the composition in $\mathbb{X}^{\mathsf{op}}$ is defined by $g^{\mathsf{op}} \bullet_{\mathbb{X}^{\mathsf{op}}} f^{\mathsf{op}} := f \bullet_{\mathbb{X}} g$.

Next, we define some important classes of maps in a category.

Definition 2.1.4. Let $\mathbb{X}$ be a category.

- A morphism $f : X \to Y$ is called **monic** in case whenever $k_1 f = k_2 f$, we have $k_1 = k_2$.

- A morphism $f : X \to Y$ is called **epic** in case whenever $f h_1 = f h_2$, we have $h_1 = h_2$.

- A morphism $f : X \to Y$ is called a **section** if there exists a morphism $f' : Y \to X$ such that $ff' = \mathrm{id}_X$. In this case, f' is called a **right inverse** of f.

- A morphism $f : X \to Y$ is called a **retraction** if there exists a morphism $f' : Y \to X$ such that $f'f = \mathrm{id}_Y$. In this case, f' is called a **left inverse** of f.

- A morphism $f : X \to Y$ is called an **isomorphism** if it is both a section and a retraction.

Note that in above k_1, k_2 are arbitrary morphisms in $\mathbb{X}$ such that $\mathrm{cod}(k_1) = \mathrm{dom}(f) = \mathrm{cod}(k_2)$. Similarly, h_1, h_2 are arbitrary morphisms in $\mathbb{X}$ such that $\mathrm{dom}(h_1) = \mathrm{cod}(f) = \mathrm{dom}(h_2)$. When the context is clear, we will not be explicit in this quantification.

In Set, the monic maps are exactly the injective functions, and dually, the epic maps are exactly the surjective functions.

We note some basic observations:

Lemma 2.1.5. The following are equivalent

(i) f is an isomorphism;

(ii) f is an epic section;

(iii) f is a monic retraction;

Proof. If f is an isomorphism, then $fk_1 = fk_2$ implies $gfk_1 = gfk_2$, where g is the left inverse of f. Then we have $k_1 = k_2$. So f is epic and f is a section by definition. The proof that f is a monic retraction is similar. Now, as these statements are dual, it suffices to show that f, an epic section, is an isomorphism. It suffices to show that f is a retraction. For that, let g be the right inverse of f, then $fgf = f = f\,\mathrm{id}$. Now, as f is epic, so $gf = \mathrm{id}$ and hence f is a retraction. □

A functor $F : \mathbb{X} \to \mathbb{Y}$ is a structure-preserving map between categories. More precisely,

Definition 2.1.6. A **functor** $F : \mathbb{X} \to \mathbb{Y}$ between two categories $\mathbb{X}$ and $\mathbb{Y}$ consists of the following data:

- An object $FX \in \mathbb{Y}$ for every $X \in \mathbb{X}$;
- A morphism $Ff : FX \to FY \in \mathbb{Y}$ for every $f : X \to Y \in \mathbb{X}$.

such that,

- For any composable pair $f : X \to Y, g : Y \to Z \in \mathbb{X}$, $F(f \bullet_{\mathbb{X}} g) = F(f) \bullet_{\mathbb{Y}} F(g)$;
- And $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$.

Sometimes a functor $F : \mathbb{X} \to \mathbb{Y}$ is called **covariant** functor to distinguish it from its **contravariant** counterpart, which is a functor $F : \mathbb{X}^{\mathsf{op}} \to \mathbb{Y}$.

A functor $F : \mathbb{X} \to \mathbb{Y}$ is said to be **faithful** if for every object $X, Y \in \mathbb{X}$, the map $\mathbb{X}(X,Y) \to \mathbb{Y}(FX,FY); f \mapsto Ff$ is injective. The functor is said to be **full** if this map is surjective, and **fully faithful** if this is a bijection.

On every category $\mathbb{X}$, we can define an identity functor, which is identity on objects and morphisms. Moreover, functors compose, so we have that the category of locally small categories and functors forms a category Cat.

A subcategory of a category $\mathbb{X}$ is a sub-collection of objects and morphisms from $\mathbb{X}$ such that it is a category in its own right. More precisely,

Definition 2.1.7. A **sub-category** $\mathbb{Y}$ of a category $\mathbb{X}$ is a subcollection of objects and morphisms of $\mathbb{X}$, such that

- if $f : X \to Y \in \mathbb{Y}$, then $X, Y \in \mathbb{Y}$;
- for every $X \in \mathbb{Y}$, $\mathrm{id}_X \in \mathbb{Y}$;
- if $f : X \to Y, g : Y \to Z \in \mathbb{Y}$, then $fg : X \to Z \in \mathbb{Y}$.

The inclusion $I : \mathbb{Y} \hookrightarrow \mathbb{X}$ is easily seen to be a functor. The subcategory $\mathbb{Y}$ is said to be a **full subcategory** if the inclusion functor is full.

One of the most important notions in category theory is that of a natural transformation,

Definition 2.1.8. Given two functors $F, G : \mathbb{X} \to \mathbb{Y}$, a **natural transformation** $\alpha : F \Rightarrow G$ is an indexed family of morphisms $\{\alpha_X : FX \to GX\}_{X \in \mathbb{X}}$ in $\mathbb{Y}$ such that for any morphism $f : X \to Y$ in $\mathbb{X}$, the following diagram commutes:

$$\begin{tikzcd} FX \arrow[r, "Ff"] \arrow[d, "\alpha_X"'] & FY \arrow[d, "\alpha_Y"] \\ GX \arrow[r, "Gf"'] & GY \end{tikzcd}$$

A natural transformation is said to be a **natural isomorphism** if for each $X \in \mathbb{X}$, $\alpha_X : FX \to GX$ is an isomorphism in $\mathbb{Y}$.

For example, the canonical evaluation map $\mathrm{ev}_V : V \to V^{**}$ is always a natural transformation for any vector space V, and it is a natural isomorphism when V is finite-dimensional. However, the isomorphism between a finite-dimensional space and its dual is not natural, as it is basis-dependent.

Definition 2.1.9. In a category $\mathbb{X}$, the **product** of two objects X and Y (if it exists) is a tuple $(X \times Y, \pi_1 : X \times Y \to X, \pi_2 : X \times Y \to Y)$ such that for any object Z and a pair of morphisms $f : Z \to X$, $g : Z \to Y$, there exists a unique morphism $\langle f, g\rangle : Z \to X \times Y$ such that the following diagram commutes:

$$\begin{tikzcd}
 & Z \arrow[dl, "f"'] \arrow[d, dashed, "\langle f,g\rangle"] \arrow[dr, "g"] & \\
X & X \times Y \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & Y
\end{tikzcd}$$

For example, in Set, the product of two sets is their Cartesian product with the usual projection map to each factor.

Definition 2.1.10. In a category $\mathbb{X}$, the **co-product** of two objects X and Y (if it exists) is a tuple $(X + Y, i_1 : X \to X + Y, i_2 : Y \to X + Y)$ such that for any object Z and a pair of morphisms $f : X \to Z$, $g : X \to Z$, there exists a unique morphism $f + g : X + Y \to Z$ such that the following diagram commutes:

$$\begin{tikzcd}
 & Z & \\
X \arrow[ur, "f"] \arrow[r, "i_1"'] & X + Y \arrow[u, dashed, "f+g"'] & Y \arrow[ul, "g"'] \arrow[l, "i_2"]
\end{tikzcd}$$

For example, in Set, the coproduct is given by the disjoint union of two sets with the usual injection maps.

Definition 2.1.11. Let $G : \mathbb{Y} \to \mathbb{X}$ be a functor. Then for an object $X \in \mathbb{X}$, $(U \in \mathbb{Y}, \eta_X : X \to G(U))$ is said to be a **universal pair** for the functor G at the object X if for any $f : X \to G(Y) \in \mathbb{X}$, there exists a unique

$f^\# : U \to Y \in \mathbb{Y}$ such that the following triangle commutes:

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, "f"'] & G(U) \arrow[d, "G(f^\#)"] \\ & G(Y) \end{tikzcd}$$

When it is the case that a functor $G : \mathbb{Y} \to \mathbb{X}$ has a universal pair for every object $X \in \mathbb{X}$, then we have,

Proposition 2.1.12. (Proposition 2.2.2 in [11]) Let $G : \mathbb{Y} \to \mathbb{X}$ be a functor such that for each $X \in \mathbb{X}$, there is a universal pair $(F(X), \eta_X)$. Then,

- $F : \mathbb{X} \to \mathbb{Y}$ is a functor with $F(g) := (g\eta_Y)^\#$ for $g : X \to Y \in \mathbb{X}$;
- $\eta_X : X \to G(F(X))$ is a natural transformation $\eta : \mathrm{id}_{\mathbb{X}} \Rightarrow FG$;
- $\varepsilon_Y := (\mathrm{id}_{G(Y)})^\# : F(G(Y)) \to Y$ is a natural transformation $\varepsilon : GF \Rightarrow \mathrm{id}_{\mathbb{Y}}$;
- The **triangle equalities** $\eta_{G(Y)} G(\varepsilon_Y) = \mathrm{id}_{G(Y)}$ and $F(\eta_X)\varepsilon_{F(X)} = \mathrm{id}_{F(X)}$ hold for all $X \in \mathbb{X}$ and $Y \in \mathbb{Y}$.

Conversely given functor $F : \mathbb{X} \to \mathbb{Y}$ and $G : \mathbb{Y} \to \mathbb{X}$ and natural transformations $\eta : \mathrm{id}_{\mathbb{X}} \Rightarrow FG$, $\varepsilon : GF \Rightarrow \mathrm{id}_{\mathbb{Y}}$, which satisfy the triangle identity, then each $(F(X), \eta_X : X \to G(F(X)))$ is a universal pair for G at X.

Definition 2.1.13. We say $F : \mathbb{X} \to \mathbb{Y}$ is **left adjoint** to $G : \mathbb{Y} \to \mathbb{X}$ (or equivalently G is **right adjoint** to F) if there are natural transformations $\eta : \mathrm{id}_{\mathbb{X}} \Rightarrow FG$, $\varepsilon : GF \Rightarrow \mathrm{id}_{\mathbb{Y}}$ satisfying the triangle identities as above. We denote this situation by

$$(\eta, \varepsilon) : F \dashv G : \mathbb{X} \to \mathbb{Y}$$

The free group construction over a set is a left adjoint to the forgetful functor from the category of groups, which sends a group to its underlying set and a group homomorphism to its underlying set function.

We also have,

Proposition 2.1.14. (Proposition 2.2.6 in [11]) If $(\eta, \varepsilon) : F \dashv G : \mathbb{X} \to \mathbb{Y}$ is an adjunction, then

(i) Each η_X is monic if and only if F is faithful.

(ii) Each ε_Y is epic if and only if G is faithful.

(iii) Each η_X is a retraction if and only if F is full.

(iv) Each ε_Y is a section if and only if G is faithful.

(v) Each η_X is an isomorphism if and only if F is fully faithful.

(vi) Each ε_Y is an isomorphism if and only if G is fully faithful.

We say a full and faithful functor $F : \mathbb{X} \to \mathbb{Y}$ is a **reflection**, in case it is a right adjoint, and a **co-reflection** in case it is a left adjoint. By the above proposition, this means that for a reflection, each ε_X is an isomorphism, and dually for a coreflection each η_X is an isomorphism. A full and faithful functor $F : \mathbb{X} \to \mathbb{Y}$ is said to be an **equivalence of categories** in case it has a left or right adjoint, which is also full and faithful. This means that both the unit and counit maps of the adjunction are isomorphisms.

2.2 Monoidal category theory

In this section, we recall some definitions from monoidal category theory that will be used throughout this thesis.

Definition 2.2.1. The **product of two categories** $\mathbb{X}$ and $\mathbb{Y}$ is a category $\mathbb{X}\times\mathbb{Y}$ whose objects are pairs (X, Y) where $X \in \mathbb{X}_0$, $Y \in \mathbb{Y}_0$ and morphisms are pairs $(f, g) : f \in \mathbb{X}_1, g \in \mathbb{Y}_1$. The composition is defined component-wise.

Definition 2.2.2. A **bifunctor** is a functor whose domain is a product category.

Definition 2.2.3. A **monoidal category** is a category $\mathbb{X}$ equipped with

- A bifunctor $\otimes : \mathbb{X} \times \mathbb{X} \to \mathbb{X}$, called the **tensor**;
- A designated object $I \in \mathbb{X}$, called the **unit**;
- A natural isomorphism (indexed here by objects), $\alpha_{\otimes}^{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$, called the **associator**. When the context is clear, we will suppress the indexing on the structure maps.
- A natural isomorphism $l_{\otimes} : I \otimes X \to X$, called the **left unitor**;
- A natural isomorphism $r_{\otimes} : X \otimes I \to X$, called the **right unitor**.

such that the following diagrams commute for all $X, Y, Z, W \in \mathbb{X}$:

(MC 1)

$$
\begin{tikzcd}
 & (W \otimes X) \otimes (Y \otimes Z) \arrow[dr, "\alpha"] & \\
((W \otimes X) \otimes Y) \otimes Z \arrow[ur, "\alpha"] \arrow[d, "\alpha\otimes \mathrm{id}"'] & & W \otimes (X \otimes (Y \otimes Z)) \\
(W \otimes (X \otimes Y)) \otimes Z \arrow[rr, "\alpha"'] & & W \otimes ((X \otimes Y) \otimes Z) \arrow[u, "\mathrm{id} \otimes \alpha"']
\end{tikzcd}
$$

(MC2)

$$\begin{tikzcd}
(X \otimes I) \otimes Y \arrow[rr, "\alpha"] \arrow[dr, "r_\otimes \otimes \mathrm{id}"'] & & X \otimes (I \otimes Y) \arrow[dl, "\mathrm{id} \otimes l_\otimes"] \\
& X \otimes Y &
\end{tikzcd}$$

These identities are called the pentagon and triangle identities, respectively. We also use $u^L_\otimes, u^R_\otimes$ interchangeably with $l_\otimes, r_\otimes$, for left and right unitors, respectively. Also for commuting diagrams (or while proving equations involving commuting diagrams), it is standard to skip the quantification over all objects of a category. Unless explicitly mentioned otherwise, the quantification in commuting diagrams is over all objects of the category in context.

A monoidal category can be thought of as a "categorified" version of a monoid, where the (bi-) functor $\otimes : \mathbb{X} \times \mathbb{X} \to \mathbb{X}$ is the multiplication. The category Set is an example of a monoidal category where the tensor product is the Cartesian product. A braided monoidal category is a monoidal category equipped with an isomorphism, which lets us "switch" objects. More precisely,

Definition 2.2.4. A **braided monoidal category** is a monoidal category $\mathbb{X}$ equipped with a natural isomorphism $B_{X,Y} : X \otimes Y \to Y \otimes X$ such that the following diagrams commute:

(BMC 1)

$$\begin{tikzcd}
(X \otimes Y) \otimes Z \arrow[r, "\alpha"] \arrow[d, "B \otimes \mathrm{id}"'] & X \otimes (Y \otimes Z) \arrow[r, "B"] & (Y \otimes Z) \otimes X \arrow[d, "\alpha"] \\
(Y \otimes X) \otimes Z \arrow[r, "\alpha"'] & Y \otimes (X \otimes Z) \arrow[r, "\mathrm{id} \otimes B"'] & Y \otimes (Z \otimes X)
\end{tikzcd}$$

(BMC 2)

$$\begin{tikzcd}
X \otimes (Y \otimes Z) \arrow[r, "\alpha^{-1}"] \arrow[d, "\mathrm{id} \otimes B"'] & (X \otimes Y) \otimes Z \arrow[r, "B"] & Z \otimes (X \otimes Y) \arrow[d, "\alpha^{-1}"] \\
X \otimes (Z \otimes Y) \arrow[r, "\alpha^{-1}"'] & (X \otimes Z) \otimes Y \arrow[r, "B \otimes \mathrm{id}"'] & (Z \otimes X) \otimes Y
\end{tikzcd}$$

A braided monoidal category is said to be **symmetric** if $B_{X,Y}B_{Y,X} = \mathrm{id}_{X\otimes Y}$. In the symmetric case, we will denote the braiding map by $c_\otimes$.

In Set, whenever we have a function of two variables, $f : X \times Y \to Z; (x, y) \mapsto f(x, y)$, we can equivalently see it as a function of one variable, with values in function of the other variable $x \mapsto (y \mapsto f(x, y))$. This gives Set the structure of a monoidal closed category. This motivates the following definition:

Definition 2.2.5. A **monoidal closed category** is a monoidal category $(\mathbb{X}, \otimes, I)$ such that for every $X \in \mathbb{X}_0$, the functor $(\_) \otimes X : \mathbb{X} \to \mathbb{X}$ has a right adjoint $X \multimap (\_) : \mathbb{X} \to \mathbb{X}$.

Now in Set, given a function $f : X \to Y$, there is not always a natural way to get a function $f' : Y \to X$, unless f is surjective. But in FdHilb, given any linear map $f : H_1 \to H_2$, we can always get a map $f^\dagger : H_2 \to H_1$, called the adjoint of f, even when f is not invertible. The adjoint of a map $f : H_1 \to H_2$ between two finite- dimensional Hilbert spaces is the unique map $f^\dagger : H_2 \to H_1$ such that for for all $v \in H_1$, and $w \in H_2$, the following equation holds:

$$\langle f(v), w\rangle = \langle v, f^\dagger(w)\rangle$$

Moreover, thinking in terms of matrices, it is not hard to see that $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$. This is to say that the tensor product structure is compatible with the adjoint structure. This motivates the following definitions:

Definition 2.2.6. [1] A category $\mathbb{X}$ is said to be a **dagger category** if it is equipped with an endofunctor $(\_)^\dagger : \mathbb{X}^{\mathsf{op}} \to \mathbb{X}$ such that it is identity on objects and is involutive, that is for every $X \in \mathbb{X}_0, X^\dagger = X$ and for every $f : X \to Y \in \mathbb{X}_1, (f)^{\dagger\dagger} = f$.

Definition 2.2.7. [22] A (symmetric) monoidal category $(\mathbb{X}, \otimes, I)$ equipped with a dagger $(\_)^\dagger : \mathbb{X}^{\mathsf{op}} \to \mathbb{X}$ is said to be a (symmetric) **dagger monoidal category** if $(\mathbb{X}, (\_)^\dagger)$ is a dagger category and the following holds:

- For any $f \in \mathbb{X}(X, Y), g \in \mathbb{X}(Z, W), (f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$
- Every structure map is a unitary isomorphism, that is, $\alpha_\otimes^{-1} = \alpha_\otimes^\dagger, l_\otimes^{-1} = l_\otimes^\dagger, r_\otimes^{-1} = r_\otimes^\dagger$, and $c_\otimes^{-1} = c_\otimes^\dagger$ in the symmetric case.

In FdHilb, to each Hilbert space H, there is a dual space of linear functionals H^*, with maps $\eta : \mathbb{C} \to H \otimes H^*$ and $\varepsilon_H : H^* \otimes H \to \mathbb{C}$ which satisfy some coherence properties. This motivates the following definitions:

Definition 2.2.8. [22] A symmetric monoidal category $(\mathbb{X}, \otimes, I)$ is said to be a **compact closed category** if every object is dualizable, that is for any $X \in \mathbb{X}_0$, there exists an object $X^* \in \mathbb{X}_0$, called the dual of X with morphisms $\eta_X : I \to X \otimes X^*$ and $\varepsilon_X : X^* \otimes X \to I$ such that the following diagrams commute:

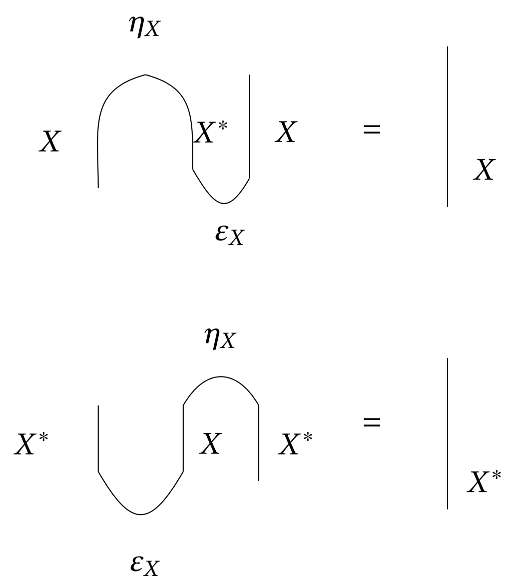
(KCC 1)

$$\begin{tikzcd}
X^* \otimes (X \otimes X^*) \arrow[d, "\alpha_\otimes^{-1}"'] & X^* \otimes I \arrow[l, "\mathrm{id}\otimes\eta_X"'] \arrow[d, "c_\otimes"] \\
(X^* \otimes X) \otimes X^* \arrow[r, "\varepsilon_X\otimes\mathrm{id}"'] & I \otimes X^*
\end{tikzcd}$$

(KCC 2)

$$\begin{tikzcd}
(X \otimes X^*) \otimes X \arrow[d, "\alpha_\otimes"'] & I \otimes X \arrow[l, "\eta_X\otimes\mathrm{id}"'] \arrow[d, "c_\otimes"] \\
X \otimes (X^* \otimes X) \arrow[r, "\mathrm{id}\otimes\varepsilon_X"'] & X \otimes I
\end{tikzcd}$$

They are also sometimes called the snake equations because of their shape in string diagrams.



(2.1)

A compact closed category $\mathbb{X}$ is a symmetric monoidal closed category if we define $X \multimap Y := Y \otimes X^*$ for any $X, Y \in \mathbb{X}$.

Definition 2.2.9. [37] A **dagger compact closed category** $\mathbb{X}$ is a dagger symmetric monoidal category that is also compact closed, and for any $X \in$

$\mathbb{X}$, the following diagram commutes:

$$\begin{tikzcd}
I \arrow[r, "\varepsilon_X^\dagger"] \arrow[dr, "\eta_X"'] & X^* \otimes X \arrow[d, "c_\otimes"] \\
& X \otimes X^*
\end{tikzcd}$$

where the unit $\eta_X : I \to X \otimes X^*$ and the counit map $\varepsilon_X : X^* \otimes X \to I$ are coming from the data of the compact closed category.

FdHilb, the prototypical category in which finite-dimensional quantum mechanics is modeled, is an example of a dagger compact closed category, where the monoidal structure is given by the usual tensor product of matrices, the dagger is given by the adjoint, and the compact structure is given by the unit and counit map defined by $\eta_H : \mathbb{C} \to H \otimes H^* := \sum_{i=1}^{n=\dim(H)} |i\rangle \otimes |i\rangle$, $\varepsilon_H : H^* \otimes H \to \mathbb{C} := \sum_{i=1}^{n=\dim(H)} \langle i| \otimes \langle i|$. Here we have picked an orthonormal basis $\{|i\rangle\}$ for H and have used the same notation for the corresponding basis of H^*. Note that the unit and the counit maps are bounded if and only if the dimension of the Hilbert space is finite; thus, the category of all Hilbert spaces (including the infinite-dimensional) is not compact.

There is a notion of a star autonomous category in which duals of objects exist, but in a weaker sense than in a compact closed category.

Definition 2.2.10. A **star autonomous category** is a symmetric monoidal closed category $(\mathbb{X}, \otimes, I, \multimap)$ with a global dualizing object $\perp$ such that the canonical morphism $d_X : X \to (X \multimap \perp) \multimap \perp$, which is the adjoint of the evaluation map $\mathrm{ev}_{X,\perp} : (X \multimap \perp) \otimes X \to \perp$, is an isomorphism for all $X \in \mathbb{X}$.

There is an equivalent definition of a star autonomous category which captures the fact that the star of the star autonomous category is indeed a (full and faithful) functor.

Definition 2.2.11. A **star autonomous category** is a symmetric monoidal category $(\mathbb{X}, \otimes, I)$ equipped with a fully faithful functor $(\_)^* : \mathbb{X}^{\mathsf{op}} \to \mathbb{X}$ such that for any $X, Y, Z \in \mathbb{X}$ there is a natural isomorphism $\mathbb{X}(X \otimes Y, Z^*) \cong \mathbb{X}(X, (Y \otimes Z)^*)$.

To see that these definitions are equivalent, first given a monoidal closed category, we define the dualizing functor $(\_)^* := (\_) \multimap \bot$. Then the morphism d_X is natural in X, so we have that there is a natural isomorphism $d : \mathrm{id}_{\mathbb{X}} \Rightarrow (\_)^{**}$. Moreover, we have,

$$\begin{aligned}
\mathbb{X}(X \otimes Y, Z^*) &\cong \mathbb{X}(X \otimes Y, Z \multimap \bot), \text{by definition of } (_)^* \\
&\cong \mathbb{X}((X \otimes Y) \otimes Z, \bot), \text{as } \mathbb{X} \text{ is monoidal closed} \\
&\cong \mathbb{X}(X \otimes (Y \otimes Z), \bot), \text{associativity natural isomorphism} \\
&\cong \mathbb{X}(X, (Y \otimes Z) \multimap \bot, \text{as } \mathbb{X} \text{ is monoidal closed} \\
&\cong \mathbb{X}(X, (Y \otimes Z)^*), \text{by definition of } (_)^*.
\end{aligned}$$

Conversely, given the dualizing functor, we define the internal hom $A \multimap B := A^* \oplus B := (A^{**} \otimes B^*)^*$.

Next, we discuss the notion of a symmetric linearly distributive category, which, as we shall see, coincides with the notion of a star autonomous category precisely when the linearly distributive category also has duals. The notion of a linearly distributive category was introduced in [13].

Definition 2.2.12. A **linearly distributive category** (LDC) consists of the following data:

- A monoidal category $(\mathbb{X}, \otimes, \top, \alpha_\otimes, l_\otimes, r_\otimes)$- called the **tensor monoidal category** .
- A monoidal category $(\mathbb{X}, \oplus, \bot, \alpha_\oplus, l_\oplus, r_\oplus)$- called the **par monoidal category** .
- Two natural transformations $\delta^L : A \otimes (B \oplus C) \to (A \otimes B) \oplus C$, δ^R : $(B \oplus C) \otimes A \to B \oplus (C \otimes A)$, called the **left and right linear distributors.**

This data is subjected to the following coherence conditions:

(LDC 1) Pentagon and triangle identities for $(\mathbb{X}, \otimes, \top, \alpha_\otimes, l_\otimes, r_\otimes)$ and $(\mathbb{X}, \oplus, \bot, \alpha_\oplus, l_\oplus, r_\oplus)$ to be monoidal categories. See Definition 2.2.3.

(LDC 2)

$$\begin{tikzcd}
\top \otimes (A \oplus B) \arrow[d, "\delta^L"'] \arrow[dr, "l_\otimes^{A \oplus B}"] & \\
(\top \otimes A) \oplus B \arrow[r, "l_\otimes^A \oplus \mathrm{id}"'] & A \oplus B
\end{tikzcd}$$

(LDC 3)

$$\begin{tikzcd}
(A \otimes B) \otimes (C \oplus D) \arrow[r, "\alpha_\otimes"] \arrow[dd, "\delta^L"'] & A \otimes (B \otimes (C \oplus D)) \arrow[d, "\mathrm{id} \otimes \delta^L"] \\
& A \otimes ((B \otimes C) \oplus D) \arrow[d, "\delta^L"] \\
((A \otimes B) \otimes C) \oplus D \arrow[r, "\alpha_\otimes \oplus \mathrm{id}"'] & (A \otimes (B \otimes C)) \oplus D
\end{tikzcd}$$

(LDC 4)

$$\begin{tikzcd}
& (A \oplus B) \otimes (C \oplus D) \arrow[dl, "\delta^L"'] \arrow[dr, "\delta^R"] & \\
((A \oplus B) \otimes C) \oplus D) \arrow[d, "\delta^R \oplus \mathrm{id}"'] & & A \oplus (B \otimes (C \otimes D)) \arrow[d, "\mathrm{id} \oplus \delta^L"] \\
(A \oplus (B \otimes C)) \oplus D \arrow[rr, "\alpha_\oplus"'] & & A \oplus ((B \otimes C) \oplus D)
\end{tikzcd}$$

The above set of generating equations produces further coherence conditions under the following symmetries of the notion of an LDC.

- swap the arrows, swap $\otimes$ and $\oplus$, swap $\top$ and $\perp$
- reverse the tensor and par $A \otimes^{\mathsf{rev}} B := B \otimes A$, $A \oplus^{\mathsf{rev}} B := B \oplus A$

For example, **LDC 1** under the symmetry will generate further coherence conditions such as $r_\otimes = \delta^R(\mathrm{id} \oplus l_\otimes)$, $\delta^R l_\oplus = l_\oplus \otimes \mathrm{id}$, and $\mathrm{id} \otimes r_\oplus = \delta^L l_\oplus$.

An LDC is said to be symmetric if both underlying monoidal structures are symmetric and the following diagram commutes:

$$\begin{tikzcd}
(B \oplus C) \otimes A \arrow[r, "c_\otimes"] \arrow[ddddr, "\delta^R"'] & A \otimes (B \oplus C) \arrow[d, "\mathrm{id} \otimes c_\oplus"] \\
& A \otimes (C \oplus B) \arrow[d, "\delta^L"] \\
& (A \otimes C) \oplus B \arrow[d, "c_\oplus"] \\
& B \oplus (A \otimes C) \arrow[d, "\mathrm{id} \oplus c_\otimes"] \\
& B \oplus (C \otimes A)
\end{tikzcd}$$

Note that by the symmetries of the data, this means that in a symmetric LDC, the left and right distributors determine each other.

Example 2.2.13.

(i) Every (symmetric) monoidal category is a (symmetric) linearly distributive category (with $\otimes = \oplus$), with distributors as the associators.

(ii) A shift monoid (regarded as a discrete category, with only identity morphisms) is a commutative monoid $(A, +, 0)$ with a designated element a which has an inverse. This induces a second operation $b \times c := (b + c) - a$. The unit for $\times$ will be a. The distributor is then given by following equality:

$$x \times (y + z) = (x + (y + z)) - a = ((x + y) - a) + z = (x \times y) + z$$

(iii) Every $*$- autonomous category is a LDC with duals. See below.

Definition 2.2.14. Suppose $\mathbb{X}$ is a LDC and $A, B \in \mathbb{X}$, then B is said to be **left linear dual** to A (or equivalently, A is said to be **right linear dual** to B), written as $(\eta, \varepsilon) : B \dashv\!\!\| A$ if there exist maps $\eta : \top \to B \oplus A$, $\varepsilon : A \otimes B \to \bot$ such that the following diagrams commute:

(Dual 1)

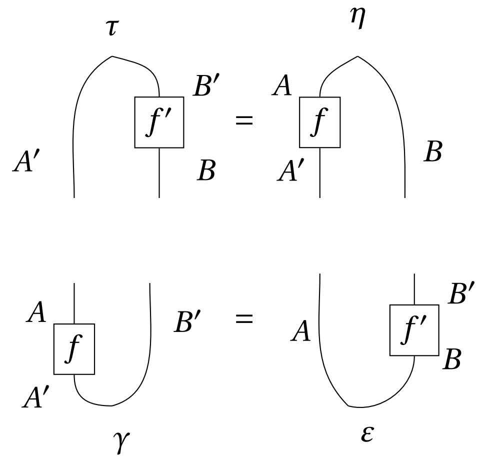
$$\begin{tikzcd}
B \arrow[r, "l_\otimes^{-1}"] \arrow[d, "\mathrm{id}"'] & \top \otimes B \arrow[r, "\eta\otimes\mathrm{id}"] & (B \oplus A) \otimes B \arrow[d, "\delta^R"] \\
B & B \oplus \bot \arrow[l, "r_\oplus"] & B \oplus (A \otimes B) \arrow[l, "\mathrm{id}\oplus\varepsilon"]
\end{tikzcd}$$

(Dual 2)

$$\begin{tikzcd}
A \arrow[r, "r_\otimes^{-1}"] \arrow[d, "\mathrm{id}"'] & A \otimes \top \arrow[r, "\mathrm{id}\otimes\eta"] & A \otimes (B \oplus A) \arrow[d, "\delta^L"] \\
A & \bot \oplus A \arrow[l, "l_\otimes"] & (A \otimes B) \oplus B \arrow[l, "\varepsilon\oplus\mathrm{id}"]
\end{tikzcd}$$

The above equations encode the triangle identities of the adjunction when the duals exist in a weaker sense than in a compact closed category. It is not difficult to see that in a symmetric LDC, $(\eta, \varepsilon) : B \multimap\!\!\| \; A$ if and only if $(\eta c_\oplus, c_\otimes \varepsilon) : A \multimap\!\!\| \; B$.

Definition 2.2.15. A **homomorphism of duals** $(f, f') : ((\eta, \varepsilon) : (A \multimap\!\!\| \; B) \to (\tau, \gamma) : (A' \multimap\!\!\| \; B'))$ is a pair of maps $f : A \to A'$ and $f' : B' \to B$ such that the following holds:



(2.2)

Note that in the string diagram language [38], the wires represent the type of the objects, and the morphisms between the objects are represented by boxes. Sequential composition is to drawn by stacking the boxes vertically (top to bottom) and the parallel composition is drawn by stacking the boxes horizontally (left to right).

In [14] (Theorem 4.3), Cockett and Seely proved that a symmetric LDC with duals is the same notion as a star autonomous category. We shall show how we can derive the left linear distributor coming from the data of a star autonomous category. First, we define the par on the LDC as $A \oplus B := A^* \multimap B \cong (A^* \otimes B^*)^*$. Then the left distributor is derived as follows:

$$\frac{(A \otimes B)^* \vdash (A \otimes B)^*}{A \otimes (A \otimes B)^* \vdash B^*} \qquad \frac{\dfrac{B^* \multimap C \vdash B^* \multimap C}{B^* \otimes (B^* \multimap C) \vdash C}}{B^* \vdash (B^* \multimap C) \multimap C}$$

$$\frac{A \otimes (A \otimes B)^* \vdash B^* \qquad B^* \vdash (B^* \multimap C) \multimap C}{A \otimes (A \otimes B)^* \vdash (B^* \multimap C) \multimap C}$$

Finally, we use the natural bijection between $X \otimes Y \to Z \multimap W$ and $X \otimes Z \to Y \multimap W$ to derive derive the distributor $\delta^L : A \otimes (B \oplus C) \to (A \otimes B) \oplus C$.

This theorem allows us to switch between the notions of an LDC with duals and that of a $*$- autonomous category. This will be important when we develop the theory of Lefschetz spaces.

2.3 Elementary topology

In this section, we recall some definitions from elementary topology.

Definition 2.3.1. A non-empty set A together with a binary relation $\leq$ is called a **directed set** when:

- $(A, \leq)$ is a preorder
- (Directed) For any $a, b \in A$, there exists $c \in A$ such that $a \leq c, b \leq c$.

Example 2.3.2. An important example of a directed set, and one that will be crucial to us is $(\mathrm{Fin}(I), \subseteq, \cup)$, where I is any indexing set, and $\mathrm{Fin}(I) := \{J \subseteq I : |J| < \infty\}$, that is $\mathrm{Fin}(I)$ is just the collection of finite subsets of I. The ordering is given by inclusion, and the direction is provided by taking the union of two sets.

Definition 2.3.3.

(a) A **net** in a topological space V is a function $v_\bullet : A \to V$, where A is a directed set.

(b) A net in a topological space V , $v_\bullet = (v_a)_{a\in A}$ is said to be **eventually in a set** $S \subseteq V$, if there exists $a \in A$ such that $v_b \in S$ for all $b \geq a$.

(c) A point v in a topological space is called a **limit point** of the net $v_\bullet$ in V if for every open neighborhood U of v, the net $v_\bullet$ is eventually in U. When a limit point for a net $v_\bullet$ exists in V, the net is said to be **convergent** (to some $v \in V$).

Definition 2.3.4. Let $(X, \lambda_1 \subseteq \mathcal{P}(X))$ and $(Y, \lambda_2 \subseteq \mathcal{P}(Y))$ be topological spaces, then a function $f : X \to Y$ is called **continuous** if it pulls back open sets to open sets, i.e. for any $t \in \lambda_2, f^{-1}(t) \in \lambda_1$.

An equivalent characterisation of continuous functions can be given as functions that preserve convergent nets. It should be noted that the reason we consider nets, and not just sequences, is that this equivalent characterisation might fail if we consider just sequences. That is to say, a $f : X \to Y$ taking every convergent sequence $x_n \to x$ in X to a convergent sequence $f(x_n) \to f(x)$ in Y might fail to be continuous. For Spaces where sequences suffice to capture continuity are called sequential spaces. Though a large class of topological spaces (including spaces where topology is induced by a metric) is sequential, we do not know if Lefschetz spaces, which we will be studying in this thesis, are sequential. Lefschetz spaces have not been studied very thoroughly in the literature, so we could not locate a reference that proves or disproves that Lefschtez spaces are sequential. In this thesis, we will not assume that Lefschtez spaces are sequential, and hence our results works in greater generality.

Lemma 2.3.5. [26] Let (X, λ_1) and (Y, λ_2) be topological spaces, then a function $f : X \to Y$ is continuous if and only if for every convergent net $x_\bullet \to x$ in X, the net $f(x_\bullet)$ converges to $f(x)$ in Y.

We next discuss neighborhoods and notions of basis for a topological space.

Definition 2.3.6.

(a) Let (X, λ) be a topological space. Then for $x \in X$, $N(x) := \{U \subseteq X : U \in \lambda \text{ and } x \in U\}$ is called the **open neighborhood system of** x.

(b) $B_x \subseteq N(x)$ is called a **local basis at** x if for every $U \in \lambda$ such that $x \in U$, there exists $B \in B_x$, such that $x \in B$ and $B \subseteq U$.

Definition 2.3.7.

(a) Let X be a set and let $\mathcal{B}$ be a collection of subsets of X such that

- $X = \cup_{V \in \mathcal{B}} V$ and
- for any $V_1, V_2 \in \mathcal{B}$ and for any $x \in V_1 \cap V_2$, there exists $W \in \mathcal{B}$ such that $x \in W$ and $W \subseteq V_1 \cap V_2$.

Then $\mathcal{B}$ is called a **basis for the topology** on X, where the topology is generated by taking arbitrary unions of the sets in $\mathcal{B}$.

(b) Let X be a set and let $\mathcal{S}$ be a collection of subsets of X such that $X = \cup_{T \in \mathcal{S}} T$.

Then $\mathcal{S}$ is called a **subbasis** for the topology on X, where the topology is generated by

- taking finite intersections of the sets in $\mathcal{S}$
- taking arbitrary unions of the sets obtained in the step above.

When we have the notion of a (sub)-basis for a topology, we can give another equivalent condition to check the continuity of a function.

Lemma 2.3.8. [31] Let (X,λ), (Y,θ) be two topological spaces and let $\mathcal{T}$ denote a (sub) basis for Y. Then a function $f : X \to Y$ is continuous if and only if for any $t \in \mathcal{T}$, $f^{-1}(t) \in \lambda$.

Definition 2.3.9. A **topological vector space** (TVS) is a vector space $(V, +, \bullet)$ (over a field say $\mathbb{C}$) endowed with a topology λ such that the vector space operations $+ : V \times V \to V$, $\bullet : \mathbb{C} \times V \to V$ are continuous. Note that the topology on the product space is the product topology, and the topology on the field is discrete.

The morphisms in the category of topological vector spaces are continuous linear maps. In any TVS X, translations $x \to x + x'$ (by any $x' \in X$), and scaling by a non-zero scalar $x \to \alpha \bullet x$ are isomorphisms in the category of topological vector spaces [39]. So, for a linear map between topological vector spaces, continuity at the origin (additive identity of the vector space structure) implies continuity everywhere. In terms of local bases, it has the following form:

Lemma 2.3.10. [36] Let X and Y be two topological vector spaces, and let B_{0_X} and C_{0_Y} be their respective local basis at their origins (which always exist; for example they could be just the neighbourhood basis at the origin).

Then a function $f : X \to Y$ is continuous if and only if for any $c \in C_{0_Y}$, there exists $b \in B_{0_X}$ such that $b \subseteq f^{-1}(c)$.

To talk about completeness in topological vector spaces, we introduce the notion of a uniform space. Uniform space is the general context in which the notion of completeness makes sense. A **uniform space** is simply a space endowed with a uniform structure. There are many equivalent definitions; we will adopt the entourage one. The definitions and results here on uniform spaces can be found in standard references on uniform spaces [9, 24].

Definition 2.3.11. Let X be a set. An **entourage** on X, γ, is a non-empty collection of relations on X, that is, $\gamma \subseteq \mathcal{P}(X \times X)$ such that

- For every $U \in \gamma$, $\mathrm{id}_X := \{(x, x) : x \in X\} \subseteq U$.
- γ is upward closed, that is if $U, V \in \mathcal{P}(X \times X)$ such that $U \subseteq V$ and $U \in \gamma$, then $V \in \gamma$.
- γ is closed to finite intersections, that is, if $U, V \in \gamma$, then $U \cap V \in \gamma$
- γ is closed to the (unary) operation of taking relational transpose, that is, if $U \in \gamma$, then $U^t \in \gamma$, where $U^t := \{(x, y) \in X \times X : (y, x) \in U\}$.
- For every $U \in \gamma$, there exists $V \in \gamma$, such that $V \circ V \subseteq U$, where '$\circ$' is relational composition (see the second example in Example 2.1.2).

Every topological vector space X comes equipped with a canonical uniform structure where we declare $M \subseteq X \times X$ to be an element of the entourage if there exists an open neighborhood around the origin, U, such that for any $(x, y) \in M$, $x - y \in U$. Henceforth, whenever we talk about notions such as Cauchy nets and completeness in a topological vector space, we will be talking in context to this canonical uniform structure.

Moreover, a uniform structure (X, γ) induces a topology on X by declaring a non-empty set $O \subseteq X$ to be open if and only if for any $o \in O$, there exists a $V \in \gamma$, such that $V[o] := \{t : (o, t) \in V\} \subseteq O$. For any topological vector space, the induced topology by its canonical uniform structure agrees with the original topology.

The nice thing about isomorphisms of uniform spaces (see below) is that they preserve uniform properties such as completeness in addition to topological properties such as Hausdorff-ness, connectedness, etc.

Definition 2.3.12. Given uniform spaces (X, γ), (Y, δ), a function $f : (X, \gamma) \to (Y, \delta)$ is said to be **uniformly continuous** if for any entourage $V \in \delta$, $(f \times f)^{-1}(V) \in \gamma$. Note that uniformly continuous functions are always continuous functions with respect to the induced topology by the uniform structure.

Lemma 2.3.13. Any linear continuous map $f : X \to Y$ between topological vector spaces X and Y is uniformly continuous (with respect to the canonical uniformity).

Proof. By definition of canonical uniformity on topological vector spaces, any entourage on Y is of the form $E_W = \{(y_1, y_2) \in Y \times Y : y_1 - y_2 \in W\}$, where W is an open neighbourhood around 0_Y. Now, by continuity, we have that there exists an open neighbourhood V around 0_X such that $f(V) \subseteq W$. Now we define $E_V := \{(x_1, x_2) \in X \times X : x_1 - x_2 \in V\}$. Clearly, E_V is an entourage on X. Then, by linearity, we have,

$$f(x_1) - f(x_2) = f(x_1 - x_2) \in f(V) \subseteq W.$$

So we have that $(f(x_1), f(x_2)) \in E_W$ and hence $(f \times f)(E_V) \subseteq E_W$ or equivalently, $E_V \subseteq (f \times f)^{-1}(E_W)$. Then, since E_V is an entourage on X, by upward closedness $(f \times f)^{-1}(E_W)$ is an entourage on Y. Thus, f is a uniformly continuous function. □

Definition 2.3.14.

(a) Let X be a TVS. And let $x_\bullet : I \to X$ be a net in X, where I is some directed set. Then the net is said to be **Cauchy** if for any open neighborhood U around the origin, there exists $i_o \in I$, such that $x_i - x_j \in U$ for all $i, j \geq i_o$.

(b) A topological vector space X is said to be **complete** if and only if every Cauchy net $x_\bullet : I \to X$ converges to some point $x \in X$.

Chapter 3

Mixed unitary categories

Having developed the background on basic category theory and topology, we define the notion of a mixed unitary category (MUC) in this chapter. The main mathematical objects of our study, that is finiteness spaces and reflexive Lefschetz spaces, are examples of mixed unitary categories. MUC were proposed in [40] as a categorical semantics for arbitrary-dimensional quantum mechanics. A MUC is a (†)- linearly distributive category (LDC) (read as "dagger LDC") with a unitary core. The unitary core is equivalent to the compact fragment of the theory. This core then extends to a full †- LDC, yielding the structure of a MUC. Thus, a MUC can be seen as capturing the idea of a finite-dimensional theory sitting within the larger setting of a †-LDC.

It is well known that dagger compact closed categories give a semantics for finite-dimensional quantum mechanics [1]. Compact closed categories are the proof theory for the compact fragment of linear logic with negation. A natural generalisation of a compact closed category is, thus, a $*$-autonomous category, where the tensor and par are not isomorphic. A further generalisation of $*$- autonomous category is a dagger ($\dagger$-) linearly distributive category (LDC), where the dagger need not necessarily come from the dualising functor.

The dagger structure also allows us to talk about unitary operators (in the core), which are of central importance in quantum mechanics.

The definitions and results in this chapter are taken from [40].

3.1 Dagger linearly distributive categories

The notion of a MUC will be developed in the setting of dagger isomix LDCs, where the unitary core is contained in the class of objects for which $A \otimes B \cong A \oplus B$, called the core of the category. This map between the tensor and the par is induced by having a map between the units of the par and the tensor.

Definition 3.1.1. A LDC $\mathbb{X}$ is said to be a **mix LDC**, if there exists a map $m : \bot \to \top$, called the *mix map*, such that the following diagram commutes for all objects $A, B \in \mathbb{X}$:

$$\begin{tikzcd}
A \otimes B \arrow[r, "\mathrm{id} \otimes u_\oplus^{L^{-1}}"] \arrow[d, "u_\oplus^{R^{-1}} \otimes \mathrm{id}"'] \arrow[rrdd, dashed, "mx_{A,B}"] & A \otimes (\bot \oplus B) \arrow[r, "\mathrm{id} \otimes (m \oplus \mathrm{id})"] & A \otimes (\top \oplus B) \arrow[d, "\delta^L"] \\
(A \oplus \bot) \otimes B \arrow[d, "\delta^R"'] & & (A \otimes \top) \oplus B \arrow[d, "u_\otimes^R"] \\
A \oplus (\bot \otimes B) \arrow[r, "\mathrm{id} \oplus (m \otimes \mathrm{id})"'] & A \oplus (\top \otimes B) \arrow[r, "\mathrm{id} \oplus u_\otimes^L"'] & A \oplus B
\end{tikzcd}$$

The map thus obtained, $mx_{A,B} : A \otimes B \to A \oplus B$, is called the **mixor**. It is a natural transformation as it is a composition of natural transformations.

Definition 3.1.2. A mix LDC is said to be an **isomix LDC** if and only if the mix map $m : \bot \to \top$ is an isomorphism.

We note that the mix map being an isomorphism does not entail that the mixor is an isomorphism. However, when the mix map is an isomorphism, the coherence required of the mixor is automatically satisfied [13]. When each mixor is an isomorphism, the LDC is said to be compact.

Definition 3.1.3. A **compact LDC** $\mathbb{X}$ is an isomix category where each mixor, $mx_{A,B} : A \otimes B \to A \oplus B$ is an isomorphism for all $A, B \in \mathbb{X}$.

We will not always be explicit in this quantification over objects. Unless further conditions are mentioned, the choice of objects is meant to be generic.

One of the ways in which compact LDCs arise is via the core of a mix category.

Definition 3.1.4. An object U is in the **core** of a mix category if and only if $mx_{U,-}$ and $mx_{-,U}$ are isomorphisms.

Thus, we can consider $\mathsf{Core}(\mathbb{X})$ to be a full subcategory of $\mathbb{X}$ which will be compact. We also have that the core is closed under tensor and par in a mix category, and if the category is isomix, then the units are in the core.

Proposition 3.1.5. [7] If $\mathbb{X}$ is a mix-LDC and $A, B \in \mathsf{Core}(\mathbb{X})$, then $A \oplus B$, $A \otimes B \in \mathsf{Core}(\mathbb{X})$. If $\mathbb{X}$ is an isomix- LDC, then $\bot, \top \in \mathsf{Core}(\mathbb{X})$.

We next need higher-dimensional cells of an LDC before we can make the notion of a †- LDC precise. For that, we first recall the definition of a monoidal functor.

Definition 3.1.6. Let $(\mathbb{X}, \otimes, I)$, $(\mathbb{Y}, \otimes, I)$ be two monoidal categories. Then a **monoidal functor** between them is a functor $F : \mathbb{X} \to \mathbb{Y}$ equipped with two natural transformations $m_\otimes : F(A) \otimes F(B) \to F(A \otimes B)$, $m_I : I \to F(I)$ such that the following diagrams commute:

(MF 1)

$$\begin{tikzcd}
(F(A) \otimes F(B)) \otimes F(C) \arrow[r, "m_\otimes \otimes \mathrm{id}"] \arrow[d, "\alpha_\otimes"'] & F(A \otimes B) \otimes F(C) \arrow[r, "m_\otimes"] & F((A \otimes B) \otimes C) \arrow[d, "F(\alpha_\otimes)"] \\
F(A) \otimes (F(B) \otimes F(C)) \arrow[r, "\mathrm{id} \otimes m_\otimes"'] & F(A) \otimes F(B \otimes C) \arrow[r, "m_\otimes"'] & F(A \otimes (B \otimes C))
\end{tikzcd}$$

(MF 2)

$$\begin{tikzcd}
F(A) \otimes I \arrow[d, "\mathrm{id} \otimes m_I"'] \arrow[rrr, "u_\otimes^R"] & & & F(A) \\
F(A) \otimes F(I) \arrow[rr, "m_\otimes"'] & & F(A \otimes I) \arrow[r, "F(u_\otimes^R)"'] & F(A)
\end{tikzcd}$$

(MF 3)

$$\begin{tikzcd}
I \otimes F(A) \arrow[d, "m_I \otimes \mathrm{id}"'] \arrow[rrr, "u_\otimes^L"] & & & F(A) \\
F(I) \otimes F(A) \arrow[rr, "m_\otimes"'] & & F(I \otimes A) \arrow[r, "F(u_\otimes^L)"'] & F(A)
\end{tikzcd}$$

A functor $F : \mathbb{X} \to \mathbb{Y}$ between two monoidal categories is called **comonoidal** if $F : \mathbb{X}^{\mathsf{op}} \to \mathbb{Y}^{\mathsf{op}}$ is monoidal. When the functor is comonoidal, we will denote the associated natural transformations by $n_\otimes(n_\oplus)$, $n_\top(n_\bot)$.

A linear functor between two LDCs is then a pair of monoidal functors satisfying some coherence conditions. More precisely,

Definition 3.1.7. A **linear functor** $F : \mathbb{X} \to \mathbb{Y}$ between two LDCs is a pair of functors $(F_\otimes, F_\oplus)$ such that $F_\otimes$ is monoidal with respect to $\otimes$ and $F_\oplus$ is comonoidal with respect to $\oplus$, and we have the following natural transformations

- $v_{\otimes}^{R} : F_{\otimes}(A \oplus B) \to F_{\oplus}(A) \oplus F_{\otimes}(B)$
- $v_{\otimes}^{L} : F_{\otimes}(A \oplus B) \to F_{\otimes}(A) \oplus F_{\oplus}(B)$
- $v_{\oplus}^{R} : F_{\otimes}(A) \otimes F_{\oplus}(B) \to F_{\oplus}(A \otimes B)$
- $v_{\oplus}^{L} : F_{\oplus}(A) \otimes F_{\otimes}(B) \to F_{\oplus}(A \otimes B)$

such that the following diagrams commute:

(LF 1)

$$\begin{tikzcd}
F_{\otimes}(\perp \oplus A) \arrow[r, "F_{\otimes}(u_{\oplus}^{L})"] \arrow[d, "v_{\otimes}^{R}"'] & F_{\otimes}(A) \\
F_{\oplus}(\perp) \oplus F_{\otimes}(A) \arrow[r, "n_{\perp} \oplus \mathrm{id}"'] & \perp \oplus F_{\otimes}(A) \arrow[u, "u_{\oplus}^{L}"']
\end{tikzcd}$$

(LF 2)

$$\begin{tikzcd}
F_{\otimes}((A \oplus B) \oplus C) \arrow[r, "F_{\otimes}(\alpha_{\oplus})"] \arrow[d, "v_{\otimes}^{R}"'] & F_{\otimes}(A \oplus (B \oplus C)) \arrow[d, "v_{\otimes}^{R}"] \\
F_{\oplus}(A \oplus B) \oplus F_{\otimes}(C) \arrow[d, "n_{\oplus} \oplus \mathrm{id}"'] & F_{\oplus}(A) \oplus F_{\otimes}(B \oplus C) \arrow[d, "\mathrm{id} \oplus v_{\otimes}^{R}"] \\
(F_{\oplus}(A) \oplus F_{\oplus}(B)) \oplus F_{\otimes}(C) \arrow[r, "\alpha_{\oplus}"'] & F_{\oplus}(A) \oplus (F_{\oplus}(B) \oplus F_{\otimes}(C))
\end{tikzcd}$$

(LF 3)

$$\begin{tikzcd}
F_{\otimes}((A \oplus B) \oplus C) \arrow[r, "F_{\otimes}(\alpha_{\oplus})"] \arrow[d, "v_{\otimes}^{L}"'] & F_{\otimes}(A \oplus (B \oplus C)) \arrow[d, "v_{\otimes}^{R}"] \\
F_{\otimes}(A \oplus B) \oplus F_{\oplus}(C) \arrow[d, "v_{\otimes}^{R} \oplus \mathrm{id}"'] & F_{\oplus}(A) \oplus F_{\otimes}(B \oplus C) \arrow[d, "\mathrm{id} \oplus v_{\otimes}^{L}"] \\
(F_{\oplus}(A) \oplus F_{\otimes}(B)) \oplus F_{\oplus}(C) \arrow[r, "\alpha_{\oplus}"'] & F_{\oplus}(A) \oplus (F_{\otimes}(B) \oplus F_{\oplus}(C))
\end{tikzcd}$$

(LF 4)

$$\begin{tikzcd}
F_\otimes(A) \otimes F_\otimes(B \oplus C) \arrow[r, "\mathrm{id} \otimes \nu_\otimes^R"] \arrow[d, "m_\otimes"'] & F_\otimes(A) \otimes (F_\oplus(B) \oplus F_\otimes(C)) \arrow[d, "\delta^L"] \\
F_\otimes(A \otimes (B \oplus C)) \arrow[d, "F_\otimes(\delta^L)"'] & (F_\otimes(A) \otimes F_\oplus(B)) \oplus F_\otimes(C) \arrow[d, "\nu_\oplus^R \oplus \mathrm{id}"] \\
F_\otimes((A \otimes B) \oplus C) \arrow[r, "\nu_\otimes^R"'] & F_\oplus(A \oplus B) \oplus F_\otimes(C)
\end{tikzcd}$$

(LF 5)

$$\begin{tikzcd}
F_\otimes(A) \otimes F_\otimes(B \oplus C) \arrow[r, "\mathrm{id} \otimes \nu_\otimes^L"] \arrow[d, "m_\otimes"'] & F_\otimes(A) \otimes (F_\otimes(B) \oplus F_\oplus(C)) \arrow[d, "\delta^L"] \\
F_\otimes(A \otimes (B \oplus C)) \arrow[d, "F_\otimes(\delta^L)"'] & (F_\otimes(A) \otimes F_\otimes(B)) \oplus F_\oplus(C) \arrow[d, "m_\otimes \oplus \mathrm{id}"] \\
F_\otimes((A \otimes B) \oplus C) \arrow[r, "\nu_\otimes^L"'] & F_\otimes(A \otimes B) \oplus F_\otimes(C)
\end{tikzcd}$$

There are other coherence conditions coming from symmetries of the data, such as simultaneously switching $\otimes$ with $\oplus$ and L with R or taking the inverse of these natural isomorphisms while switching $m_\top$ with $n_\perp$ and so on. We will not give a full list here. But the full definition can be found in [15, 40]. For example, see Chapter-2 , Section 2 of [40].

A linear functor between symmetric LDCs is said to be **symmetric** if and only if the following additional diagrams commute:

(LFS 1)

$$\begin{tikzcd}
F_\otimes(A) \otimes F_\otimes(B) \arrow[r, "m_\otimes"] \arrow[d, "c_\otimes"'] & F_\otimes(A \otimes B) \arrow[d, "F_\otimes(c_\otimes)"] \\
F_\otimes(B) \otimes F_\otimes(A) \arrow[r, "m_\otimes"'] & F_\otimes(B \otimes A)
\end{tikzcd}$$

(LFS 2)

$$\begin{tikzcd}
F_\oplus(A \oplus B) \arrow[r, "n_\oplus"] \arrow[d, "F_\oplus(c_\oplus)"'] & F_\oplus(A) \oplus F_\oplus(B) \arrow[d, "c_\oplus"] \\
F_\oplus(B \oplus A) \arrow[r, "n_\oplus"'] & F_\oplus(B) \oplus F_\oplus(A)
\end{tikzcd}$$

We will be particularly interested in Frobenius functors, which are a degenerate form of a linear functor, having the same monoidal and comonoidal components.

Definition 3.1.8. A linear functor $F : \mathbb{X} \to \mathbb{Y}$ is called a **Frobenius functor** if and only if

(FF 1) $F_\otimes = F_\oplus$,

(FF 2) $m_\otimes = v_\oplus^L = v_\oplus^R$, and

(FF 3) $n_\oplus = v_\otimes^L = v_\otimes^R$.

These Frobenius functors will be called (iso) mix if they also preserve the mix structure.

Definition 3.1.9. Suppose $\mathbb{X}$ and $\mathbb{Y}$ are mix categories. Then a Frobenius functor $F : \mathbb{X} \to \mathbb{Y}$ is called a **mix functor** if the following diagram commutes:

$$\begin{tikzcd}
F(\bot) \arrow[r, "n_\bot"] \arrow[rrr, bend right, "F(m)"'] & \bot \arrow[r, "m"] & \top \arrow[r, "m_\top"] & F(\top)
\end{tikzcd}$$

Definition 3.1.10. Suppose $\mathbb{X}$ and $\mathbb{Y}$ are isomix categories. Then a mix functor $F : \mathbb{X} \to \mathbb{Y}$ is called an **isomix functor** if the following additional diagram commutes:

$$
\begin{tikzcd}
\top \arrow[rrr, bend left, "m^{-1}"] \arrow[dr, "m_\top"'] & & & \bot \\
& F(\top) \arrow[r, "F(m^{-1})"'] & F(\bot) \arrow[ur, "n_\bot"'] &
\end{tikzcd}
$$

When either $n_\bot : F(\bot) \to \bot$ or $m_\top : \top \to F(\top)$ is an isomorphism, then any mix functor between isomix categories is automatically an isomix functor (See Lemma 2.26 of [40]).

Now, for the 2- cells, we first recall the definition of a monoidal transformation.

Definition 3.1.11. A **monoidal transformation** $\alpha : F \Rightarrow G$ between two monoidal functors is a natural transformation $\alpha : F \Rightarrow G$ such that the following diagrams commute:

(MN 1)

$$
\begin{tikzcd}
F(A) \otimes F(B) \arrow[r, "\alpha(A)\otimes\alpha(B)"] \arrow[d, "m_\otimes^F"'] & G(A) \otimes G(B) \arrow[d, "m_\otimes^G"] \\
F(A \otimes B) \arrow[r, "\alpha_{A\otimes B}"'] & G(A \otimes B)
\end{tikzcd}
$$

(MN 2)

$$
\begin{tikzcd}
I \arrow[d, "m_I^F"'] \arrow[dr, "m_I^G"] & \\
F(I) \arrow[r, "\alpha_I"'] & G(I)
\end{tikzcd}
$$

Again, a linear natural transformation will be a pair of natural transformations with some coherence conditions.

Definition 3.1.12. A **linear natural transformation** $\alpha : F \Rightarrow G$ between two linear functors consists of a pair of natural transformations $(\alpha_\otimes, \alpha_\oplus)$

such that $\alpha_\otimes : F_\otimes \Rightarrow G_\otimes$ and $\alpha_\oplus : F_\oplus \Rightarrow G_\oplus$ are monoidal transformations such that the following diagrams commute:

(LN 1)

$$\begin{tikzcd}
F_\otimes(A \oplus B) \arrow[rr, "\alpha_\otimes"] \arrow[d, "v_\otimes^R"'] & & G_\otimes(A \oplus B) \arrow[d, "v_\otimes^R"] \\
F_\oplus(A) \oplus F_\otimes(B) \arrow[dr, "\mathrm{id} \oplus \alpha_\otimes"'] & & G_\oplus(A) \oplus G_\otimes(B) \arrow[dl, "\alpha_\oplus \oplus \mathrm{id}"] \\
& F_\oplus(A) \oplus G_\otimes(B) &
\end{tikzcd}$$

(LN 2) $\alpha_\otimes v_\otimes^L(\mathrm{id} \oplus \alpha_\oplus) = v_\otimes^L(\alpha_\otimes \oplus \mathrm{id})$

(LN 3) $(\mathrm{id} \otimes \alpha_\otimes) v_\oplus^L(\alpha_\oplus) = (\alpha \oplus \otimes \mathrm{id}) v_\oplus^L$

(LN 4) $(\alpha_\otimes \otimes \mathrm{id}) v_\oplus^R \alpha_\oplus = (\mathrm{id} \otimes \alpha_\oplus) v_\oplus^R$

An **adjunction of linear functors** $(\eta, \varepsilon) : F \dashv G$ is an adjunction in the usual sense in the 2- category of LDCs with linear functors and linear transformations. In particular, such an adjunction leads to a pair of monoidal adjunctions [29], $(\eta_\otimes, \varepsilon_\otimes) : F_\otimes \dashv G_\otimes$ and $(\eta_\oplus, \varepsilon_\oplus) : F_\oplus \dashv G_\oplus$. An equivalence is a linear adjunction in which both the unit and counit are linear natural isomorphisms.

Remark 3.1.13. For any isomix category $\mathbb{X}$, we always have two linear functors, $Mx_\downarrow : (\mathbb{X}, \otimes, \otimes) \to (\mathbb{X}, \otimes, \oplus)$ and $Mx_\uparrow : (\mathbb{X}, \oplus, \oplus) \to (\mathbb{X}, \otimes, \oplus)$. For example, the monoidal and comonoidal components of $Mx_\downarrow$ are given by $(\mathrm{id}, \mathrm{id}, \mathrm{id})$, $(\mathrm{id}, mx, m^{-1})$ respectively. When $\mathbb{X}$ is a compact LDC, then $\mathsf{Core}(\mathbb{X}) = \mathbb{X}$ and hence the mixor will be an isomorphism. Thus, a compact LDC is (linearly) equivalent to a monoidal category.

Now we are ready to state the definition of a †- LDC, and later of a †-isomix category in which the notion of a MUC is developed. We first note that the dagger on a LDC can not be stationary on objects. By reverse of a map $f : A \to B$, we simply mean a map $f^{\mathsf{r}} : B \to A$. The reverse of the (left) linear distributor $\delta^{L^{\mathsf{r}}} : (A \otimes B) \oplus C \to A \otimes (B \oplus C)$ is not a valid map in LDCs. By $\delta^{L^{\mathsf{r}}}$ not being valid, we mean that this map is not derivable in the sequent calculus of LDCs. So if there were a dagger on LDC which is stationary on objects then the dagger of δ^{L} will give a map of the type $\delta^{L^{\mathsf{r}}}$, which is not valid in an LDC. Therefore, the dagger must minimally be non-stationary on objects in LDCs. See [8, 40] for further details.

Definition 3.1.14.

(a) Let $(\mathbb{X}, \otimes, \top, \oplus, \bot)$ be a LDC. Then by an **opposite linearly distributive category** $(\mathbb{X}, \otimes, \top, \oplus, \bot)^{\mathsf{op}}$, we mean $(\mathbb{X}^{\mathsf{op}}, \oplus, \bot, \otimes, \top)$, where $\mathbb{X}^{\mathsf{op}}$ is the usual opposite category and $\otimes^{\mathsf{op}} := \oplus$, $\oplus^{\mathsf{op}} := \otimes$, $\top^{\mathsf{op}} := \bot$, $\bot^{\mathsf{op}} := \top$.

(b) Let $(F_\otimes, F_\oplus) : (\mathbb{X}, \otimes, \top, \oplus, \bot) \to (\mathbb{Y}, \otimes, \top, \oplus, \bot)$ be a linear functor. Then the **opposite linear functor** is given by $(F_\oplus^{\mathsf{op}}, F_\otimes^{\mathsf{op}})$.

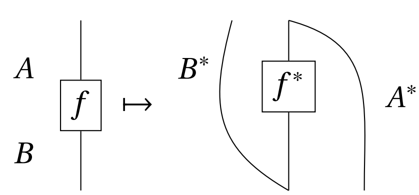
Now, though the dagger is non-stationary on objects in an LDC, it is still an involutive equivalence. More precisely,

Definition 3.1.15. [12] A **dagger linearly distributive category** (†- LDC), is a LDC $\mathbb{X}$, with a contravariant Frobenius linear functor $(\_)^\dagger : \mathbb{X}^{\mathsf{op}} \to \mathbb{X}$ which is a linear involutive equivalence $(\_)^\dagger \dashv\!\!\| \ (\_)^{\dagger\mathsf{op}} : \mathbb{X}^{\mathsf{op}} \to \mathbb{X}$

Now observe that since the dagger is an involutive equivalence and the domain category is $\mathbb{X}^{\mathsf{op}}$, this means that the unit and counit map is the same, which we will simply denote by $\iota : X \to X^{\dagger\dagger}$. By definition of linear functors and linear transformations, it means that we have laxors $\lambda_\otimes : A^\dagger \otimes B^\dagger \to (A \oplus B)^\dagger$, $\lambda_\oplus : A^\dagger \oplus B^\dagger \to (A \otimes B)^\dagger$, $\lambda_\top : \top \to \bot^\dagger$, $\lambda_\bot : \bot \to \top^\dagger$. The first two are called tensor laxors, and the latter two are called unit laxors. Recall that for a LDC, $(\mathbb{X}, \otimes, \top, \oplus, \bot)^{\mathsf{op}}$, the opposite LDC is defined to be $(\mathbb{X}^{\mathsf{op}}, \oplus, \bot, \otimes, \top)$, and therefore these laxors flip between the tensor and the par. These laxors satisfy various coherence conditions, of which we will list a few here; the full list can be found in [40].

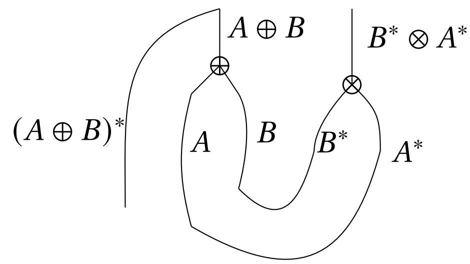
- $(\mathrm{id} \otimes \lambda_\otimes)\lambda_\otimes \alpha_\oplus^\dagger = \alpha_\otimes^{-1}(\lambda_\otimes \otimes \mathrm{id})\lambda_\otimes$
- $u_\otimes^R (u_\oplus^R)^\dagger = (\lambda_\top \otimes \mathrm{id})\lambda_\otimes$
- $(\mathrm{id} \otimes \lambda_\oplus)\lambda_\otimes = \delta^L(\lambda_\otimes \oplus \mathrm{id})\lambda_\oplus$
- $(\iota \oplus \iota)\lambda_\oplus = \iota \lambda_\otimes^\dagger$
- $\lambda_\bot = \iota \lambda_\top^\dagger$
- $\iota_{A^\dagger} = (\iota_A^{-1})^\dagger$

Note that the last condition is the triangle identity of the adjunction, $(\iota, \iota) : (\_)^\dagger \dashv (\_)^{\dagger\mathsf{op}} : \mathbb{X}^{\mathsf{op}} \to \mathbb{X}$. When the LDC structure is inherited from the $*$- autonomous category structure, the $(\_)^*$ functor gives us a Frobenius linear functor on the LDC thus inherited, $(\_)^* : \mathbb{X}^{\mathsf{oprev}} \to \mathbb{X}$. We define $(\mathbb{X}, \otimes, \top, \oplus, \bot)^{\mathsf{rev}} := (\mathbb{X}, \otimes^{\mathsf{rev}}, \top, \oplus^{\mathsf{rev}}, \bot)$, where $A \otimes^{\mathsf{rev}} B := B \otimes A$, $A \oplus^{\mathsf{rev}} B := B \oplus A$. The dual assignment extends as a Frobenius functor as follows:

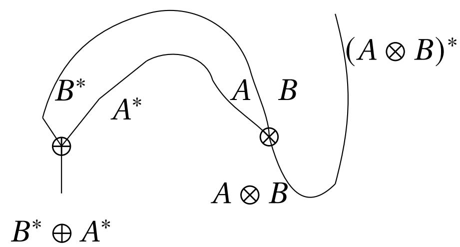


(3.1)

And the tensor laxors $m_\otimes : B^* \otimes A^* \to (A \oplus B)^*, \eta_\oplus : (A \otimes B)^* \to B^* \oplus A^*$ are given as follows

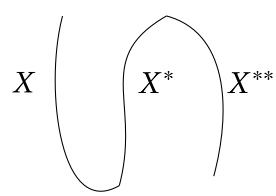


(3.2)



(3.3)

Similarly, the unit laxors are given by using the unit and counit map of the dual adjunction. Moreover, the dualizing functor becomes an equivalence of Frobenius linear functors. The unit and the counit is given by the snake diagram



(3.4)

And we know that $\iota : X \to X^{**}$ is an isomorphism by the definition of a $*$-autonomous category.

Thus, we have,

Lemma 3.1.16. Suppose $(\mathbb{X}, \otimes, I, \multimap, \bot)$ is a $*$- autonomous category. Then $(\iota, \iota) : (\_)^* \dashv (\_)^{*^{\mathsf{oprev}}} : \mathbb{X}^{\mathsf{oprev}} \to \mathbb{X}$ is a Frobenius involutive equivalence.

This tells us that if the LDC structure is inherited from a star autonomous category, then we can define the dagger to be the dualizing functor to get an example of a $\dagger$- LDC.

A †- (iso) mix category is then a †- LDC, where the dagger also preserves the (iso) mix structure.

Definition 3.1.17.

(a) A †- LDC $\mathbb{X}$ is said to be a †- **mix** category if $\mathbb{X}$ is a mix category and the dagger functor $(\_)^\dagger : \mathbb{X}^{\mathsf{op}} \to \mathbb{X}$ is a mix functor.

(b) A †- LDC $\mathbb{X}$ is said to be a †- **isomix** category if $\mathbb{X}$ is an isomix category and the dagger functor $(\_)^\dagger : \mathbb{X}^{\mathsf{op}} \to \mathbb{X}$ is a isomix functor.

The maps between †- LDCs are given by linear functors which also preserve the dagger structure. More precisely,

Definition 3.1.18. A linear functor $F : \mathbb{X} \to \mathbb{Y}$ between †- LDCs is said to be a †- **linear functor** if F is equipped with linear natural isomorphisms $\rho_\otimes : F_\otimes(A^\dagger) \to F_\oplus(A)^\dagger, \rho_\oplus : F_\oplus(A^\dagger) \to F_\otimes(A)^\dagger$ such that the following diagrams commute:

$$\begin{tikzcd}
F_\otimes(A) \arrow[r, "\iota"] \arrow[d, "F_\otimes(\iota)"'] & F_\otimes(A)^{\dagger\dagger} \arrow[d, "\rho_\oplus^\dagger"] \\
F_\otimes(A^{\dagger\dagger}) \arrow[r, "\rho_\otimes"'] & F_\oplus(A^\dagger)^\dagger
\end{tikzcd}
\qquad
\begin{tikzcd}
F_\oplus(A) \arrow[r, "F_\oplus(A)^{\dagger\dagger}"] \arrow[d, "F_\oplus(\iota)"'] & F_\oplus(A)^{\dagger\dagger} \arrow[d, "\rho_\otimes^\dagger"] \\
F_\oplus(A^{\dagger\dagger}) \arrow[r, "\rho_\oplus"'] & F_\otimes(A^\dagger)^\dagger
\end{tikzcd}$$

Now the dagger in a LDC often could be just the dualizing functor, but at times, we compose this with a conjugative functor, which becomes our dagger structure on the LDC. The conjugative functor categorifies the conjugation of the complex numbers (seen as a discrete category). More precisely,

Definition 3.1.19. [18] A **conjugative monoidal category** is a monoidal category $(\mathbb{X}, \otimes, I)$, equipped with a functor $\overline{(\_)} : \mathbb{X}^{\mathsf{rev}} \to \mathbb{X}$, with natural isomorphisms $\xi : \overline{A} \otimes \overline{B} \to \overline{B \otimes A}, \varepsilon_A : \overline{\overline{A}} \to A$ such that the following diagrams commute:

(CM 1) $\overline{\varepsilon_A} = \varepsilon_{\overline{A}}$

(CM 2)

$$\begin{tikzcd}
(\overline{A} \otimes \overline{B}) \otimes \overline{C} \arrow[rr, "\alpha_\otimes"] \arrow[d, "\xi \otimes \mathrm{id}"'] & & \overline{A} \otimes (\overline{B} \otimes \overline{C}) \arrow[d, "\mathrm{id} \otimes \xi"] \\
(\overline{B \otimes A}) \otimes \overline{C} \arrow[d, "\xi"'] & & \overline{A} \otimes \overline{C \otimes B} \arrow[d, "\xi"] \\
\overline{C \otimes (B \otimes A)} \arrow[rr, "\overline{\alpha_\otimes^{-1}}"'] & & \overline{(C \otimes B) \otimes A}
\end{tikzcd}$$

(CM 3)

$$\begin{tikzcd}
\overline{\overline{A}} \otimes \overline{\overline{B}} \arrow[rr, "\xi"] \arrow[d, "\varepsilon \otimes \varepsilon"'] & & \overline{\overline{B} \otimes \overline{A}} \arrow[d, "\overline{\xi}"] \\
A \otimes B & & \overline{\overline{A \otimes B}} \arrow[ll, "\varepsilon"]
\end{tikzcd}$$

A symmetric monoidal category which is conjugative is called **symmetric conjugative** if the following additional diagram commutes:

$$\begin{tikzcd}
\overline{A} \otimes \overline{B} \arrow[r, "\xi"] \arrow[d, "c_\otimes"'] & \overline{B \otimes A} \arrow[d, "\overline{c_\otimes}"] \\
\overline{B} \otimes \overline{A} \arrow[r, "\xi"'] & \overline{A \otimes B}
\end{tikzcd}$$

Definition 3.1.20. A LDC $(\mathbb{X}, \otimes, \top, \oplus, \bot)$ together with a functor $\overline{(\_)} : \mathbb{X} \to \mathbb{X}$ is said to be a **conjugative LDC** if and only if $(\mathbb{X}, \otimes, \top, \xi_\otimes, \varepsilon)$, $(\mathbb{X}, \oplus, \bot, \xi_\oplus, \varepsilon)$ are conjugative monoidal categories and the following coherence conditions hold:

(CLDC 1) $(\xi_\oplus \otimes \mathrm{id})\delta^R(\mathrm{id} \oplus \xi_\oplus) = \xi_\otimes \overline{\delta^L} \xi_\oplus$.

(CLDC 2) $\xi_\oplus \overline{\delta^R} \xi_\oplus = (\mathrm{id} \otimes \xi_\oplus)\delta^L(\xi_\otimes \oplus \mathrm{id})$.

It is clear that the conjugation is a Frobenius functor $((\_)_\otimes = (\_)_\oplus)$ and moreover an equivalence because by definition $\varepsilon : \overline{\overline{A}} \to A$ is a natural isomorphism.

3.2 Unitary and mixed unitary categories

In a †- category, an isomorphism $f : A \to B$ is said to be unitary if and only if $ff^\dagger = \mathrm{id}_A$ and $f^\dagger f = \mathrm{id}_B$. But in a LDC, the dagger is not stationary on objects, so we can not compose f and $f^\dagger$, as their types do not match. But minimally, if we have $A \cong A^\dagger$ and $B \cong B^\dagger$ and these isomorphisms behave nicely with the linear structure, then we can proceed to have a coherent theory. This is the motivation to develop the unitary structure in a †- isomix category.

Definition 3.2.1. A †- isomix category $\mathbb{X}$ is said to have **unitary structure** if there is small class of objects $\mathcal{U} \subseteq \mathbb{X}$, called the **unitary objects** of $\mathbb{X}$ such that :

(U 1) For all $A \in \mathcal{U}$, $A \in \mathrm{Core}(\mathbb{X})$ and A is equipped with an isomorphism $\varphi_A : A \to A^\dagger$, called the **unitary structure map** of A.

(U 2) $\mathcal{U}$ is closed under $(\_)^\dagger$, and for all $A \in \mathcal{U}$, $\varphi_{A^\dagger} = (\varphi_A^{-1})^\dagger$.

(U 3) For all $A \in \mathcal{U}$, the following diagram commutes:

$$\begin{tikzcd}
A \arrow[d, "\varphi_A"'] \arrow[dr, "\iota"] & \\
A^\dagger \arrow[r, "\varphi_{A^\dagger}"'] & A^{\dagger\dagger}
\end{tikzcd}$$

(U 4) $\top, \bot \in \mathcal{U}$ and they satisfy

$$\begin{tikzcd}
\bot \arrow[r, "\varphi_\top"] \arrow[d, "\lambda_\bot"'] \arrow[dr, "m"] & \bot^\dagger \arrow[d, "\lambda_\top^{-1}"] \\
\top^\dagger \arrow[r, "\varphi_\top^{-1}"'] & \top
\end{tikzcd}$$

(U 5) If $A, B \in \mathcal{U}$, then $A \otimes B, A \oplus B \in \mathcal{U}$ and they satisfy

$$\begin{tikzcd}
A \otimes B \arrow[r, "\varphi_A \otimes \varphi_B"] \arrow[rrr, bend right, "mx"'] & A^\dagger \otimes B^\dagger \arrow[r, "\lambda_\oplus"] & (A \oplus B)^\dagger \arrow[r, "\varphi_{A \oplus B}^{-1}"] & A \oplus B
\end{tikzcd}$$

$$\begin{tikzcd}
A \otimes B \arrow[r, "\varphi_{A \otimes B}"] \arrow[rrr, bend right, "mx"'] & (A \otimes B)^\dagger \arrow[r, "\lambda_\otimes^{-1}"] & A^\dagger \oplus B^\dagger \arrow[r, "\varphi_A^{-1} \oplus \varphi_B^{-1}"] & A \oplus B
\end{tikzcd}$$

It is important to note that there is no demand of naturality on the unitary structure map. Thus, the unitary structure is really a structure we choose to put on a $\dagger$- isomix category and is not a property inherent to the category. Any $\dagger$- dagger monoidal category $\mathbb{X}$ where the dagger is stationary on objects (See Definition 2.2.7), can be given a unitary structure by defining $\varphi_A := \mathrm{id}_A$ for any $A \in \mathbb{X}$.

Lemma 3.2.2. If A is a unitary object in a $\dagger$- isomix category, then $\varphi_{A^{\dagger\dagger}} = \varphi_A^{\dagger\dagger}$

Proof. Note the following

$$\begin{aligned}
\varphi_{A^{\dagger\dagger}} &= (\varphi_{A^\dagger}^{-1})^\dagger \\
&= ((((\varphi_A)^{-1})^\dagger)^{-1})^{-1} \\
&= ((((\varphi_A)^{-1})^{-1})^\dagger)^\dagger \\
&= \varphi_A^{\dagger\dagger},
\end{aligned}$$

where the first two equalities are by (**U 2**) and the third equality is by functoriality of dagger. ◻

Now, when we have a unitary structure, we can define the notion of a unitary morphism.

Definition 3.2.3. If $f : A \to B$ is an isomorphism between two unitary objects, then it is said to be **a unitary isomorphism** if and only if the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "\varphi_A"] \arrow[d, "f"'] & A^\dagger \\ B \arrow[r, "\varphi_B"'] & B^\dagger \arrow[u, "f^\dagger"'] \end{tikzcd}$$

We note down a few lemmas about the unitary isomorphisms in a †-isomix category.

Lemma 3.2.4. Unitary isomorphisms compose, that is, if $f : A \to B$, $g : B \to C$ are unitary isomorphisms, then their composite, $fg : A \to C$, is unitary.

Proof. Note down the following series of equalities,

$$\begin{aligned} \varphi_A &= f \varphi_B f^\dagger \\ &= f g \varphi_C g^\dagger f^\dagger \\ &= f g \varphi_C (fg)^\dagger, \end{aligned}$$

where the first two equalities use the unitarity of f and g respectively, and the last equality is using functoriality of the dagger. $\square$

Lemma 3.2.5. An isomorphism in a †- isomix category $f : A \to X$ between two unitary objects is unitary if and only if $f^\dagger : X^\dagger \to A^\dagger$ is unitary.

Proof. First when f is unitary, then $\varphi_X = f^{-1}\varphi_A(f^\dagger)^{-1}$, so we have

$$\begin{aligned} \varphi_{X^\dagger} &= ((f^{-1}\varphi_A(f^\dagger)^{-1})^{-1})^\dagger \\ &= (f^\dagger\varphi_A^{-1}f)^\dagger \\ &= f^\dagger(\varphi_A^{-1})^\dagger f^{\dagger\dagger} \\ &= f^\dagger\varphi_{A^\dagger}f^{\dagger\dagger}. \end{aligned}$$

So $f^\dagger$ is unitary. And when $f^\dagger$ is unitary, we have

$$\begin{aligned} (\varphi_A^{-1})^\dagger &= (f^\dagger)^{-1}\varphi_{X^\dagger}(f^{\dagger\dagger})^{-1} \\ &= (f^\dagger)^{-1}(\varphi_X^{-1})^\dagger(f^{\dagger\dagger})^{-1} \\ &= ((f^\dagger)^{-1}\varphi_X^{-1}f^{-1})^\dagger. \end{aligned}$$

So by faithfulness of † functor, we have $\varphi_A^{-1} = (f^\dagger)^{-1}\varphi_X^{-1}f^{-1}$, and hence $\varphi_A = f\varphi_X f^\dagger$, so f is unitary. □

We also have that the unitary structure map itself is a unitary isomorphism.

Lemma 3.2.6. The unitary structure map $\varphi_A : A \to A^\dagger$ is a unitary isomorphism for any $A \in \mathcal{U}$.

Proof. For the unitary structure map, we have:

$$\varphi_A\varphi_{A^\dagger}\varphi_A^\dagger = \varphi_A(\varphi_A^{-1})^\dagger\varphi_A^\dagger = \varphi_A(\varphi_A\varphi_A^{-1})^\dagger = \varphi_A.$$

So $\varphi_A : A \to A^\dagger$ is a unitary isomorphism. □

Lemma 3.2.7. For any $A \in \mathcal{U}$, $\varphi_A = \iota_A\varphi_A^\dagger$.

Proof. We have

$$\begin{aligned} \iota_A &= \varphi_A \varphi_{A^\dagger} \\ &= \varphi_A (\varphi_A^{-1})^\dagger, \end{aligned}$$

where the first equality is by (**U 3**) and the second equality is by (**U 2**). Now this implies,

$$\begin{aligned} \varphi_A &= \iota((\varphi_A^{-1})^\dagger)^{-1} \\ &= \iota((\varphi_A^{-1})^{-1})^\dagger \\ &= \iota\varphi_A^\dagger, \end{aligned}$$

where the first equality uses the fact that the structure map is a unitary isomorphism, and the second equality comes from the functoriality of the dagger. □

Similarly, the involutor is also a unitary isomorphism.

Lemma 3.2.8. The involutor $i_A : A \to A^{\dagger\dagger}$ is a unitary isomorphism for any $A \in \mathcal{U}$.

Proof. For the involutor we have,

$$\iota_A \varphi_{A^{\dagger\dagger}} \iota_A^\dagger = \iota_A (\varphi_{A^\dagger}^{-1})^\dagger \iota_A^\dagger = \iota_A \varphi_A^\dagger = \varphi_A,$$

where the last equality is using Lemma 3.2.7. □

We say that a †- isomix category is a **unitary category** if $\mathcal{U} = \mathbb{X}$. When a category is unitary, then we can indeed define a dagger, $(\_)^*$, which is stationary on objects as

$$\begin{tikzcd} B \arrow[r, "f^*"] \arrow[d, "\varphi_B"'] \arrow[dr, phantom, ":="] & A \\ B^\dagger \arrow[r, "f^\dagger"'] & A^\dagger \arrow[u, "\varphi_A^{-1}"'] \end{tikzcd}$$

This new dagger $(\_)^*$ we claim extends as a contravariant involutive functor, making $\mathbb{X}$ a $\dagger$- category (See Definition 2.2.6). Thus, in a unitary category, we can always define a contravariant dagger that is stationary on objects.

Proposition 3.2.9. In a unitary category $\mathbb{X}$, if we define the dagger of a morphism $f : A \to B$ as $(f)^* := \varphi_B f^\dagger \varphi_A^{-1}$, then $(\_)^* : \mathbb{X}^{\mathsf{op}} \to \mathbb{X}$ is a contravariant involutive functor.

Proof. For functoriality, for any $A \in \mathbb{X}$, observe that we have

$$\begin{aligned} \mathrm{id}_A^* : &= \varphi_A \,\mathrm{id}_A^\dagger\, \varphi_A^{-1} \\ &= \varphi_A \,\mathrm{id}_A\, \varphi_A^{-1} \\ &= \mathrm{id}_A, \end{aligned}$$

where the second equality is using functoriality of $(\_)^\dagger$. So $(\_)^*$ preserves identities. For composition, for any $f : A \to B$ and $g : B \to C$, we have

$$\begin{aligned} g^* f^* &= \varphi_C g^\dagger \varphi_B^{-1} f^\dagger \varphi_A^{-1} \\ &= \varphi_C g^\dagger f^\dagger \varphi_A^{-1} \\ &= \varphi_C (fg)^\dagger \varphi_A^{-1} \\ &= (fg)^*, \end{aligned}$$

where the first equality and last are by definition, and the penultimate equality is by contravariance of $\dagger$. So we have that $(\_)^* : \mathbb{X}^{\mathsf{op}} \to \mathbb{X}$ is a contravariant functor. For involution, we have,

$$\begin{aligned} f^{**} &:= \varphi_A (f^*)^\dagger \varphi_B^{-1} \\ &:= \varphi_A (\varphi_B f^\dagger \varphi_A^{-1})^\dagger \varphi_B^{-1} \\ &= \varphi_A (\varphi_A^{-1})^\dagger f^{\dagger\dagger} \varphi_B^\dagger \varphi_B^{-1} \\ &= \iota f^{\dagger\dagger} \iota^{-1} = f. \end{aligned}$$

Thus $(\_)^* : \mathbb{X}^{\mathsf{op}} \to \mathbb{X}$ is a contravariant involution. □

Now, when the category is unitary, this new definition of dagger gives us the following

Lemma 3.2.10. An isomorphism $u : X \to Y$ in a unitary category $\mathbb{X}$ is unitary if and only if $u^{-1} = u^*$.

Proof. Now $u : X \to Y$ is unitary if and only if $\varphi_X = u \varphi_Y u^\dagger$. This implies $u^{-1} = \varphi_Y u^\dagger \varphi_X^{-1}$, but this is exactly u^* by definition. □

In fact more is true, recall the $Mx_\downarrow : (\mathbb{X}, \otimes, \otimes) \to (\mathbb{X}, \otimes, \oplus)$ in Remark 3.1.13. We have the following proposition.

Proposition 3.2.11. [40] Unitary categories are $\dagger$- linearly equivalent via the mix functor $Mx_\downarrow$ to the underlying dagger monoidal category on the tensor. Furthermore, closed unitary categories under this equivalence become dagger compact closed categories.

Now there exists a construction, where starting from *any* $\dagger$- isomix category, we can extract a subcategory which is unitary.

Definition 3.2.12. In a $\dagger$- isomix category, a **pre-unitary object** is an object $U \in \mathrm{Core}(\mathbb{X})$ equipped with an isomorphism $\alpha : U \to U^\dagger$ such that $\alpha(\alpha^{-1})^\dagger = \iota$

Definition 3.2.13. Suppose $\mathbb{X}$ is a $\dagger$- isomix category, then **the canonical unitary core** of $\mathbb{X}$, $\mathsf{Unitary}(\mathbb{X})$, is defined as follows

- Objects: Pre- unitary objects (U,α);
- Morphisms: $f : (U,\alpha) \to (V,\beta)$, where $f : U \to V$ is any map.

Then we have that $\mathsf{Unitary}(\mathbb{X})$ is a compact $\dagger$- LDC. The dagger of (U,α) is defined to be, $(U,\alpha)^\dagger := (U^\dagger, (\alpha^{-1})^\dagger)$. The unitary structure map for the tensor and par is given as follows

$$U \otimes V \xrightarrow{mx} U \oplus V \xrightarrow{\alpha\oplus\beta} U^\dagger \oplus V^\dagger \xrightarrow{\lambda_\oplus} (U \otimes V)^\dagger$$

$$U \oplus V \xrightarrow{mx^{-1}} U \otimes V \xrightarrow{\alpha\otimes\beta} U^\dagger \otimes V^\dagger \xrightarrow{\lambda_\otimes} (U \oplus V)^\dagger$$

The unit for tensor and par is still $\top$ and $\bot$ respectively, with their structure maps defined by

$$\top \xrightarrow{m^{-1}} \bot \xrightarrow{\lambda_\bot} \top^\dagger$$

$$\bot \xrightarrow{m} \top \xrightarrow{\lambda_\top} \bot^\dagger$$

We also have $\mathsf{Unitary}(\mathbb{X})$ is a unitary category.

Proposition 3.2.14. For any $\dagger$- isomix category $\mathbb{X}$, $\mathsf{Unitary}(\mathbb{X})$ is a unitary category with a full and faithful embedding via a $\dagger$- isomix functor to the $\dagger$- isomix category, $U : \mathsf{Unitary}(\mathbb{X}) \to \mathbb{X}$

Proof. The laxors are all identity maps, so the underlying functor is immediately a $\dagger$- mix functor, the unitary structure map of (U,α) is set to be $\alpha : (U,\alpha) \to (U^\dagger, (\alpha^{-1})^\dagger)$. Then (**U 1 − 5**) follows by construction. $\square$

The notion of a mixed unitary category generalises this construction, where we embed a unitary category into a †- isomix category.

Definition 3.2.15. [40] A mixed unitary category (MUC) is a †- isomix category $\mathbb{X}$, equipped with a strong †- isomix functor $M : \mathbb{U} \to \mathbb{X}$ from a unitary category $\mathbb{U}$ to $\mathbb{X}$ such that there exist natural transformations

- $mx' : M(U) \oplus X \to M(U) \otimes X$ with $mx\,mx' = \mathrm{id}$ and $mx'\,mx = \mathrm{id}$
- $mx'' : X \oplus M(U) \to X \otimes M(U)$ with $mx\,mx'' = \mathrm{id}$ and $mx''\,mx = \mathrm{id}$

Note that the requirement of mx' and mx'' being inverses to the mix map ensures that the functor $M : \mathbb{U} \to \mathbb{C}$ factors through $\mathsf{Core}(\mathbb{X})$.

An immediate example of an MUC will be the embedding of the canonical unitary core into the ambient category. Here $M : \mathsf{Unitary}(\mathbb{X}) \to \mathbb{X}$ is just the identity map, which is immediately a strong †- isomix functor. It also factors through the core by construction.

We finish this chapter by giving a couple of examples of mixed unitary categories. Consider the discrete category of groups where the tensor product of two elements of a group g, h is given by $g \otimes h := gh$ and $g^* := g^{-1}$. Suppose we fix the group to be the additive group of complex numbers, $(\mathbb{C}, +, 0)$. We define $(a + \iota b)^\dagger := \overline{(a + \iota b)^*} = \overline{(-a - \iota b)} = -a + \iota b$. Thus, the pre-unitary objects of $(\mathbb{C}, +, 0)$ are precisely the purely complex numbers, that is, $a + \iota b$ such that the real part is 0, that is, $a = 0$. The unitary structure map is then simply the identity map between a purely complex number and itself.

We next consider the group $\mathbb{M}_2(\mathbb{C})$, that is the group of 2×2 matrices over $\mathbb{C}$. The conjugation is defined entry-wise, then it is clear that the pre-unitary objects of $(\mathbb{M}_2(\mathbb{C}), \bullet, I)$ will be the unitary matrices. In both cases, we can embed this canonical unitary core into the ambient category to get examples of mixed unitary categories.

Chapter 4

Lefschetz spaces

4.1 Background

We want to extend the correspondence between the bases of Hilbert spaces and Frobenius algebras on them to the setting of a mixed unitary category, where this extension can be seen as from the unitary core to a full dagger linearly distributive category (†- LDC). The finite sets (interpreted as the basis of Hilbert space on them) in this correspondence are replaced by finiteness spaces (Chapter 5), whose core is exactly that of finite sets and relations. The finite-dimensional Hilbert spaces get replaced by reflexive Lefschetz spaces. The core of reflexive Lefschetz spaces is of finite-dimensional spaces, which is equivalent to that of finite-dimensional matrices, which, in turn, is equivalent to the category of finite-dimensional Hilbert spaces. Thus, the extension could be seen from the core to a full †- LDC.

In this chapter, we introduce the first leg of this extension, namely, the Lefschetz spaces. The study of Lefschetz spaces was initiated by Solomon Lefschetz [28]. It was later studied by people like Barr [4], Ehrhard [19] and

Blute [5]. It is well known that when V is an infinite-dimensional vector space, then the space of linear functionals on its dual V^* has a larger dimension than that of the space of functionals induced by the elements of V via the natural evaluation map. That is to say that the canonical evaluation map $\mathrm{ev}_V : V \to V^{**}$ is in general not surjective. Barr wanted to study if there is a suitable topology we can put on the dual space, such that the space of continuous linear functionals on the dual space is still in bijection with those that are induced by the elements of V. He showed that the Lefschetz topology is what makes the evaluation map $\mathrm{ev}_V : V \to V^{**}$ always a continuous bijection. The full subcategory of those spaces for which the evaluation map is also open and thus is an isomorphism are called reflexive. The category of reflexive spaces with continuous linear maps forms a star $(*)$- autonomous category.

We follow Barr [4] to develop the theory of reflexive Lefschetz spaces. This will be up to Section 4.2. Then we will use the equivalence between the notion of an LDC with duals and of a $*$- autonomous category [14] to develop further the theory of reflexive Lefschetz space within the framework of LDCs. We ultimately want to show that the category of reflexive Lefschetz spaces with a basis and linear continuous maps forms a mixed unitary category (MUC).

4.2 The category of Lefschetz spaces

A Lefschetz space is a topological vector space that has a local basis at the origin of open linear subspaces. The keyword here is linear. Any topological vector space can be given a basis at the origin of open sets; we demand that these open sets are now linear subspaces of the vector space. For this reason, they have also been referred to as linearly topologised vector spaces [5, 6].

Definition 4.2.1.

(a) A **Lefschetz space** V is a topological vector space $(V, +, \tau)$ that has a local basis at the origin consisting of open linear subspaces of V.

(b) The category of Lefschetz spaces, Lef, has

Objects: Lefschetz spaces;

Morphisms: continuous linear maps;

Composition: usual composition of functions;

Identity: the identity continuous linear transformation.

We will first define an endofunctor, called the internal hom functor. For any $V \in \mathsf{Lef}$, we define, $V \multimap (\_) : \mathsf{Lef} \to \mathsf{Lef}$. On objects, $V \multimap W$ is the space of all linear continuous functions from V to W, and the subbasis for the topology for $V \multimap W$ is given by subspaces of the form,

$$\{f : V \to W : f(v) \in W_0\}_{v \in V, W_0 \in N_0(W)}$$

where $f : V \to W$ is a continuous linear map and v, W_0 runs over the elements of V and open linear subspaces around 0_W respectively (the collection is denoted by $N_0(W)$).

Lemma 4.2.2. $V \multimap W$ is a Lefschetz space.

Proof. To prove that $V \multimap W$ is a Lefschetz space, we need to give a local basis at $0_{V \multimap W}$ (the constant function 0 from V to W) consisting of open linear subspaces of $V \multimap W$. We claim that the local basis at $0_{V \multimap W}$ is given by the family

$$\{f : V \to W : f(v) \in W_0\}_{v \in V, W_0 \in N_0(W)}$$

where by $N_0(W)$ we mean the open linear subspaces around 0_W.

To see that the above family is indeed a local basis at $0_{V \multimap W}$, note that by definition (recall Definition 2.3.7), the topology on $V \multimap W$ is given by taking arbitrary unions of finite intersections of the sets in this family. In particular, each set in this family is open. Moreover, $0_{V \multimap W}$ belongs to each set in this family. Thus, any open set generated by this subbasis which contains $0_{V \multimap W}$ contains a set from this family. Hence we have that the local basis at $0_{V \multimap W}$ is given by open subspaces of the form

$$\{f : V \to W : f(v) \in W_0\},$$

and we are done. □

Hence, the internal hom functor $V \multimap (\_) : \mathsf{Lef} \to \mathsf{Lef}$ is well defined on objects. On a map $f : W \to U$, we define the functor by post-composition. That is $V \multimap f : (V \multimap W) \to (V \multimap U)$, where for any $g \in V \multimap W$, $(V \multimap f)(g) := gf$. This preserves identity and composition. Hence $V \multimap (\_) : \mathsf{Lef} \to \mathsf{Lef}$ is a functor. And so we have,

Lemma 4.2.3. For any $V \in \mathsf{Lef}$, $V \multimap (\_) : \mathsf{Lef} \to \mathsf{Lef}$ is a functor.

Barr then proved that for any $V \in \mathsf{Lef}$, $V \multimap (\_) : \mathsf{Lef} \to \mathsf{Lef}$ has a left adjoint $(\_) \otimes V : \mathsf{Lef} \to \mathsf{Lef}$ [4], that is ,

$$(\_) \otimes V \dashv V \multimap (\_)$$

and thus for any $U, V, W \in \mathsf{Lef}$, we have the following natural isomorphism

$$\mathrm{Lef}(U \otimes V, W) \cong \mathrm{Lef}(U, V \multimap W). \tag{4.1}$$

This makes Lef into a monoidal closed category.

Remark 4.2.4. An explicit description of the tensor product in the category of Lefschetz spaces can be found in [6]. By explicit description, we mean knowing the basis of the topology on the tensor product of two Lefschetz spaces, given the topology on each of them. The basis for topology on $V_1 \otimes V_2$ is given by a family of subspaces of the form $B_1 \otimes V_2 + B_2 \otimes V_1$, where B_1, B_2 are open subspaces around the origin of V_1 and V_2 respectively. This topology on the tensor product can equivalently be described as the strongest (finest) topology that makes the tensor multiplication map $V_1 \times V_2 \to V_1 \otimes V_2$ uniformly continuous. For more details, see Section 24, Chapter-V of [6].

We also have that the tensor product in Lef is symmetric.

Theorem 4.2.5. [4] The tensor product on Lefschetz spaces $(\_) \otimes V : \mathsf{Lef} \to \mathsf{Lef}$ is symmetric.

Proof. Let $F : U \otimes V \to W$ be a continuous map, then via the adjunction above (Equation 4.1), F corresponds to a map $G : U \to V \multimap W$. This means that for any fixed $u \in U$, $F(u, -) : V \to W$ is a continuous map. Moreover, the map G itself must be continuous. Now recall that an open subspace around $0_{V \multimap W}$ is given by a set of the form,

$$\{f : V \to W : f(v) \in W_0\}_{v \in V, W_0 \in N_0(W)}.$$

Hence G is continuous if and only if for any fixed $v \in V$, G pulls back an open subspace around 0_W to an open subspace in U, that is, $F(-, v)$ is continuous for any $v \in V$. Hence the bilinear map $F : U \otimes V \to W$ is continuous if and only if the linear maps it determines are continuous. That is to say, for any fixed $u \in U, v \in V$, $F(u, -) : V \to W$, $F(-, v) : U \to W$ are continuous. This is exactly the condition for a bilinear map $F : V \otimes U \to W$ to be continuous, and thus the tensor here is symmetric. □

So we have that Lef is a symmetric closed monoidal category. We denote by V^*, the space of linear functionals on V, that is $V^* := V \multimap \mathbb{C}$. The $*$ map is a contravariant functor by defining its action on maps by precomposition. That is for any $f : V \to W$, $f^* : W^* \to V^*$, defined by $f^*(g) := fg$. It is immediate to check that this preserves composition and identity and hence is functorial. We now note that the natural map $|\operatorname{ev}_V| : |V| \to |V^{**}|$ (where by $|.|$, we mean the underlying set (set map)), defined by $v \mapsto \operatorname{ev}(v)$, where for any $\phi \in V^*$, $\operatorname{ev}(v)(\phi) := \phi(v)$ is realised in Lef. This can be seen by

$$\frac{\dfrac{\dfrac{V^* \xrightarrow{\text{id}} V^*}{V^* \otimes V \to \mathbb{C}}}{V \otimes V^* \to \mathbb{C}}}{V \xrightarrow{\text{ev}} V^{**}}$$

Now Barr showed that this natural evaluation map is always a continuous bijection [4]. When the evaluation map $\operatorname{ev}_V : V \to V^{**}$ is also open, and thus an isomorphism, we say that $V \in \mathsf{Lef}$ is **reflexive**. We denote the full subcategory of reflexive Lefschetz spaces by RLef.

We want to show that RLef is also a monoidal closed category. For that, we need a couple of lemmas:

Lemma 4.2.6. [4] For any $V \in \mathsf{Lef}$, V^* is reflexive.

Proof. We know by Barr that the natural evaluation map for any Lefschetz space (in particular for any reflexive Lefschetz space) is always a continuous bijection [4]. So we have that $\mathrm{ev}_{V^*} : V^* \to V^{***}$ is a continuous bijection. We claim that $(\mathrm{ev}_V)^* : V^{***} \to V^*$ is the continuous inverse of ev_{V^*}. First note that $(\mathrm{ev}_V)^*$ is continuous as ev_V is continuous and $(\_)^* : \mathsf{Lef} \to \mathsf{Lef}$ is a functor. Now we need to prove that $\mathrm{ev}_{V^*}(\mathrm{ev}_V)^* = \mathrm{id}_{V^*}$ and $(\mathrm{ev}_V)^*\,\mathrm{ev}_{V^*} = \mathrm{id}_{V^{***}}$.

Let $\varphi \in V^*$, then observe

$$\begin{aligned}(\mathrm{ev}_V)^*(\mathrm{ev}_{V^*}(\varphi)) &= (\mathrm{ev}_V)^*(\mathrm{ev}_{V^*}(\varphi)) \\ &= \mathrm{ev}_V\,\mathrm{ev}_{V^*}(\varphi).\end{aligned}$$

Then for any $v \in V$,

$$\begin{aligned}\mathrm{ev}_{V^*}(\varphi)(\mathrm{ev}_V(v)) &= \mathrm{ev}(\varphi)(\mathrm{ev}(v)) \\ &= \varphi(v).\end{aligned}$$

So we have $\mathrm{ev}_{V^*}(\mathrm{ev}_V)^* = \mathrm{id}_{V^*}$. For $(\mathrm{ev}_V)^*\,\mathrm{ev}_{V^*} = \mathrm{id}_{V^{***}}$, let $\psi \in V^{***}$, then,

$$\begin{aligned}\mathrm{ev}_{V^*}((\mathrm{ev}_V)^*(\psi)) &= \mathrm{ev}_{V^*}(\mathrm{ev}_V\,\psi) \\ &= \mathrm{ev}_{V^*}(\mathrm{ev}_V\,\psi).\end{aligned}$$

Now any $\phi \in V^{**}$ is of the form $\mathrm{ev}_V(v)$ for some $v \in V$, and so,

$$\mathrm{ev}_{V^*}(\mathrm{ev}_V\,\psi)(\mathrm{ev}_V(v)) = \psi(\mathrm{ev}_V(v)).$$

And so we have, $(\mathrm{ev}_V)^*\,\mathrm{ev}_{V^*} = \mathrm{id}_{V^{***}}$. Hence $\mathrm{ev}_{V^*} : V^* \to V^{***}$ is an isomorphism, so V^* is reflexive. □

Using this lemma, we can now show the following:

Lemma 4.2.7. [4] For any $V \in \mathsf{Lef}$ and $W \in \mathsf{RLef}$, $V \multimap W$ is reflexive.

Proof. Note the following series of isomorphisms,

$$V \multimap W \cong V \multimap W^{**} \cong V \multimap (W^* \multimap \mathbb{C}) \cong (V \otimes W^*) \multimap \mathbb{C} = (V \otimes W^*)^*.$$

Note that the third isomorphism is a valid natural isomorphism, as in any monoidal closed category, we have not only $\mathbb{X}(U \otimes V, W) \cong \mathbb{X}(U, V \multimap W)$, but also $U \otimes V \multimap W \cong U \multimap (V \multimap W)$ [30]. And hence we have $V \multimap W \cong (V \otimes W^*)^*$. Then by Lemma 4.2.6, $V \multimap W$ is reflexive, as $V \multimap W$ is (isomorphic) to a dual space. □

Next, using Lemma 4.2.6, we prove that the inclusion functor $I : \mathsf{RLef} \to \mathsf{Lef}$ has a left adjoint.

Lemma 4.2.8. [4] The inclusion functor $I : \mathsf{RLef} \to \mathsf{Lef}$ has a left adjoint given by $(\_)^{**} : \mathsf{Lef} \to \mathsf{RLef}$.

Proof. First note that by Lemma 4.2.6, for any $V \in \mathsf{Lef}$, $V^* \in \mathsf{RLef}$, so the $(\_)^{**}$ is well-defined. Now we will prove the adjunction by giving the universal pair. For any $V \in \mathsf{Lef}$, we claim that the universal pair is given by $(V^{**}, n_V := \mathrm{ev}_V : V \to V^{**})$. Then we must prove that for any map $f : V \to W$ in Lef, where W is a reflexive space, there exists a unique map $h : V^{**} \to W$ such that the following triangle commutes.

$$\begin{tikzcd} V \arrow[r, "\mathrm{ev}_V"] \arrow[dr, "f"'] & V^{**} \arrow[d, "h"] \\ & W \end{tikzcd}$$

We define $h := f^{**}(\mathrm{ev}_W)^{-1}$. Note that this is a valid map as W is reflexive. Now note that for any $v \in V$, $f^{**}(\mathrm{ev}_V(v)) \in W^{**}$ and for any $\varphi \in W^*$

$$\mathrm{ev}_V(f^*(\varphi)) \quad = \quad \mathrm{ev}_V(f\varphi),$$

which for any $v \in V$, takes the value $\varphi(f(v))$. Hence when we apply ev_W^{-1} to it we get $f(v)$. And hence the triangle commutes. For uniqueness, let $g : V^{**} \to W$ such that $\mathrm{ev}_V\, g = f$, then we have,

$$\mathrm{ev}_V\, g = \mathrm{ev}_V\, f^{**}\, \mathrm{ev}_W^{-1}\,.$$

This implies that $g = f^{**}\, \mathrm{ev}_W^{-1}$ on all $\varphi \in V^{**}$, such that $\varphi = \mathrm{ev}_V(v)$ for some $v \in V$. But we know that the evaluation map is always a bijection for any Lefschetz space, and hence we have $g = f^{**}\, \mathrm{ev}_W^{-1}$. So we have the required adjunction. □

Putting Lemma 4.2.6, 4.2.7, 4.2.8 together, we have that for any $V \in \mathsf{RLef}$,

$$((\_) \otimes V)^{**} \dashv V \multimap (\_)$$

and thus we have for any $U, V, W \in \mathsf{RLef}$, we have the following natural isomorphism

$$\mathrm{RLef}((U \otimes V)^{**}, W) \cong \mathrm{RLef}(U, V \multimap W)$$

This makes RLef into a symmetric monoidal closed category. For the star autonomous structure, the global dualising object is given by the base field ($\mathbb{C}$), and as the evaluation map $\mathrm{ev}_V : V \to V^{**}$ is an isomorphism for any reflexive Lefschetz space, we have that RLef is a $*$- autonomous category.

Remark 4.2.9. Note that even in RLef, maps out of the tensor product still correspond to a pair of continuous linear maps $F_1(u, -) : V \to W$ and $F_2(-, v) : U \to W$. Moreover, we have $|U \otimes V| \cong |(U \otimes V)^{**}|$ (where recall the $(|.|)$ means the underlying set of these spaces), so for brevity of notation we will simply denote the tensor product of two reflexive spaces U, V by $U \otimes V$.

By the discussion in the background chapter on equivalence between the notion of a $*$- autonomous category and a LDC with duals [14], we know that RLef is an LDC with dual. That means that there are maps $\eta : \mathbb{C} \to V^* \oplus V$ and $\varepsilon : V \otimes V^* \to \mathbb{C}$ which satisfy the snake equations. For exercise, we will now derive the map $\varepsilon : V \otimes V^* \to \mathbb{C}$. By definition, it is the transpose of the evaluation map $\mathrm{ev}_V : V \to V^{**}$. So ε is the unique map that makes the following triangle commute:

$$\begin{tikzcd} V \arrow[r, "n_V"] \arrow[dr, "\mathrm{ev}_V"'] & V^* \multimap (V \otimes V^*) \arrow[d, "V^* \multimap (\mathrm{ev}_V)^{\#}"] \\ & V^* \multimap \mathbb{C} \end{tikzcd}$$

Here the map $n_V : V \to V^* \multimap (V \otimes V^*)$ on basis vectors is defined as follows $n_V(v_i) := \sum_{k \in K} \delta_{v_k \to v_i \otimes \delta_{v_k}}$, where δ_{v_k} denotes the functional on V which takes the value 1 at v_k and 0 elsewhere. And $\delta_{v_k \to v_i \otimes \delta_{v_k}}$ denotes the continuous linear function from V^* to $V \otimes V^*$ which takes δ_{v_k} to $v_i \otimes \delta_{v_k}$ (and every other basis vector to 0). Now the evaluation map is given by sending v_i to $\mathrm{ev}(v_i)$, which in this notation can be denoted by $\delta_{v_i \to 1}$. Then $(\mathrm{ev}_V)^{\#} : V \otimes V^* \to \mathbb{C}$ is given by sending $v_i \otimes \delta_{v_j}$ to 1, if $i = j$, and 0, otherwise. It is easy to check that $n_V V^* \multimap (\mathrm{ev}_V)^{\#} = \mathrm{ev}_V$, as

$$\begin{aligned} V^* \multimap (\mathrm{ev}_V)^{\#}(n_V(v_i)) &= V^* \multimap (\mathrm{ev}_V)^{\#}(\sum_{k \in K} \delta_{v_k \to v_i \otimes \delta_{v_k}}) \\ &= \delta_{v_i \to 1}. \end{aligned}$$

Since all of these maps are linear, this means that the colloquial map we use $\varepsilon : V \otimes V^* \to \mathbb{C}$ by sending $v \otimes \varphi \to \varphi(v)$ is the actual map we get from the adjunction. So we have $\varepsilon = (\mathrm{ev}_V)^{\#} : V \otimes V^* \to \mathbb{C}$.

4.3 Reflexive Lefschetz spaces as dagger isomix categories

Since RLef is a $*$- autonomous category, we know via the equivalence between the notion of a $*$- autonomous category and that of an LDC (with duals) [14] that RLef is an LDC with duals. We will now develop the rest of the theory of reflexive Lefschetz spaces as an LDC. In this section, we will define a $\dagger$ structure on RLef and prove that RLef is a $\dagger$- isomix category.

Lemma 4.3.1. RLef is an isomix LDC.

Proof. To show that RLef is an isomix LDC, we will first show that $\mathbb{C}$ is a unit for the par. Recall that by definition of par induced by the $*$-autonomous structure, we have for any $V, W \in$ RLef, $V \oplus W := (V^* \otimes W^*)^*$. It is clear that $\mathbb{C}^*$ is a one-dimensional space, as the space of linear functionals on $\mathbb{C}$ is completely determined by the value it takes on $1 \in \mathbb{C}$. But the only one dimensional space is $\mathbb{C}$ itself, so we have for any $V \in$ RLef,

$$\begin{aligned} V \oplus \mathbb{C} &:= (V^* \otimes \mathbb{C}^*)^* \\ &\cong (V^* \otimes \mathbb{C})^* \\ &\cong V^{**} \\ &\cong V, \end{aligned}$$

where to get the third isomorphism we have used the fact that $\mathbb{C}$ is a unit for the tensor [4], and the last isomorphism follows as V is a reflexive space. And hence the mix map is simply the identity map, so we have that RLef is an isomix category. □

We then claim the following:

Proposition 4.3.2. The core of RLef is exactly the full subcategory of finite-dimensional reflexive spaces.

We need a couple of definitions and lemmas to prove this proposition.

Definition 4.3.3. Let $\mathbb{X}$ be a category. Then the biproduct of $X_1, X_2 \in \mathbb{X}$ if it exists, is a tuple $(X_1 \star X_2, \pi_1 : X_1 \star X_2 \to X_1, \pi_2 : X_1 \star X_2 \to X_2, i_1 : X_1 \to X_1 \star X_2, i_2 : X_2 \to X_1 \star X_2)$ such that

- $(X_1 \star X_2, \pi_1 : X_1 \star X_2 \to X_1, \pi_2 : X_1 \star X_2 \to X_2)$ is a product tuple,
- $(X_1 \star X_2, i_1 : X_1 \to X_1 \star X_2, i_2 : X_2 \to X_1 \star X_2)$ is a coproduct tuple, and
- $i_1\pi_1 = \mathrm{id}$, $i_2\pi_2 = \mathrm{id}$, $\pi_2 i_2 \pi_1 i_1 = \pi_1 i_1 \pi_2 i_2$.

Then we claim

Lemma 4.3.4. RLef has finite biproducts, where the biproduct of two reflexive Lefschetz spaces V_1 and V_2 is given by $V_1 \star V_2$. Here $V_1 \star V_2$ is the direct sum of V_1 and V_2 in the category of vector spaces, and the topology is given coordinate-wise.

Proof. We first give the maps $\pi_1 : V_1 \star V_2 \to V_1$ by defining for any $v_1 \in V_1$ and $v_2 \in V_2$, $\pi_1(v_1, v_2) := v_1$ and $\pi_2(v_1, v_2) := v_2$. Now $\pi_1 : V_1 \star V_2 \to V_1$ is continuous as it pulls back an open set U in V_1 to $(U, V_2) \subseteq V_1 \star V_2$ which is open by our definition. Similarly, π_2 is continuous. Then the unique map required from $Z \in \mathsf{RLef}$ to $V_1 \star V_2$ for a given pair of maps $f : Z \to V_1, g : Z \to V_2$ is given by $\langle f, g\rangle : Z \to V_1 \star V_2$ by defining $\langle f, g\rangle(z) := \langle f(z), g(z)\rangle$. Now $\langle f, g\rangle$ is continuous as it pulls back an open set (O_1, O_2) to $f^{-1}(O_1) \cap g^{-1}(O_2)$, which is open as f and g are continuous. It is easy to check that $\langle f, g\rangle\pi_1 = f$, $\langle f, g\rangle\pi_2 = g$ and $\langle f, g\rangle$ is the unique such map, as for any other map if $h : Z \to V_1 \star V_2$ $\pi_1 h = f$ and $\pi_2 h = g$, then $h = \langle f, g\rangle$. So $(V_1 \star V_2, \pi_1 : V_1 \star V_2 \to V_1, \pi_2 : V_1 \star V_2 \to V_2)$ is a product tuple.

Now we give the injection maps by defining for any $v_1 \in V_1$ and $v_2 \in V_2$, $i_1 : V_1 \to V_1 \star V_2$, $i_1(v_1) := (v_1, 0)$, and $i_2 : V_2 \to V_1 \star V_2$ by $i_2(v_2) := (0, v_2)$. Now i_1 is continuous as it pulls back an open set (O_1, O_2) in $V_1 \star V_2$ to O_1. Similarly, i_2 is continuous. Then for a pair of maps $f : V_1 \to Z$, $g : V_2 \to Z$, we define the map $f + g : V_1 \star V_2 \to Z$ by setting $f + g(v_1, v_2) := f(v_1) + g(v_2)$. Observe that any $(v_1, v_2) \in V_1 \star V_2$ can be written as $(v_1, 0) + (0, v_2)$, and continuity of $f + g$ on vectors of the form $(v_1, 0)$, $(0, v_2)$ follows from continuity of f, g respectively. Since $f + g$ is the coordinate wise sum of f and g, it follows that $f + g$ is continuous. It is easy to check that $i_1 f + g = f$, $i_2 f + g = g$. And $f + g$ is the unique such map as any $(v_1, v_2) \in V_1 \star V_2$ could be written as $(v_1, 0) + (0, v_2)$. So $(V_1 \star V_2, i_1 : V_1 \to V_1 \star V_2, i_2 : V_2 \to V_1 \star V_2)$ is a coproduct tuple.

Now all that is left to check are the identities, for that,

$$\begin{aligned} \pi_1(i_1(v_1)) &= \pi_1(v_1, 0) \\ &= v_1. \end{aligned}$$

So $i_1\pi_1 = \mathrm{id}$, similarly $i_2\pi_2 = \mathrm{id}$. And $\pi_2 i_2 \pi_1 i_1 = 0 = \pi_1 i_1 \pi_2 i_2$. So we have that RLef has (finite) biproducts. □

Now we have the following proposition:

Lemma 4.3.5. Let $B, C \in \mathsf{Core}(\mathbb{X})$, where $\mathbb{X}$ is a symmetric LDC with duals and with biproducts, then $B \star C \in \mathsf{Core}(\mathbb{X})$, where $B \star C$ denotes the biproduct of B and C.

Proof. Let $A \in \mathbb{X}$, then

$$\begin{aligned} A \otimes (B \star C) &\cong (A \otimes B) \star (A \otimes C) \\ &\cong (A \oplus B) \star (A \oplus C) \\ &\cong A \oplus (B \star C), \end{aligned}$$

where in the first and last equality follows from the fact that tensor and par are left and right adjoints [13], so they preserve colimits and limits, respectively. As the tensor and the par are symmetric, it follows that $(B \star C) \otimes A \cong (B \star C) \oplus A$, so $B \star C \in \mathsf{Core}(\mathbb{X})$. □

Now in RLef, we know that $\mathbb{C} \in \mathsf{Core}(\mathsf{RLef})$ as it is isomix by Proposition 3.1.5, moreover, the topology on the direct sum is coordinate-wise, so any finite-dimensional space is just n copies of $\mathbb{C}$ put together. Then it follows from Lemma 4.3.5 that any finite-dimensional reflexive Lefschetz space is in $\mathsf{Core}(\mathsf{RLef})$. Now, for any infinite-dimensional space V, it is generally not the case that $V \oplus V := (V^* \otimes V^*)^* \cong V \otimes V$, unless the category collapses to a compact closed category [17]. So we have that $\mathsf{Core}(\mathsf{RLef}) = \mathsf{Fin}(\mathsf{RLef})$, where $\mathsf{Fin}(\mathsf{RLef})$ is the full subcategory of finite-dimensional Lefschetz spaces. This completes the proof of Proposition 4.3.2.

As we saw, RLef is a $*$- autonomous category, where the V^* is the space of continuous linear functions on V. Thus RLef becomes a $\dagger$-LDC where the dagger is just the dualising functor by Lemma 3.1.16.

For reasons coming from physics, we want, whenever we have a dual system $(V, V^*, \mathrm{ev} : V \times V^* \to \mathbb{C})$, the evaluation to be linear in one variable (often the first variable), but *conjugate* linear in the other. That is, $\mathrm{ev}(v, -) : V^* \to \mathbb{C}$ is linear but $\mathrm{ev}(-, \varphi) : V \to \mathbb{C}$ is conjugate linear, $\mathrm{ev}(v, c \bullet \varphi) =$

$\overline{c} \bullet \varphi(v)$; where $\bullet$ denotes the action of scalars from the field. So we change the definition of the $\dagger$ on RLef to accommodate this.

In RLef, we define the conjugation functor [18], as follows:

$$\mathsf{RLef} \xrightarrow{\overline{(\_)}} \mathsf{RLef}$$

$$\begin{array}{ccccc} V & \qquad & \overline{V} & \qquad & c\,\overline{\bullet}\,v := \overline{c} \bullet v \\ \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle \overline{f}} & & \overline{f}(v) := f(v) \\ W & & \overline{W} & & \end{array}$$

That is, on objects, the conjugation functor changes the scalar action to the conjugate scalar action (we are working over fields with a notion of conjugation such as $\mathbb{C}$), and on morphisms, the underlying set maps remain the same. Note that $|\overline{V}| = |V|$. Then $\overline{f}$ is now a linear (continuous) homomorphism between $\overline{V}$ and $\overline{W}$. Importantly, note that the conjugation functor does not affect the topology of the space.

Now we can easily check that this defines a conjugation functor as in Definition 3.1.19. We define the structure maps as, $\xi_{\otimes} := c_{\otimes}$, $\xi_{\oplus} := c_{\oplus}$, $\varepsilon := \mathrm{id}$. That is, we define the conjugative laxor simply to be the symmetry map of RLef and the natural isomorphism $\varepsilon : \overline{\overline{V}} \to V := \mathrm{id}$. These are clearly linear and continuous maps. For the coherence conditions, we have for any $v_1, v_2, v_3 \in V$, where V is a reflexive Lefschetz space,

$$\begin{aligned} c_{\otimes}(\mathrm{id} \otimes c_{\otimes})(\alpha_{\otimes}((v_1 \otimes v_2) \otimes v_3)) &= c_{\otimes}(\mathrm{id} \otimes c_{\otimes})(v_1 \otimes (v_2 \otimes v_3)) \\ &= c_{\otimes}(v_1 \otimes (v_3 \otimes v_2)) \\ &= (v_3 \otimes v_2) \otimes v_1, \end{aligned}$$

and,

$$\begin{aligned}
\overline{\alpha_\otimes^{-1}}(c_\otimes(c_\otimes \otimes \mathrm{id})((v_1 \otimes v_2) \otimes v_3))) &= \overline{\alpha_\otimes^{-1}}(c_\otimes((v_2 \otimes v_1) \otimes v_3)) \\
&= \overline{\alpha_\otimes^{-1}}(v_3 \otimes (v_2 \otimes v_1)) \\
&= (v_3 \otimes v_2) \otimes v_1.
\end{aligned}$$

Note that $|\overline{\alpha_\otimes^{-1}}| = |\alpha_\otimes^{-1}|$, as conjugation has no effect on the underlying (set) map. Now, for the second coherence condition, we have

$$\begin{aligned}
\overline{c_\otimes}(c_\otimes(v_1 \otimes v_2)) &= \overline{c_\otimes}(v_2 \otimes v_1) \\
&= v_1 \otimes v_2 \\
&= \mathrm{id}((\mathrm{id} \otimes \mathrm{id})(v_1 \otimes v_2)).
\end{aligned}$$

To be symmetric conjugative, we have

$$\begin{aligned}
\overline{c_\otimes}(c_\otimes(v_1 \otimes v_2)) &= \overline{c_\otimes}(v_2 \otimes v_1) \\
&= v_1 \otimes v_2 \\
&= c_\otimes(c_\otimes(v_1 \otimes v_2)).
\end{aligned}$$

Similarly, RLef is conjugative with respect to $\oplus$ and the coherence conditions required for the conjugative LDC simply reduce to the coherence required of the symmetry map in a symmetric LDC. Thus, we have that RLef is a conjugative LDC.

Now both the dualzing functor and the conjugation functors are equivalences of Frobenius linear functors, so they compose in the 2- category of LDCs with linear functors and linear natural transformations to give us that RLef is a $\dagger$-LDC, where $\dagger := (\_)^*\overline{(\_)}$.

To conclude this, we note,

Proposition 4.3.6. RLef is a †-LDC with $\dagger := (\_)^* \overline{(\_)}$.

Thus, we have a dagger structure on RLef.

The notion of a mixed unitary category is developed within a †- isomix category. So we will prove that the dagger functor on RLef is isomix. First, for the dualising functor, note that $(\mathbb{C})^* = \mathbb{C}$, so the mix map and all the unit laxors are identity maps, therefore $(\_)^*$ is an isomix functor. For the conjugation, we have for any $a + b\iota \in \mathbb{C}$,

$$\begin{aligned} m_{\mathbb{C}}(m(\eta_{\mathbb{C}}(a+b\iota))) &= m_{\mathbb{C}}(m(a-b\iota)) \\ &= m_{\mathbb{C}}(a-b\iota) \\ &= a+b\iota \\ &= \overline{\mathrm{id}}(a+b\iota) \\ &= a+b\iota. \end{aligned}$$

So $\overline{(\_)}$ is a mix functor. To check if it is isomix, note that for any $a + b\iota \in \mathbb{C}$,

$$\begin{aligned} n_{\mathbb{C}}(\overline{(\mathrm{id}^{-1})}(m_{\mathbb{C}}(a+b\iota))) &= n_{\mathbb{C}}(\overline{(\mathrm{id}^{-1})}(a-b\iota)) \\ &= n_{\mathbb{C}}(a-b\iota) \\ &= a+b\iota \\ &= \overline{(\mathrm{id}^{-1})}(a+b\iota) \\ &= (\mathrm{id}^{-1})(a+b\iota) \\ &= \mathrm{id}(a+b\iota) \\ &= (a+b\iota). \end{aligned}$$

So we have that $\overline{(\_)} : \mathsf{RLef} \to \mathsf{RLef}$ is an isomix functor.

Thus both $(\_)^*$ and $\overline{(\_)}$ are isomix functors on RLef, so we have RLef is a †-isomix category where $\dagger := (\_)^* \overline{(\_)}$. So we have,

Lemma 4.3.7. RLef is a $\dagger$- isomix category.

We will not always be explicit in quantification over objects while proving coherence conditions, as it is standard to skip quantification when it comes to commuting diagrams or while proving equations involving them. Unless explicitly stated otherwise, the quantification is over all objects of the category in context.

Next, we want to show that the category of reflexive Lefschetz space forms a mixed unitary category. But to give a unitary structure to the spaces in $\mathsf{Core}(\mathsf{RLef})$, which we know are finite-dimensional spaces by Proposition 4.3.2, we need to attach the data of a basis to our spaces. Though for this chapter we only need the notion of a basis for finite-dimensional spaces, we develop it more generally here, for completeness. This will be of crucial importance when we define a functor from the category of reflexive Lefschetz spaces with a basis to the category of finiteness spaces by studying the support of summable families over the basis.

4.4 Notion of a basis in reflexive Lefschetz spaces

There exist various notions of a basis. The most familiar ones are of a Hamel basis [35], where we are only allowed to take finite linear combinations, and of a Schauder basis [10], where we are allowed to take infinite linear combinations when working with spaces with a topology. But the Schauder basis comes with a fixed order on it and is also countable in size. For reflexive Lefschetz spaces, we will be working with a different notion of summability and hence a different notion of a basis. Before we define what we mean by a Lefschetz basis, we need some terminology.

Definition 4.4.1. Let V be a topological vector space and $(v_i)_{i\in I}$ be any arbitrary family of elements of V. Then we say this family is **summable** if the following net is convergent:

$$v_\bullet : \mathrm{Fin}(I) \to V$$

where for any $A \in \mathrm{Fin}(I)$, $v_A := \sum_{i\in A} v_i$. That is, we say $(v_i)_{i\in I}$ is summable if the following limit exists:

$$\mathrm{Lim}_{A\in\mathrm{Fin}(I)} \sum_{i\in A} v_i.$$

And we denote the limit by

$$\sum_{i\in I} v_i.$$

Now we can proceed to define an appropriate notion of a basis in Lefschetz spaces.

Definition 4.4.2.

(a) Let V be a reflexive Lefschetz space. Then a collection of elements of V $\{v_i\}_{i\in I}$ is said to be **linearly independent** if and only if

$$\sum_{i\in I} \alpha_i v_i = 0 \implies \alpha_i = 0\,,\ \forall i \in I.$$

where α_i are scalars from the field and the sum should be interpreted as in Definition 4.4.1 and the equality should be interpreted as saying converging to 0.

(b) A collection of elements of V, $\{v_i\}_{i\in I}$, that is not linearly independent is called **linearly dependent.**

Lemma 4.4.3. Let $\{v_i\}_{i\in I}$ be a linearly independent set in a reflexive Lefschetz space V. Then for any $J \subseteq I$, $\{v_j\}_{j\in J}$ is also linearly independent.

Proof. We extend the sum $\sum_{j\in J} \alpha_j v_j$ by adding $\alpha_{i'} = 0$, for all $i' \in I - J$. Clearly, then the sum $\sum_{i\in I} \alpha_i v_i$ still converges to 0. As $\{v_i\}_{i\in I}$ is a linearly independent set, so this implies $\alpha_i = 0$, for all $i \in I$. In particular $\alpha_j = 0$, for all $j \in J$, so $\{v_j\}_{j\in J}$ is linearly independent. □

So we have that any subset of a linearly independent set is also linearly independent. We also have that any re-indexed sum converges to the same limit.

Lemma 4.4.4. Let $\{v_i\}_{i\in I}$ be a collection of elements from a reflexive Lefschetz space V. We can see this indexing as a function $i : I \to V$. Let $\sum_{i\in I} v_i$ be convergent to some $v \in V$. Now let $\sigma : I \to I$ be a bijection. Then the re-indexed sum $\sum_{\sigma i\in I} v_{\sigma i}$ is also convergent. Moreover, it is convergent to the same limit v.

Proof. Let $\sum_{i\in I} v_i = v$. By definition, this means that, for any open neighbourhood U around v, there exists $A_0^{U_i} \in \mathrm{Fin}(I)$ such that $\sum_{i\in A} v_i \in U$ for every $A \in \mathrm{Fin}(I)$ so that $A_0^{U_i} \subseteq A$. We want to prove that $\sum_{\sigma i\in I} v_{\sigma i}$ is also convergent to v. For this note that, as finite sums are independent of the order of addition, so we have that for for any open neighbourhood U around v, we have that $\sum_{\sigma i\in A} v_{\sigma i} \in U$ for every $A \in \mathrm{Fin}(I)$, so that $A_0^{U_i} \subseteq A$, and so $\sum_{\sigma i\in I} v_{\sigma i}$ is convergent to v. □

Note that this in particular implies that any re-indexing of a linearly independent set is also independent, and so we have that the notion of convergence and independence is agnostic to indexing.

Remark 4.4.5. Observe that Lemma 4.4.4 shows that our notion of convergence is indeed different from the one used in the definition of Schauder basis [10], as it is well known that a convergent sum might diverge if the elements are re-indexed.

Definition 4.4.6.

(a) Let $S := \{v_i : i \in I\}$ be a collection of vectors in a reflexive Lefschetz space V. Then the **span of** S is defined to be

$$\operatorname{Span}[S] := \{v \in V : v = \operatorname{Lim}_{A \in \operatorname{Fin}(J)} \sum_{a \in A} \alpha_a v_a,\ \alpha_a \in \mathbb{C}, J \subseteq I\}.$$

(b) Let V be a reflexive Lefschetz space. Then a collection of elements of V, $S = \{v_i : i \in I\}$, is called a **Lefschetz basis** for V, if S is linearly independent and if S spans V. That is, every vector in v is a convergent limit of finite linear combinations over (some subset of) I.

Henceforth, we will be referring to a Lefschetz basis simply as a basis. We next prove that every vector is a unique linear combination of the basis vectors. And the main idea of this proof is that since linear operations are continuous with respect to the topology, we can take distinct representations of a vector over the basis to get a new sum which converges to 0, and then the desired result follows from linear independence.

Proposition 4.4.7. Let V be a reflexive Lefschetz space and S a basis for V. Then every vector in V is the limit of a *unique* linear combination over V.

Proof. Every vector is limit of some linear combination over S by definition, as S spans V. We will now prove uniqueness.

For some $v \in V$, let $v = \sum_{j \in J \subseteq S} \alpha_j v_j$ and $v = \sum_{k \in K \subseteq S} \alpha_k v_k$. Then as linear operations are continuous with respect to the topology, we have that $\sum_{j \in J \subseteq S} \alpha_j v_j - v = 0$, and $\sum_{k \in K \subseteq S} \alpha_k v_k - v = 0$. Then by definition, for any open linear subspace U around 0 in V, there exists $A_0 \in \mathrm{Fin}(J)$ such that $\sum_{a \in A \subseteq J \subseteq S} \alpha_a v_a - v \in U$, for all $A \in \mathrm{Fin}(J)$, so that $A_0 \subseteq A$. And there exists $B_0 \in \mathrm{Fin}(K)$ such that $\sum_{b \in B \subseteq K \subseteq S} \alpha_b v_b - v \in U$, for all $B \in \mathrm{Fin}(K)$, so that $B_0 \subseteq B$. Since U is a linear subspace, so $(\sum_{a \in A \subseteq J \subseteq S} \alpha_a v_a - v) - (\sum_{b \in B \subseteq K \subseteq S} \alpha_b v_b - v) \in U$, for all $A \in \mathrm{Fin}(J)$, such that $A_0 \subseteq A$, and for all $B \in \mathrm{Fin}(K)$, such that $B_0 \subseteq B$.

So if we define $\sum_{jk : jk \in J \cup K} \alpha_{jk} v_{jk}$ as the following:

- $\alpha_{jk} := \alpha_j$ if $v_{jk} \in J$ and $v_{jk} \notin K$
- $\alpha_{jk} := \alpha_k$ if $v_{jk} \in K$ and $v_{jk} \notin J$
- $\alpha_{jk} := (\alpha_j - \alpha_k)$ if $v_{jk} \in J$ and $v_{jk} \in K$

Then $\sum_{w : w \in W \subseteq J \cup K} \alpha_w v_w \in U$, for all $W \in \mathrm{Fin}(J \cup K)$, so that $A_0 \cup B_0 \subseteq W$. As the choice of the open subspace was arbitrary, we have that $\sum_{jk : jk \in J \cup K} \alpha_{jk} v_{jk}$, as defined above, converges to 0.

But then $J \cup K \subseteq S$, and S is linearly independent, so by Lemma 4.4.3, $\{v_{jk} : jk \in J \cup K\}$ is also linearly independent. So, by the definition of linear independence, we have $\alpha_{jk} = 0$, $jk \in J \cup K$. This gives us by the definition of scalars in $\sum_{jk : jk \in J \cup K} \alpha_{jk} v_{jk}$ that, $\alpha_k = 0$ when $v_{jk} \in K$ and $v_{jk} \notin J$, $\alpha_j = 0$ when $v_{jk} \in J$ and $v_{jk} \notin K$, and $\alpha_j = \alpha_k$, whenever $v_{jk} \in J$ and $v_{jk} \in K$. Hence $v \in V$ is the limit of a unique linear combination over S. □

As in the finite-dimensional case, we have that linear dependence of a set of vectors is the same notion as a vector being a (non-trivial) sum of the other vectors in that set.

Proposition 4.4.8. Let $\{v_i\}_{i\in I}$ be a collection of elements from V, a reflexive Lefschetz space. Then $\{v_i\}_{i\in I}$ is linearly dependent if and only if there exists a vector v_{i_0}, $i_0 \in I$ such that $v_{i_0} = \mathrm{Lim}_{A\in\mathrm{Fin}(I-\{i_0\})}\sum_{a\in A}\alpha_i v_i$.

Proof. As $\{v_i\}_{i\in I}$ is linearly dependent, so there exists some choice of scalars α_i such that $\sum_{i\in I}\alpha_i v_i = 0$, but not all $\alpha_i = 0$. Without loss of generality, let α_{i_0} for some $i_0 \in I$ not equal to 0. Then as linear operations are continuous, and as $\mathrm{Lim}_{A\in\mathrm{Fin}(I)}\sum_{a\in A}\alpha_a v_a = 0$, so $\mathrm{Lim}_{A\in\mathrm{Fin}(I)}\sum_{a\in A}\frac{\alpha_a}{\alpha_{i_0}} v_a = 0$. Again, as linear operations are continuous, $\mathrm{Lim}_{A\in\mathrm{Fin}(I)}\sum_{a\in A}(\frac{\alpha_a}{\alpha_{i_0}} v_a) - v_{i_0} = -v_{i_0}$. Multiplying by -1 on both sides we have, $\mathrm{Lim}_{A\in\mathrm{Fin}(I)}\sum_{a\in A}(-\frac{\alpha_a}{\alpha_{i_0}} v_a) + v_{i_0} = v_{i_0}$. But then observe that the limit on the left-hand side is just the limit $\mathrm{Lim}_{A\in\mathrm{Fin}(I-\{i_0\})}\sum_{a\in A}\alpha'_a v_a$, where $\alpha'_a := \frac{-\alpha_a}{\alpha_{i_0}}$. And hence $v_{i_0} = \mathrm{Lim}_{A\in\mathrm{Fin}(I-\{i_0\})}\sum_{a\in A}\alpha'_a v_a$.

For the converse, note that, if $v_{i_0} = \mathrm{Lim}_{A\in\mathrm{Fin}(I-\{i_0\})}\sum_{a\in A}\alpha_i v_i$, then subtracting v_{i_0} from both sides, we have $\sum_{i\in I}\alpha_i v_i = 0$, but $\alpha_{i_0} \neq 0$, so we have that $\{v_i\}_{i\in I}$ is a linearly independent set. □

The above definitions and theorems in this section show that there is a nice notion of basis in arbitrary-dimensional reflexive Lefschetz spaces. Moreover, this is an appropriate extension from the finite-dimensional case, as our general notion of basis reduces to giving that any sum over a finite set converges, which should be the case, as we will soon see that every finite-dimensional Lefschetz space is discrete. Now we will attach the data of a basis to finite-dimensional reflexive Lefschetz spaces to provide it with a unitary structure.

4.5 Reflexive Lefschetz spaces as mixed unitary categories

In this section, we define the category of reflexive Lefschetz spaces with a basis, as defined in Section 4.4, and provide a unitary structure to the finite-dimensional spaces. We then completely characterise the unitary maps in this category as the isomorphisms that take our chosen bases to the chosen bases.

Definition 4.5.1. The category of reflexive Lefschetz spaces with basis, RLef_B, has

Objects: Reflexive Lefschetz spaces with a chosen basis;

Morphisms: continuous linear maps independent of the basis;

Composition and Unit: same as RLef.

We must be explicit in giving the data of the basis of various constructions present in the category RLef_B. For this, suppose that $(U, \mathcal{B} = \{b_i\}_{i\in I})$ and $(V, C = \{c_j\}_{j\in J})$ are two reflexive spaces with given bases, then the basis of various constructs are given as follows:

- $(U \otimes V, \mathcal{B} \times C := \{b_i \otimes c_j\}_{i\in I, j\in J})$;
- $U \multimap V, \mathcal{B} \times C := \{\phi_{b_i \to c_j}\}_{i\in I, j\in J}$, where by $\phi_{b_i\to c_j}$, we mean the continuous linear map that takes b_i to c_j, and every other basis vector to 0;
- $(\mathbb{C}, \mathcal{D} = \{*\})$;

- $(U^*, \mathcal{B}\times\{*\} = \{\phi_{b_i\to *}\}_{i\in I})$, where $\phi_{b_i\to *}$ denotes the linear functional on U, which takes b_i to 1, and every other basis vector to 0. Upon identifying this functional with b_i, we have that basis of the dual space is simply given by the same basis;
- $(\overline{U}, \mathcal{B} = \{b_i\}_{i\in I})$.

It is then immediate that, like RLef, $\mathsf{RLef_B}$ is also a †- isomix category, as we haven't changed anything except for adding the data of basis to the objects.

By Proposition 4.3.2, we know that $\mathsf{Core}(\mathsf{RLef_B})$ is the full subcategory of finite-dimensional reflexive Lefschetz spaces. To these spaces, we want to provide it with unitary structure. To do so, let $(V, \{e_i\}_{i=1}^n)$ be a finite dimensional reflexive Lefschetz space, then the dagger of $(V, \{e_i\}_{i=1}^n)$ is by definition $(V, \{e_i\}_{i=1}^n)^\dagger = (\overline{V^*}, \{\varphi_{e_i\to 1}\}_{i=1}^n)$, where recall $\varphi_{e_i\to 1}$ is the linear functional on V that takes the value 1 at e_i and 0 on every other basis vector. For simplicity, we define $\delta_i := \varphi_{e_i\to 1}$. Then we give the unitary structure map for V by $\alpha_V : V \longrightarrow \overline{V^*}$ by defining $\alpha_V(e_i) := \delta_i$ and extending it linearly. This is continuous as finite-dimensional spaces have a discrete topology. To show that it is indeed the unitary structure map, we need to verify $\alpha(\alpha^{-1})^\dagger = \mathrm{ev}_V$.

Lemma 4.5.2. For a finite-dimensional reflexive Lefschetz space V, α : $(V, \{e_i\}_{i=1}^n) \longrightarrow (\overline{V^*}, \{\varphi_{e_i\to 1}\}_{i=1}^n)$ defined by (linearly extending) $\alpha_V(e_i) := \delta_i$ satisfies $\alpha(\alpha^{-1})^\dagger = \mathrm{ev}_V$ and thus provides a unitary structure for $(V, \{e_i\}_{i=1}^n)$.

Proof. We verify this on the basis vectors, as these maps are linear by construction.

$$\begin{aligned}(\alpha^{-1})^\dagger(\alpha(e_i)) &= (\alpha^{-1})^\dagger(\delta_i) \\ &= \alpha^{-1}\delta_i.\end{aligned}$$

Now for any $\delta_j \in \overline{V^*}$, we have

$$\delta_i(\alpha^{-1}(\delta_j)) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases}$$

And similarly for $\mathrm{ev}_V : V \to \overline{\overline{V^{**}}}$, we have

$$\mathrm{ev}_V(e_i)(\delta_j) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases}$$

So we have that $\alpha(\alpha^{-1})^\dagger = \mathrm{ev}_V$.

$$\begin{tikzcd} V \arrow[d, "\alpha"'] \arrow[dr, "\mathrm{ev}_V"] & \\ \overline{V^*} \arrow[r, "(\alpha^{-1})^\dagger"'] & \overline{\overline{V^{**}}} \end{tikzcd}$$

So α provides a unitary structure for V. $\square$

Hence, the canonical unitary core of $\mathsf{RLef_B}$ is given by the finite-dimensional spaces. $\mathsf{Unitary}(\mathsf{RLef_B}) = \mathsf{Fin}(\mathsf{RLef_B})$. Then by embedding it in the ambient category RLef, $\mathsf{Unitary}(\mathsf{RLef_B}) = \mathsf{Fin}(\mathsf{RLef_B}) \hookrightarrow \mathsf{RLef_B}$, we get the structure of a mixed unitary category. We will simply say that $\mathsf{RLef_B}$ is a MUC.

Proposition 4.5.3. Let $f : (V, \{e_i\}_{i=1}^n) \to (W, \{b_i\}_{i=1}^n)$ be an isomorphism between finite-dimensional spaces in $\mathsf{RLef_B}$, then f is unitary if and only if it takes the chosen basis to the chosen basis.

Proof. Let $f : (V, \{e_i\}_{i=1}^n) \to (W, \{b_i\}_{i=1}^n)$ be taking the chosen basis to the chosen basis. Then we want to show that the following diagram commutes.

$$\begin{tikzcd} V \arrow[r, "\alpha_V"] \arrow[d, "f"'] & \overline{V^*} \\ W \arrow[r, "\alpha_W"'] & \overline{W^*} \arrow[u, "f^\dagger"'] \end{tikzcd}$$

Now for any basis vector $e_i \in V$, we have

$$\begin{aligned} f^\dagger(\alpha_W(f(e_i))) &= f^\dagger(\alpha_W(b_j)) \\ &= f^\dagger(\delta_{b_j}) \\ &= f\delta_{b_j}, \end{aligned}$$

which for any $e_i \in V$ evaluates to

$$\delta_{b_j}(f(e_i)) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases}$$

This is because $f(e_i) = b_j$ as f is an isomorphism which takes our chosen basis to the chosen basis. Here by δ_{b_j}, we mean the linear functional $\varphi_{b_j \to 1}$ on W.

And $\alpha_V(e_i) = \delta_{e_i}$, which does

$$\delta_{e_i}(e_j) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases}$$

This proves that when f takes our chosen basis to our chosen basis, then f is unitary. For the converse, let $f : V \to W$ be such that for some chosen basis $e_i \in I$, $f(e_i) = \beta b_j + \gamma b_k$, such that $j \neq k$. This is to say, there exists some element of our basis which does not evaluate to an element of our chosen basis. (More generally we can take $f(e_i) = \sum_{i \in I} \beta_i b_i$, such that it has at least two distinct vectors with non-zero scalars, but that will not change our reasoning). Then for e_i, we have

$$\begin{aligned} f^\dagger(\alpha_W(f(e_i))) &= f^\dagger(\alpha_W(\beta b_j + \gamma b_k)) \\ &= f^\dagger(\beta\delta_{b_j} + \gamma\delta_{b_k}) \\ &= f(\beta\delta_{b_j} + \gamma\delta_{b_k}), \end{aligned}$$

which evaluates $e_i \in V$ to

$$\begin{aligned}(\beta\delta_{b_j} + \gamma\delta_{b_k})(f(e_i)) &= (\beta\delta_{b_j} + \gamma\delta_{b_k})(b_j + b_k) \\ &= \beta\delta_{b_j}(b_j + b_k) + \gamma\delta_{b_k}(b_j + b_k) \\ &= \beta + \gamma.\end{aligned}$$

But $\alpha_V(e_i) = \delta_{e_i}$, which does

$$\delta_{e_i}(e_j) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases}$$

So the diagram does not commute, hence f is not unitary. So we have that $f : (V, \{e_i\}_{i=1}^n) \to (W, \{b_i\}_{i=1}^n)$ is unitary if and only if it takes the chosen basis to the chosen basis. $\square$

This proposition gives a complete characterisation of unitary maps in $\mathsf{RLef}_{\mathsf{B}}$.

Chapter 5

The support functor

5.1 Introduction

In the last chapter, we introduced the category of reflexive Lefschetz spaces with morphisms as continuous linear maps. In the finite-dimensional case, we construct a free Hilbert space on a finite set, which acts as the basis for the space thus constructed. This construction will now be replaced by constructing a reflexive Lefschetz space on a finiteness space. In this chapter, we will introduce finiteness spaces and give the construction of a (reflexive) Lefschetz space on a finiteness space. This construction is due to Ehrhard [19], and hence will be referred to as Ehrhard's construction.

In the finite-dimensional case, one can simply forget the Hilbert space structure of a given Hilbert space with a basis to extract a finite set. Analogously, we first observe that the web of the finiteness space acts as a basis (in the sense of Section 4.4) for the Lefschetz space constructed on it using Ehrhard's construction. Then we prove that the finitary sets are encoded in the Lefschetz spaces arising out of this construction as the support of convergent nets of finite sums over this basis. We use this observation to give a functorial construction of a finiteness space starting with the data of a reflexive Lefschetz space with a basis. We prove that, on objects, this construction is inverse to Ehrhard's construction to classify the topologies on reflexive Lefschetz spaces as finiteness topologies.

5.2 Finiteness spaces

A finiteness space is a set with a family of subsets that, in some ways, behaves like a collection of finite subsets.

Definition 5.2.1. [19] Let X be a set and $\mathcal{U} \subseteq \mathcal{P}(X)$ be a family of subsets of X. We then define $\mathcal{U}^{\perp} := \{S \subseteq X : |S \cap u| < \infty, \forall u \in \mathcal{U}\}$. $\mathcal{U}^{\perp}$ is called the **finitary complement** of $\mathcal{U}$.

Here are some basic observations on the finitary complementation:

Lemma 5.2.2. [40] Let X be any set, and let $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(X)$. Then the following facts hold:

(i) $\mathcal{A} \subseteq \mathcal{B} \implies \mathcal{B}^{\perp} \subseteq \mathcal{A}^{\perp}$

(ii) $\mathcal{A} \subseteq \mathcal{A}^{\perp\perp}$

(iii) $\mathcal{A}^{\perp} = \mathcal{A}^{\perp\perp\perp}$

(iv) $\mathcal{A}^{\perp}$ is downset closed and closed under finite unions.

(v) $\mathcal{A} = \mathcal{A}^{\perp\perp} \implies \mathcal{P}_f(X) \subseteq \mathcal{A}$, where $\mathcal{P}_f(X)$ is the set of all finite subsets of X.

Proof.

(i) By definition, $\mathcal{B}^\perp = \{S \subseteq X : |S \cap u| < \infty, \forall b \in \mathcal{B}\}$, but then $\mathcal{A} \subseteq \mathcal{B}$, so $\mathcal{B}^\perp \subseteq \mathcal{A}^\perp$.

(ii) Observe that the definition is symmetrical, and as $\mathcal{A}^\perp = \{S \subseteq X : |S \cap u| < \infty, \forall b \in \mathcal{A}\}$, so $|S \cap a| < \infty$ for any $S \in \mathcal{A}^\perp$ and $a \in \mathcal{A}$, so $\mathcal{A} \subseteq \mathcal{A}^{\perp\perp}$.

(iii) We have $\mathcal{A} \subseteq \mathcal{A}^{\perp\perp}$ from (ii), so from (i), we have, $\mathcal{A}^{\perp\perp\perp} \subseteq \mathcal{A}^\perp$. And from (ii) again by substituting $\mathcal{A}^\perp$, we have, $\mathcal{A}^\perp \subseteq \mathcal{A}^{\perp\perp\perp}$, so we have $\mathcal{A}^\perp = \mathcal{A}^{\perp\perp\perp}$.

(iv) Immediate by definition.

(v) It is clear that $\mathcal{A}^{\perp\perp}$ contains all finite subsets of X, as any finite set intersects finitely with any set. Now as $\mathcal{A} = \mathcal{A}^{\perp\perp}$, so it is finitary complement of some family. Hence we have $\mathcal{P}_f(X) \subseteq \mathcal{A}$.

Definition 5.2.3.

(a) [5] Let X be a set and let $\mathcal{U}$ be a family of subsets of X. Then $(X, \mathcal{U})$ is called a **pre-finiteness space** if the following holds:

- The empty set is in $\mathcal{U}$.
- For each $x \in X$, $\{x\} \in \mathcal{U}$.
- $\mathcal{U}$ is downward closed, that is, if $u_1, u_2 \in \mathcal{P}(X)$, $u_1 \subseteq u_2$ and $u_2 \in \mathcal{U}$, then $u_1 \in \mathcal{U}$.
- $\mathcal{U}$ is closed to finite unions, that is, if $u_1, u_2 \in \mathcal{U}$, then $u_1 \cup u_2 \in \mathcal{U}$.

(b) [19] Let X be a set and let $\mathcal{U}$ be a family of subsets of X. Then $(X, \mathcal{U})$ is called a **finiteness space** if and only if $\mathcal{U} = \mathcal{U}^{\perp\perp}$.

Remark 5.2.4. A finiteness space can be equivalently defined as a triple $(X, \mathcal{A}, \mathcal{B})$, where X is a set, called the **web**, and $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(X)$ such that $\mathcal{A} = \mathcal{B}^{\perp}$ and $\mathcal{B} = \mathcal{A}^{\perp}$. The elements of $\mathcal{A}$ are called **finitary sets** of X, and the elements of $\mathcal{B}$ are called **co-finitary sets** of X. Note that the finitary and cofinitary sets determine each other. Every set X carries two trivial finitary structures $(X, \mathcal{P}_f(X), \mathcal{P}(X))$, $(X, \mathcal{P}(X), \mathcal{P}_f(X))$. Moreover, by Lemma 5.2.2 (v), for any finite set X, the only finitary structure possible on it is the trivial one $(X, \mathcal{P}(X), \mathcal{P}(X))$.

We also have by Lemma 5.2.2 (iii), that for any set X and any family of subsets $\mathcal{U} \subseteq \mathcal{P}(X)$, $(X, \mathcal{U}^{\perp})$ is always a finiteness space. And by Remark 5.2.4, it is the case that every finiteness space is of this form. Thus, we can generate a class of examples of finiteness space by defining our finitary sets to be the complement of a given family.

A non-trivial example of a finiteness space is given by taking X a set with a total order on it and defining $\mathcal{U} := \{S \subseteq X : |S \cap \{x \in X : x \leq y\}| < \infty, \forall y \in X\}$. Since $\mathcal{U}$ is of the form $\mathcal{V}^{\perp}$, where $\mathcal{V} = \{x : x \leq y\}_{y \in X}$. So $(X, \mathcal{U})$ is a finiteness space. It was also proved in [8] that for any topological space X, $(X, \mathcal{B})$, where $\mathcal{B}$ is the collection of bounded subsets of X is a pre-finiteness space.

Remark 5.2.5. Observe that given any set X, taking finitary complements of $\mathcal{A} \subseteq \mathcal{P}(X)$ induces a Galois connection [11] on $\mathcal{P}(\mathcal{P}(X))$ ordered by inclusion.

$$\begin{tikzcd}
(\mathcal{P}(\mathcal{P}(X)), \subseteq) \arrow[r, bend left, "(_)^{\perp}"] \arrow[r, phantom, "\perp"] & (\mathcal{P}(\mathcal{P}(X)), \subseteq) \arrow[l, bend left, "(_)^{\perp}"]
\end{tikzcd}$$

Finiteness structures on X, $(X, \mathcal{U})$, then exactly are the subsets $\mathcal{U} \subseteq \mathcal{P}(X)$ for which this adjunction is an equivalence.

Lemma 5.2.6. Every finiteness space is a pre-finiteness space.

Proof. Immediate using Lemma 5.2.2. As $\mathcal{U} = (\mathcal{U}^{\perp})^{\perp}$, so $\mathcal{U}$ is downset closed and closed under finite unions. Moreover, $\mathcal{U}$ contains all finite sets, in particular, singletons and the null set. $\square$

To emphasize the full structure of a finiteness space, we will at times write $(X, \mathcal{U})$ as $(X, \mathcal{U}, \mathcal{U}^{\perp})$. The morphisms of finiteness spaces are given by relations that preserve finitary sets and reflect cofinitary sets.

Definition 5.2.7. [19] A **finiteness relation** $R : (X, \mathcal{U}, \mathcal{U}^{\perp}) \to (Y, \mathcal{V}, \mathcal{V}^{\perp})$ between two finiteness spaces is a relation $R : X \to Y$ such that

(i) For all $A \in \mathcal{U}$, $A \rhd R := \{y \in Y : \exists x \in A \text{ such that } xRy\} \in \mathcal{V}$, and

(ii) For all $B \in \mathcal{V}^{\perp}$, $R \lhd B := \{x \in X : \exists y \in B \text{ such that } xRy\} \in \mathcal{U}^{\perp}$.

So a relation between finiteness spaces is a relation that preserves finitary sets and pulls back co-finitary sets. In fact, it is the case that in the presence of (i), a weaker condition suffices.

Lemma 5.2.8. [19] Let $R : (X, \mathcal{U}, \mathcal{U}^{\perp}) \to (Y, \mathcal{V}, \mathcal{V}^{\perp})$ be a relation between two finiteness spaces such that

(i) For all $A \in \mathcal{U}$, $A \rhd R := \{y \in Y : \exists x \in A : xRy\} \in \mathcal{V}$, and

(ii)' For all $y \in Y$, $R \lhd \{y\} := \{x \in X : xRy\} \in \mathcal{U}^{\perp}$.

Then R is a finiteness relation.

Proof. We need to prove that for all $B \in \mathcal{V}^{\perp}, R \lhd B \in \mathcal{U}^{\perp}$, that is for any $A \in \mathcal{U}$, $|(R \lhd B) \cap A| < \infty$. Now as $A \in \mathcal{U}$, so $A \rhd R \in \mathcal{V}$, and so $|(A \rhd R) \cap B| < \infty$. Now observe,

$$\begin{aligned} (R \lhd B) \cap A &= \{x \in A : \exists y \in B : xRy\} \\ &= \cup_{y \in B \cap (R \rhd A)} (R \lhd \{y\}) \cap A \end{aligned}$$

which is a finite union of finite sets, as $R \lhd \{y\} \in \mathcal{V}^{\perp}$ and so $|(R \lhd B) \cap A| < \infty$. □

We have that finiteness relations compose, and the identity is a finiteness relation, making finiteness spaces into a category with morphisms as finiteness relations.

Lemma 5.2.9. Finiteness relations compose, and the identity for composition is given by the identity in the category of relations, that is, $\mathrm{id}_X := \{(x, x) : x \in X\}$

Proof. Let $R : X \to Y$ and $S : Y \to Z$ be two finiteness relations. Then for any $A \in \mathcal{U}$, $A \rhd (RS) = (A \rhd R) \rhd S$ and so $A \rhd (RS) \in \mathcal{W}^{\perp}$. Similarly for any $B \in \mathcal{W}^{\perp}$, $(RS) \lhd B = S \lhd (R \lhd B)$ and so $(RS) \lhd B \in \mathcal{U}^{\perp}$. So finiteness relations compose. It is then immediate that the identity relation (in Rel) is a finiteness relation. □

So we define the category of finiteness spaces and relations FinRel as

Objects: Finiteness spaces $(X, \mathcal{U}, \mathcal{U}^{\perp})$;

Morphisms: $R : (X, \mathcal{U}, \mathcal{U}^{\perp}) \to (Y, \mathcal{V}, \mathcal{V}^{\perp})$, a finiteness relation;

Composition: as in Rel;

identity: the identity relation.

We can define an involution on FinRel by

$$\mathsf{FinRel} \xrightarrow{(\_)^*} \mathsf{FinRel}$$

$$\begin{array}{ccc} (X, \mathcal{U}, \mathcal{U}^\perp) & & (X, \mathcal{U}^\perp, \mathcal{U}^{\perp\perp} = \mathcal{U}) \\ {\scriptstyle R}\big\downarrow & & \big\uparrow{\scriptstyle R^t} \\ (Y, \mathcal{V}, \mathcal{V}^\perp) & & (Y, \mathcal{V}^\perp, \mathcal{V}^{\perp\perp} = \mathcal{V}) \end{array}$$

where R^t denotes the transpose of the relation R. It is easy to see that R^t is a finiteness relation between $(Y, \mathcal{V}^\perp, \mathcal{V})$ and $(X, \mathcal{U}^\perp, \mathcal{U})$, as for any $B \in \mathcal{V}^\perp$, $B \rhd R^t = R \lhd B \in \mathcal{U}^\perp$ and for any $A \in \mathcal{U}$, $R^t \lhd A = A \rhd R \in \mathcal{V}$. So $R^t : (Y, \mathcal{V}^\perp, \mathcal{V}) \to (X, \mathcal{U}^\perp, \mathcal{U})$ is a finiteness relation. It is an involution as on objects we have that $\mathcal{U}^{\perp\perp} = \mathcal{U}$ and $\mathcal{V}^{\perp\perp} = \mathcal{V}$ and on morphism $R^{tt} = R$.

Remark 5.2.10. We will not delve further into the categorical structure of FinRel, but we shall mention that the involution operation mentioned above $(\_)^* : \mathsf{FinRel} \to \mathsf{FinRel}$ gives FinRel the structure of a $*$- autonomous category, where the tensor product of two finiteness spaces $(X, F(X) := \mathcal{U}, \mathcal{U}^\perp)$ and $(Y, F(Y) := \mathcal{V}, \mathcal{V}^\perp)$ is defined to have the web as the cartesian product of X and Y, with the finitary sets given by $F(X \otimes Y) :=\downarrow \{A \times B : A \in \mathcal{U}, B \in \mathcal{V}\}$. That is finitary set of the tensor of two finiteness space $(X, F(X) := \mathcal{U})$ and $(Y, F(Y) := \mathcal{V})$ is given by the downward closure of the cartesian product of finitary sets of X and Y. The unit for the tensor is the singleton set with the trivial finiteness structure. The finiteness structure on internal hom is given by $F(X \multimap Y) := F(X \otimes Y^\perp)^\perp$, where the "perp-ing" operation is the same as the dualizing operation mentioned above. As is the case with any $*$-

autonomous category [14], FinRel gets a second tensor product, called the par product, where the finiteness structure on the par product is given by $F(X \oplus Y) := F(X^\perp \otimes Y^\perp)^\perp$. Note that the unit for the tensor is the same as the unit for the par. This makes FinRel an isomix LDC.

The core of FinRel is given by the category of finite sets and relations, and their embedding into FinRel gives us the structure of a mixed unitary category. For details, one can look at [19, 40]. There are also other choices of morphisms available for finiteness spaces, such as functions (which are similarly required to preserve finitary sets and reflect co-finitary sets) FinF, partial functions FinPar, and even matrices FinMat [40].

For FinF, finiteness spaces with functions, we have,

Lemma 5.2.11. Let $f : (X, \mathcal{U}, \mathcal{U}^\perp) \to (Y, \mathcal{V}, \mathcal{V}^\perp)$ be a function between two finiteness spaces such that

(i) For all $A \in \mathcal{U}$, $f(A) \in \mathcal{V}$, and

(ii)' For all $y \in Y$, $f^{-1}(y) \in \mathcal{U}^\perp$.

Then f is a finiteness function, that is for any $A \in \mathcal{U}$, $f(A) \in \mathcal{V}$ and for any $B \in \mathcal{V}^\perp$, $f^{-1}(B) \in \mathcal{U}^\perp$

Proof. Exactly analogous to Lemma 5.2.8. □

5.3 From finiteness spaces to Lefschetz spaces

In this section, we give a construction of a Lefschetz space starting from a finiteness space due to Ehrhard [19]. We will describe the construction, generalizing it in a few ways. First, we give the construction from a pre-finiteness space, as described in [5]. Second, we consider sets with arbitrary cardinality, rather than just the countable ones. And lastly, while proving theorems about these spaces, we do not assume that Lefschetz spaces are sequential spaces (that is, spaces whose topology is determined by studying the convergent sequences in them), so we consider nets, rather than sequences. We will consider the construction of space over $\mathbb{C}$, rather than arbitrary rings. We will refer to this construction as Ehrhard's construction.

Let $(X, \mathcal{U})$ be a pre-finiteness space. Consider the set $\mathbb{C}(X, \mathcal{U}) := \{f : X \to \mathbb{C} : \sup(f) \in \mathcal{U}\}$, where $\sup(f) := \{x \in X : f(x) \neq 0\}$ is the support of the function $f : X \to \mathbb{C}$ (Not to be confused with supremum; we will use "supp" for supremum). As the finitary sets are closed under finite unions and are downward closed, this gives $\mathbb{C}(X, \mathcal{U})$ the structure of an Abelian group with pointwise addition.

Lemma 5.3.1. $\mathbb{C}(X, \mathcal{U})$ is an Abelian group with pointwise addition.

Proof.

- Binary operation: Observe that $\sup(f_1 + f_2) \subseteq \sup(f_1) \cup \sup(f_2)$, and since $\mathcal{U}$ is closed under finite unions and is downset closed, so $\sup(f_1 + f_2) \in \mathcal{U}$.

- Associativity: As addition is associative in $\mathbb{C}$.

- Commutativity: As addition is commutative in $\mathbb{C}$.

- Identity: is given by the constant function $f(x) = 0_{\mathbb{C}}$, for all $x \in X$. Note that $\sup(0_{\mathbb{C}}) = \emptyset \in \mathcal{U}$.

- Inverse: for any $f \in \mathbb{C}(X, \mathcal{U})$, its inverse is given by $-f$, where $(-f)(x) := -f(x)$, for all $x \in X$. Note that $\sup(-f) = \sup(f) \in \mathcal{U}$.

□

And with the scalar action coming from the base field $\mathbb{C}$, $\mathbb{C}(X, \mathcal{U})$ is in fact a vector space.

Lemma 5.3.2. $\mathbb{C}(X, \mathcal{U})$ is a vector space, when we define the action of the scalar as $(\alpha \bullet f)(x) := (\alpha f)(x) := \alpha f(x)$, for all $x \in X$.

Proof. Observe that $\sup(\alpha f) = \sup(f)$, when α is non-zero. Otherwise, $\sup(\alpha f) = \emptyset$. In both cases, we have that it is a valid action. Next, we have that $1 \bullet f = f$, and for any $f, g \in \mathbb{C}(X, \mathcal{U})$ and $\alpha, \beta \in \mathbb{C}$, we have that, $\alpha \bullet (\beta \bullet f) = (\alpha \circ_{\mathbb{C}} \beta) \bullet f$, $\alpha \bullet (f + g) = \alpha \bullet f + \alpha \bullet g$, and $(\alpha + \beta) \bullet f = \alpha \bullet f + \beta \bullet f$. These laws are an immediate consequence of the point-wise definition of the addition and scalar multiplication operation and unitality and distributivity laws in $\mathbb{C}$. □

We want to put a topology on $\mathbb{C}(X, \mathcal{U})$. For this, we need to make a couple of definitions.

Definition 5.3.3. [19] We define the topology on $\mathbb{C}(X, \mathcal{U})$ as follows. First, we define for any $u' \in \mathcal{U}^{\perp}$, the set $V_{u'} := \{f \in \mathbb{C}(X, \mathcal{U}) : \sup(f) \cap u' = \emptyset\}$. Note that for any $f \in \mathbb{C}(X, \mathcal{U})$, $|\sup(f) \cap u'| < \infty$, as $\sup(f) \in \mathcal{U}$ by construction.

And we say that a subset $S \subseteq \mathbb{C}(X, \mathcal{U})$ is open if and only if for any $s \in S$, there exists $u' \in \mathcal{U}^{\perp}$, such that $s + V_{u'} \subseteq S$. This is analogous to the definition of an open set in any topological space, where the set is said to be open if we can put a neighbourhood around each point contained in the set. The role of neighbourhood for $\mathbb{C}(X, \mathcal{U})$ will be played by $V_{u'}$.

With this definition, we confirm that it is indeed a topology.

Lemma 5.3.4. Definition 5.3.3 provides a topology on $\mathbb{C}(X, \mathcal{U})$.

Proof.

- The emmpty set $(\emptyset)$ is vacuously open.

- $\mathbb{C}(X, \mathcal{U})$ is open as for any $x \in \mathbb{C}(X, \mathcal{U})$, any $u' \in \mathcal{U}^{\perp}$ works.

- If $\{U_i : i \in \{1, ...n\}\}$ is any finite collection of open sets in $\mathbb{C}(X, \mathcal{U})$, then for any $x \in \cap_{i=1}^{n} U_i$, there exists $V_{u_1}, ...V_{u_n}$, such that $x + V_{u_i} \subseteq U_i,\ i \in \{1, ...n\}$. Now we must find $u' \in \mathcal{U}^{\perp}$, such that $x + V_{u'} \subseteq \cap_{i=1}^{n} U_i$. Now observe that $\cap_{i=1}^{n} V_{u_i} = V_{\cup_{i=1}^{n} u_i}$, and as $\mathcal{U}^{\perp}$ is closed under finite unions, so we define $u' := \cup_{i=1}^{n} u_i$, and as $x + V_{u_i} \subseteq U_i$, so we have $x + V_{u'} \subseteq \cap_{i=1}^{n} U_i$.

- $\{U_i : i \in I\}$ is any arbitrary collection of open sets in $\mathbb{C}(X, \mathcal{U})$, then if $x \in \cup_{i=1}^{n} U_i$, then $x \in U_{i_0}$, for some $i_0 \in I$, and as U_{i_0} is open, so there exists $u_{i_0} \in \mathcal{U}^{\perp}$, such that $x + V_{u_{i_0}} \subseteq U_{i_0}$ and hence $x + V_{u_{i_0}} \subseteq \cup_{i=1}^{n} U_i$.

□

We want to show that $\mathbb{C}(X, \mathcal{U})$ is a Lefschetz space; we need to provide a local basis at the origin of open linear subspaces. These open linear subspaces are given by $V_{u'}$.

Lemma 5.3.5. $V_{u'}$ is an open linear subspace of $\mathbb{C}(X, \mathcal{U})$

Proof. $V_{u'}$ being a linear subspace is immediate, and for any $x \in V_{u'}$, $x+V_{u'} \subseteq V_{u'}$, so $V_{u'}$ is open. □

Lemma 5.3.6. $\{V_{u'}\}_{u' \in \mathcal{U}^\perp}$ forms a neighbourhood basis around the origin in $\mathbb{C}(X, \mathcal{U})$.

Proof. Let U be any open subset of $\mathbb{C}(X, \mathcal{U})$ such that $0 \in U$, then by definition, there exists $u' \in \mathcal{U}^\perp$, such that $0 + V_{u'} \subseteq U$, that is, $V'_u \subseteq U$, so $\{V_{u'}\}_{u' \in \mathcal{U}^\perp}$ forms an open neighborhood basis at $0_{\mathbb{C}(X,\mathcal{U})}$ consisting of linear subspaces of $\mathbb{C}(X, \mathcal{U})$ □

It is not difficult to see that the scalar multiplication map $\mathbb{C} \times \mathbb{C}(X, \mathcal{U}) \to \mathbb{C}(X, \mathcal{U})$ and the addition map $\mathbb{C}(X, \mathcal{U}) \times \mathbb{C}(X, \mathcal{U}) \to \mathbb{C}(X, \mathcal{U})$ are continuous, where $\mathbb{C}$ is given the discrete topology. So $\mathbb{C}(X, \mathcal{U})$ is a topological vector space with a neighbourhood system of open linear subspaces around the origin. Hence, we have:

Proposition 5.3.7. $\mathbb{C}(X, \mathcal{U})$ is a Lefschetz Space.

Moreover by Ehrhard's classification [19], we have that the topology on the Lefschetz space constructed from a (pre-) finiteness space, $\mathbb{C}(X, \mathcal{U})$ is finer than the product topology ($\mathbb{C}$ with the discrete topology and $\mathbb{C}(X, \mathcal{U})$ being considered as the product $\mathbb{C}^{|X|}$), and coarser than the discrete topology (on $\mathbb{C}^{|X|}$).

We next prove that the topology on $\mathbb{C}(X, \mathcal{U})$ as in Definition 5.3.3 is Hausdorff and totally disconnected. For Hausdorffness, the opens separating distinct functions f, g in $\mathbb{C}(X, \mathcal{U})$ are given by $V_{\{x\}}$, where x is a point where $f(x) \neq g(x)$.

Lemma 5.3.8. [5] The topology on $\mathbb{C}(X,\mathcal{U})$ as in Definition 5.3.3 is Hausdorff.

Proof. First note that $\mathbb{C}(X,\mathcal{U})$ is a topological vector space, so translations are homeomorphisms, and as $V_{u'}$ is open around $0_{\mathbb{C}(X,\mathcal{U})}$, so for any $f \in \mathbb{C}(X,\mathcal{U})$ and $u' \in \mathcal{U}^{\perp}$, $f + V_{u'}$ is open. Now let $f, g \in \mathbb{C}(X,\mathcal{U})$, such that $f \neq g$, then there exists $x \in X$, such that $f(x) \neq g(x)$. Now any finite set is in $\mathcal{U}^{\perp}$, in particular $\{x\} \in \mathcal{U}^{\perp}$. Then observe $f \in f + V_{\{x\}}$, as $0 \in V_{\{x\}}$, similarly $g \in g + V_{\{x\}}$ and $f + V_{\{x\}} \cap g + V_{\{x\}} = \emptyset$ (as at x, the value that functions in these sets take are $f(x)$ and $g(x)$ respectively). So any two distinct points can be separated using disjoint open sets. So $\mathbb{C}(X,\mathcal{U})$ is a Hausdorff space. $\square$

We next show that every open set in $\mathbb{C}(X,\mathcal{U})$ is also closed, making the topology totally disconnected.

Lemma 5.3.9. The topology on $\mathbb{C}(X,\mathcal{U})$ as in Definition 5.3.3 is totally disconnected.

Proof. First note that it is enough to prove that each open neighbourhood around $0_{\mathbb{C}(X,\mathcal{U})}$ is both open and closed. The sets $\{V_{u'}\}_{u' \in \mathcal{U}^{\perp}}$ form a neighbourhood around $0_{\mathbb{C}(X,\mathcal{U})}$ and they are open by Lemma 5.3.5. We will now prove that it is also closed. Let $u' \in \mathcal{U}^{\perp}$ and let $f \in \mathbb{C}(X,\mathcal{U}) - V_{u'}$, then there exists $x \in \sup(f) \cap u'$. Then $f + V_{\{x\}} \subseteq \mathbb{C}(X,\mathcal{U}) - V_{u'}$, and as the choice of f was arbitrary, so $\mathbb{C}(X,\mathcal{U}) - V_{u'}$ is open, and hence $V_{u'}$ is closed. $\square$

For $(X,\mathcal{U})$, a finiteness space, $\mathbb{C}(X,\mathcal{U})$ is a Lefschetz space, so we can take its dual in Lef (the category of Lefschetz spaces and continuous linear maps) to get $\mathbb{C}(X, \mathcal{U})^*$. We then prove that this is exactly the space we get if we apply Ehrhard's construction to the dual finiteness space $(X, \mathcal{U}^{\perp}, \mathcal{U})$.

The key idea used in this proof is that in $\mathbb{C}(X, \mathcal{U})$, where $(X, \mathcal{U})$ is a finiteness space, the limit of finite sums of functions of point support over a finitary set converges, thus any continuous linear functional on $\mathbb{C}(X, \mathcal{U})$ must intersect finitely with any finitary set, as the topology on the base field is discrete.

Proposition 5.3.10. Let $\mathbb{C}(X, \mathcal{U})$ be the Lefschetz space constructed (using Ehrhard's construction) from the finiteness space $(X, \mathcal{U}, \mathcal{U}^{\perp})$. Then the dual space $\mathbb{C}(X, \mathcal{U})^* := \mathbb{C}(X, \mathcal{U}) \multimap \mathbb{C}$ arises from Ehrhard's construction on the dual finiteness space $(X, \mathcal{U}^{\perp}, \mathcal{U})$.

Proof. First we will prove that $|(\mathbb{C}(X, \mathcal{U}))^*| \cong |\mathbb{C}(X, \mathcal{U}^{\perp})|$, that is to say they have same underlying set .

First we claim that $\sum_{x \in S, S \subseteq X, S \in \mathcal{U}} f_x$, where

$$f_x(x') = \begin{cases} 1, & \text{if } x' = x, \\ 0, & \text{if } x' \neq x, \end{cases}$$

is convergent in $\mathbb{C}(X, \mathcal{U})$. To see this, first note that, the open neighbourhoods around $0_{\mathbb{C}(X,\mathcal{U})}$ are of the form $\{V_{u'}\}_{u' \in \mathcal{U}^{\perp}}$. Since $\mathcal{U} \subseteq \mathcal{U}^{\perp\perp}$, so for any $u' \in \mathcal{U}^{\perp}$, $|S \cap u'| < \infty$. Now for any $u' \in \mathcal{U}^{\perp}$, we define $S_0 := S \cap u'$. Then $S_0 \in \mathrm{Fin}(S)$, and $\sum_{x \in T, T \subseteq S} f_x - \sum_{x \in W, W \subseteq S} f_x \in V_{u'}$, for all $T, W \in \mathrm{Fin}(S)$, such that $S_0 \subseteq T$, $S_0 \subseteq W$. Hence the net determined by the finite sums of $\sum_{x \in S, S \subseteq X, S \in \mathcal{U}} f_x$ is Cauchy.

Now consider the function defined as follows:

$$g(x) = \begin{cases} 1, & \text{if } x \in S, \\ 0, & \text{if } x \notin S. \end{cases}$$

Then clearly $\sup(g) = S \in \mathcal{U}$. Now observe that $\mathrm{Lim}_{T\in \mathrm{Fin}(S)} \sum_{x\in T\subseteq S, S\in\mathcal{U}} f_x - g = 0_{\mathbb{C}(X,\mathcal{U})}$, and as $\mathbb{C}(X,\mathcal{U})$ is a topological vector space, so $\mathrm{Lim}_{T\in \mathrm{Fin}(S)} \sum_{x\in T\subseteq S, S\in\mathcal{U}} f_x = g$.

Now let $\phi : \mathbb{C}(X,\mathcal{U}) \to \mathbb{C}$ be any linear map. Since $\mathrm{Lim}_{T\in \mathrm{Fin}(S)} \sum_{x\in T\subseteq S, S\in\mathcal{U}} f_x$ is convergent in $\mathbb{C}(X,\mathcal{U})$, so for ϕ to be continuous, $\mathrm{Lim}_{T\in \mathrm{Fin}(S)} \sum_{x\in T\subseteq S, S\in\mathcal{U}} \phi(f_x)$ must be convergent in $\mathbb{C}$. But $\mathbb{C}$ has the discrete topology, so any convergent sequence must be constant, so this means that there exists $W \in \mathrm{Fin}(S)$, such that $\sum_{x\in Y\subseteq S, S\in\mathcal{U}} \phi(f_x) = z$, for all $Y \in \mathrm{Fin}(S)$, so that $Y \supseteq W$. This precisely means $|\sup(\phi) \cap S| \leq |W| < \infty$, and since this happens for any $S \in \mathcal{U}$, so this implies $\sup(\phi) \in \mathcal{U}^{\perp}$. Note that here we are identifying ϕ as a function from X to $\mathbb{C}$, as ϕ is a linear map, so its values are fixed by $\phi(f_x), x \in X$.

Now, for the other way round, we will prove that if $\phi : \mathbb{C}(X,\mathcal{U}) \to \mathbb{C}$ is a linear map such that $\sup(\phi) \in \mathcal{U}^{\perp}$, then ϕ is continuous. For this, let $f_\bullet : I \to \mathbb{C}(X,\mathcal{U})$ be a convergent net in $\mathbb{C}(X,\mathcal{U})$, and let $f_\bullet \to f$. Then by definition, for any $u' \in \mathcal{U}^{\perp}$, there exists $i_0^{u'} \in I$, such that $(f_i - f)|_{u'} = 0$, for all $i \geq i_0^{u'}$. In particular, as $\sup(\phi) \in \mathcal{U}^{\perp}$, so there exists $i_0^{\phi} \in I$, such that $(f_i - f)|_{u'} = 0$, for all $i \geq i_0^{\phi}$. Hence the net $\phi(f_\bullet) - \phi(f)$ is eventually constant in $\mathbb{C}$, since $\mathbb{C}$ has the discrete topology, so this implies $\phi(f_\bullet)$ converges to $\phi(f)$ in $\mathbb{C}$. So $\phi : \mathbb{C}(X,\mathcal{U}) \to \mathbb{C}$ is continuous.

So we have $|(\mathbb{C}(X,\mathcal{U}))^*| \cong |\mathbb{C}(X,\mathcal{U}^{\perp})|$. Finally, the topology on $|(\mathbb{C}(X,\mathcal{U}))^*|$ is given by the pointwise hom, but the only open linear subspace of $\mathbb{C}$ (apart from $\mathbb{C}$ itself) is $0_{\mathbb{C}}$, the trivial subspace. So, identifying ϕ as a function of X to $\mathbb{C}$, we have the neighborhood around $0_{(\mathbb{C}(X,\mathcal{U}))^*}$ is given by sets of the form $V_u := \{f : X \to \mathbb{C} : \sup(f) \cap u = \emptyset\}_{u \in \mathcal{U}}$. But that is exactly the space $\mathbb{C}(X, \mathcal{U}^\perp)$ □

We next observe that Lefschetz spaces arising out of Ehrhard's construction on a pre-finiteness space $(X, \mathcal{U})$ are complete if and only if $(X, \mathcal{U})$ is a finiteness space. This was observed in [5], we provide an alternative proof for the fact that when $(X, \mathcal{U})$ is not a finiteness space, then $\mathbb{C}(X, \mathcal{U})$ is not complete. This relies on the observation that functions of point support (by which we mean f_x, as defined above) over some set in $\mathcal{U}^{\perp\perp}$ are still Cauchy, but not convergent.

Proposition 5.3.11. [5] $\mathbb{C}(X, \mathcal{U})$, that is, the Lefschetz space constructed from a pre-finiteness space is complete (see Definition 2.3.14) if and only if $(X, \mathcal{U})$ is a finiteness space.

Proof. ($\Longrightarrow$)First, we will prove that when $(X, \mathcal{U})$ is a finiteness space, then the Lefschetz space constructed from it, $\mathbb{C}(X, \mathcal{U})$, is complete. For this, let $f_\bullet : I \to \mathbb{C}(X, \mathcal{U})$ be any Cauchy net. Since $\{x\} \in \mathcal{U}^\perp$ for any $x \in X$, therefore, by definition there exists $i_0^x \in I$, such that $(f_i - f_j)|_x = 0$, for all $i, j \in I$, so that $i, j \geq i_0^x$. In other words, the net $f_\bullet$ is eventually constant for any $x \in X$. We now define $f : X \to \mathbb{C}$ as $f(x) := f_{i_o^x}(x)$. Now we will prove that $\sup(f) \in \mathcal{U}$. Now as $\{V_{u'}\}_{u' \in \mathcal{U}^\perp}$ are open linear subspaces around $0_{\mathbb{C}(X,\mathcal{U})}$, so for any $u' \in \mathcal{U}^\perp$, there exists $i_0^{u'}$, such that $f_i|_{u'} = f_j|_{u'}$, for all $i, j \geq i_0^{u'}$. Now fix a $u' \in \mathcal{U}^\perp$, and $x \in u'$. Since I is a directed set, we can take i, such that $i \geq i_0^x, i_0^{u'}$. Then observe $f(x) := f_{i_0^x}(x) = f_i(x) = f_{i_0^{u'}}(x)$. Hence $|\sup(f) \cap u'| = |\sup(f_{i_0^{u'}}) \cap u'| < \infty$, as $\sup(f_{i_0^{u'}}) \in \mathcal{U}$. Since the choice of u' was arbitrary, we have that $|\sup(f) \cap u'| < \infty$, for all $u' \in \mathcal{U}^{\perp}$. Hence $\sup(f) \in \mathcal{U}^{\perp\perp} = \mathcal{U}$, as $(X, \mathcal{U})$ is a finiteness space. Note that this also proves that $f_{\bullet} \to f$, as for any $x \in u'$, $f(x) = f_{i_0^{u'}}(x)$.

($\Longleftarrow$)Now, for the other way round, we will prove that when $(X, \mathcal{U})$ is not a finiteness space, but only a pre-finiteness space, then the space obtained from Ehrhard's construction $\mathbb{C}(X, \mathcal{U})$ is not complete. As $\mathbb{C}(X, \mathcal{U})$ is not a finiteness space, so there exists $S \subseteq X$, such that $S \in \mathcal{U}^{\perp\perp}$, but $S \notin \mathcal{U}$. Now we will prove that $\sum_{x \in S} f_x$ is Cauchy in $\mathbb{C}(X, \mathcal{U})$. For this, observe that for any $u' \in \mathcal{U}^{\perp}$, if we define, $S_0 := S \cap u'$, then $S_0 \in \text{Fin}(S)$ and $\sum_{x \in T} f_x - \sum_{x \in W} f_x \in V_{u'}$, for all $T, W \in \text{Fin}(S)$, so that $S_0 \subseteq T, S_0 \subseteq W$, and hence the net determined by the finite sums of $\sum_{x \in S} f_x$ is Cauchy. Now suppose that $\sum_{x \in S} f_x$ is convergent to some limit $f \in \mathbb{C}(X, \mathcal{U})$. Then as $\mathbb{C}(X, \mathcal{U})$ is a topological vector space, so $\text{Lim}_{T \in \text{Fin}(S)} \sum_{x \in T \subseteq S} f_x - f = 0_{\mathbb{C}(X, \mathcal{U})}$. Now as $\{x\} \in \mathcal{U}^{\perp}$ for any $x \in X$, so this implies that there exists $W \in \text{Fin}(S)$, such that $(\sum_{x \in V} f_x - f)|_x = 0$, for all $V \in \text{Fin}(S)$, so that $V \supseteq W$. So we must define $f(x) := 1$, if $x \in S$, and $f(x) := 0$, otherwise. But then $\sup(f) = S$, which by assumption does not belong to $\mathcal{U}$. So $f \notin \mathbb{C}(X, \mathcal{U})$, which is a contradiction. And hence $(X, \mathcal{U})$ is not complete.

So we have that the Lefschetz space constructed from a pre-finiteness space is complete if and only if $(X, \mathcal{U})$ is a finiteness space. $\square$

Now we note an interesting property of $\mathbb{C}(X, \mathcal{U})$, when $(X, \mathcal{U})$ is a finiteness space. The following proposition (Proposition 5.3.12) says that the finitary sets of a finiteness space get encoded in the Lefschetz spaces on it as the support of convergent nets of finite sums over the functions of point support.

This observation will be crucial in the next section, when we will construct a (pre-) finiteness space starting from a Lefschetz space.

Proposition 5.3.12. Let $(X,\mathcal{U})$ be a finiteness space, and let $\mathbb{C}(X,\mathcal{U})$ be the Lefschetz space constructed on $(X,\mathcal{U})$ using Ehrhard's construction. Then $\sum_{x\in S\subseteq X} f_x$ converges in $\mathbb{C}(X,\mathcal{U})$ if and only if $S\in\mathcal{U}$.

Proof. $(\Longrightarrow)$First, let $\sum_{x\in S\subseteq X} f_x$ be convergent in $\mathbb{C}(X,\mathcal{U})$ to some $f\in\mathbb{C}(X,\mathcal{U})$. Then, as $\mathbb{C}(X,\mathcal{U})$ is a topological vector space, so this means $\mathrm{Lim}_{A\in\mathrm{Fin}(S)}\sum_{x\in A} f_x - f = 0_{\mathbb{C}(X,\mathcal{U})}$. Now open sets around $0_{\mathbb{C}(X,\mathcal{U})}$ are of the form $\{V_{u'}\}_{u'\in\mathcal{U}^\perp}$, so this means for any $u'\in\mathcal{U}^\perp$, there exists $A_0\in\mathrm{Fin}(S)$, such that $(\sum_{x\in A} f_x)|_{u'} = f|_{u'}$, for all $A\in\mathrm{Fin}(S)$, so that $A\supseteq A_0$. Now recall $\mathrm{sup}(f_s)=\{s\}$, so $\cup_{x\in S}\mathrm{sup}(f_x)=S$. So this means $|S\cap u'|\leq|A_0|<\infty$. Now the choice of u' was arbitrary, so we have that $|S\cap u'|\leq|A_0|<\infty$, for all $u'\in\mathcal{U}^\perp$, so $S\in\mathcal{U}^{\perp\perp}=\mathcal{U}$.

$(\Longleftarrow)$The other way round, consider for any $S\in\mathcal{U}$, $\sum_{x\in S\subseteq X} f_x$. We define $f: X\to\mathbb{C}$ as $f(x):=1$, if $x\in S$ and $f(x)=0$, if $x\notin S$. We claim $\sum_{x\in S\subseteq X} f_x = f$. For this observe that for any $u'\in\mathcal{U}^\perp$, $|S\cap u'|<\infty$, as $\mathcal{U}\subseteq\mathcal{U}^{\perp\perp}$. Define $S_0:=S\cap u'$, then we have $(\sum_{x\in T\subseteq S} f_x - f)|_{u'} = 0_{\mathbb{C}(X,\mathcal{U})}$, for all $T\in\mathrm{Fin}(S)$, so that $T\supseteq S_0$. Since choice of u' was arbitrary, so we have $\sum_{x\in S\subseteq X} f_x - f = 0_{\mathbb{C}(X,\mathcal{U})}$, so $\sum_{x\in S\subseteq X} f_x = f$, and we are done. $\square$

Remark 5.3.13. Now, observe that $\{f_x\}_{x\in X}$ forms a basis for $\mathbb{C}(X,\mathcal{U})$ as clearly $\{f_x\}_{x\in X}$ are linearly independent and any function in $\mathbb{C}(X,\mathcal{U})$ with a finitary support can be expressed as the convergent sum over these functions of point support by Proposition 5.3.12. This solidifies the notion that the web of the finiteness space acts as a basis for the Lefschetz space on it.

We also observe that in the above proposition, the only thing that mattered for the limit of the finite sums to converge or not was their support, and not the scalar coefficients. This suggests the following:

Proposition 5.3.14. Let $(X, \mathcal{U})$ be a finiteness space. Then if $\sum_{s\in S\subseteq X} \alpha_s f_s$ converges in $\mathbb{C}(X, \mathcal{U})$ for some choices of non-zero $\alpha_s \in \mathbb{C}$, then $\sum_{s\in S\subseteq X} \alpha'_s f_s$ converges for any other choice of $\alpha'_s \in \mathbb{C}$, $\alpha'_s \neq 0$ for any $s \in S$.

Proof. First, as $\sum_{s\in S\subseteq X} \alpha_s f_s$ is convergent so by Proposition 5.3.12, we have that $S \in \mathcal{U}$. Then a proof similar to Proposition 5.3.12 will show that $\sum_{s\in S\subseteq X} \alpha_s f_s = f$, where $f(x) := \alpha_s$, if $x \in S$, and $f(x) = 0$, if $x \notin S$. Similarly, a proof similar to that of the above theorem will show that $\sum_{s\in S\subseteq X} \alpha'_s f_s = f'$, where $f'(x) := \alpha'_s$, if $x \in S$, and $f'(x) = 0$, if $x \notin S$. □

Thus, the convergence over the basis vectors in $\mathbb{C}(X, \mathcal{U})$ is independent of the choice of (non-zero) scalar coefficients. Note that this property is stronger than what we have for general topological vector spaces, where we are only allowed to scale up all the terms of a series by the same constant.

We will say that a topological vector space has **scalar invariant convergence** if and only if the summable families over the basis vectors in the space are independent of the choice of (non-zero) scalar coefficients.

We already saw that the $\mathbb{C}(X, \mathcal{U})$, when $(X, \mathcal{U})$ is a finiteness space, has scalar invariant convergence. Now a basis for $\mathbb{C}(X, \mathcal{U})$ can be given by functions of point support $\{f_x\}_{x\in X}$. We now generalise this theorem to any complete Lefschetz space with a given basis. The key idea in the following proposition is that a given net of finite sums over the basis vectors with some choice of non-zero scalars is Cauchy if and only if it is Cauchy with any other choice of scalar coefficients. The result then follows from the completeness of the space.

Proposition 5.3.15. Complete Lefschetz spaces with a given basis have scalar invariant convergence.

Proof. Let V be a complete Lefschetz space over $\mathbb{C}$ with a given basis $\{v_i\}_{i\in I}$. Then we will prove that if $\sum_{j\in J\subseteq I} \alpha_j v_j$ is Cauchy for some choice of non-zero scalars $\alpha_j \in \mathbb{C}$, then $\sum_{j\in J\subseteq I} \alpha'_j v_j$ is Cauchy for any other choice of (non-zero) scalars α'_j.

Let $\sum_{j\in J\subseteq I} \alpha_j v_j$ be Cauchy, then for any open neighbourhood V_0 around 0_V, there exists $K \in \mathrm{Fin}(I)$ such that $\sum_{l\in L\subseteq J} \alpha_l v_l - \sum_{m\in M\subseteq J} \alpha_m v_m \in V_0$ for any $L, M \in \mathrm{Fin}(I)$, so that $L, M \supseteq K$. In particular $\sum_{k\in K\subseteq I} \alpha_k v_k - \sum_{k'\in K\cup\{v_m\}\subseteq J} \alpha'_k v'_k \in V_0$. Since V_0 is a linear subspace, so this implies for any $v_m \in I - K$, $v_m \in V_0$.

Then observe that $\sum_{j\in J\subseteq I} \alpha'_j v_j$ is Cauchy as, for any $L, M \in \mathrm{Fin}(I)$, so that $L, M \supseteq K$, $\sum_{l\in L\subseteq J} \alpha'_l v_l - \sum_{m\in M\subseteq J} \alpha'_m v_m$ belongs to V_0, as there is no vector v_k, from the set $K \subseteq I$, and as for any $v_m \in I - K$, $v_m \in V_0$. So this sum is a finite linear combination of vectors already in V_0 and hence it belongs to V_0, V_0 being a linear subspace. Hence we have $\sum_{l\in L\subseteq J} \alpha'_l v_l - \sum_{m\in M\subseteq J} \alpha'_m v_m \in V_0$ for any $L, M \in \mathrm{Fin}(I)$, so that $L, M \supseteq K$. So $\sum_{j\in J\subseteq I} \alpha'_j v_j$ is Cauchy as well.

Then the result follows from the completeness of V. □

We now want to prove that any reflexive Lefschetz space has the property of scalar invariant convergence. This will be proven by first proving that any reflexive Lefschetz space is indeed complete and so it has scalar invariant convergence by Proposition 5.3.15.

To prove that any reflexive space is complete, we first observe that for a Lefschetz space V, and a complete (respectively Hausdorff) space W, $V \multimap W$ is complete (respectively Hausdorff). Thus, the topology on the internal hom is determined by the topology of the codomain.

Lemma 5.3.16. Let V be any Lefschetz space and W a Hausdorff Lefschetz space. Then $V \multimap W$ is Hausdorff.

Proof. Lef $f, g \in V \multimap W$, such that $f \neq g$. Then there exists $v \in V$, such that $f(v) \neq g(v)$. Let $w_1 := f(v)$, and , $w_2 := g(v)$. Since W is Hausdorff, so there exists open sets U_1 and U_2 in W such that $w_1 \in U_1, w_2 \in U_2$ and $U_1 \cap U_2 = \emptyset$. And now, $f \in \{\phi : V \to W : \phi(v) \in U_1\} := S_1$ and $g \in \{\phi : V \to W : \phi(v) \in U_2\} := S_2$ and $S_1 \cap S_2 = \emptyset$. Note that S_1, S_2 are open in $V \multimap W$, as they are translates of the open sets around $0_{V \multimap W}$, where, for example S_1 will be a translate of the open subspace U_0 around 0_W by some vector $w \in W$, such that $U_0 + w = U_1$. Hence, distinct points in $V \multimap W$ can be separated, thus $V \multimap W$ is Hausdorff. □

A similar proof, which depends on the topology of the co-domain, gives us the following:

Lemma 5.3.17. Let V be any Lefschetz space and W a complete Lefschetz space. Then $V \multimap W$ is complete.

Proof. Let $f_\bullet : I \to V \multimap W$ be a Cauchy net in $V \multimap W$. Then, by definition, for any open linear subspace U, around $0_{V \multimap W}$, there exists $i_0 \in I$, such that $f_i - f_j \in U$, for all $i, j \geq i_0$. Fix $v_1 \in V$, then we get a Cauchy sequence $f_\bullet^{v_1} : I \to W$ in W and since W is complete so f^{v_1} converges in W. That means for any open linear subspace W_0 around 0_W, there exists $i_{v_1} \in I$, such that $f_i(v) \in W_0$, for all $i \geq i_{v_1}$. Now observe that an open linear subspace U' around $0_{V \multimap W}$ is given by finite intersection of sets of the form $\{h : V \to W : h(v) \in W_0\}$. Now since I is a directed set, we can take i', such that $i \geq i_{v_i}$, for all $i \in \{1, 2, ...n\}$. Then we have that $f_i \in U'$ for all $i \geq i'$. Hence, $f_\bullet : I \to V \multimap W$ converges in $V \multimap W$ and so $V \multimap W$ is complete. □

Corollary 5.3.18. Reflexive Lefschetz spaces are complete and Hausdorff.

Proof. Let V be a reflexive Lefschetz space. As V is reflexive, so we have $V \cong (V \multimap \mathbb{C}) \multimap \mathbb{C}$. Now $\mathbb{C}$ is a one-dimensional discrete space, so it is complete and Hausdorff. Moreover, the isomorphism between V and V^{**} is mediated by the evaluation map, which is a linear continuous isomorphism, so it also preserves uniform properties such as completeness (in addition to Hausdorffness), so V is complete and Hausdorff. □

So we have an important corollary.

Corollary 5.3.19. Reflexive Lefschetz spaces have scalar invariant convergence.

Proof. Any reflexive space is complete by Corollary 5.3.18 and so by Proposition 5.3.15, we have that any reflexive space has the property of scalar invariant convergence. □

To end this section, we give an alternative proof of Lemma 9 from [19]:

Lemma 5.3.20. Let $(X, \mathcal{U})$ be a finiteness space where X is a countable set, and $\mathbb{C}(X, \mathcal{U})$ be the Lefschetz space with basis $\mathcal{B} = \{f_x\}_{x \in X}$ constructed on the finiteness space $(X, \mathcal{U})$ using Ehrhard's construction. Let $\{f_n\}_{n \in \mathbb{N}}$ be a family of basis elements of $\mathbb{C}(X,\mathcal{U})$. Then the series $\sum_{n=1}^{\infty} f_n$ converges in $\mathbb{C}(X,\mathcal{U})$ if and only if $\mathrm{Lim}_{n\to\infty} f_n = 0$.

Proof. First note that the definition of convergence of $\sum_{n=1}^{\infty} f_n$ is the usual definition of convergence of a sequence of partial sums that we are familiar with from analysis. We will first prove that this definition of convergence agrees with the definition of convergence that we have been working with, that is, Definition 4.4.1 for the sum $\sum_{n=1}^{\infty} f_n$.

Let $\sum_{n=1}^{\infty} f_n$ be convergent to some limit $f \in \mathbb{C}(X,\mathcal{U})$. Then by definition, this means for any $u' \in \mathcal{U}^{\perp}$, there exists $N \in \mathbb{N}$ such that $(\sum_{n=1}^{p} f_n)|_{u'} = f|_{u'}$, for all $p \in \mathbb{N}$, such that $p \geq N$. Now call the set $\{f_n\}_{n\in\mathbb{N}} = S \subseteq \mathcal{B}$. Then this precisely means that for any $u' \in \mathcal{U}^{\perp}$, there exists $S_0 \in \mathrm{Fin}(S)$, such that $(\sum_{t\in T} f_t)|_{u'} = f|_{u'}$, for all $T \in \mathrm{Fin}(S)$, so that $T \supseteq S_0$. So $\mathrm{Lim}_{T\in\mathrm{Fin}(S)} \sum_{t\in T} f_t = f$. And hence the definition agrees. Note that this also implies that $\sum_{\sigma(n)=1}^{\infty} f_n = f$, where σ is any permutation of $\mathbb{N}$ by Lemma 4.4.4

Now by Proposition 5.3.12 , we have $\sum_{n=1}^{\infty} f_n = \sum_{s\in S} f_s$ converges if and only if $S \in \mathcal{U}$. Since S is a countable set, so this implies that there exists $N \in \mathbb{N}$, such that $(f_p)|_{u'} = 0$, for all $p \in \mathbb{N}$, such that $p \geq N$. This exactly is the condition that $\mathrm{Lim}_{n\to\infty} f_n = 0$, and we are done. $\square$

Our proof is simpler than one in [19], as we only need to see that the sum over the basis vectors converges if and only if the support is a finitary set (by Proposition 5.3.12), and when the web is countable, it exactly means that the sequence converges to 0 in $\mathbb{C}(X,\mathcal{U})$.

We will also see in Chapter 6 that Ehrhard's construction can be made functorial; this was observed in [5].

5.4 From reflexive Lefschetz spaces to finiteness spaces

Now we will use the observation made in Proposition 5.3.12 to construct a finiteness space out of a reflexive Lefschetz space. We will then prove that this construction is inverse to Ehrhard's construction. Our proof strategy will be as follows. We will start with a reflexive Lefschetz space with a given basis, and construct a pre-finiteness space using this data. We will then construct a Lefschetz space on this pre-finiteness space thus obtained, and prove that it is uniformly homeomorphic to the original Lefschetz space we started with. Then, using Proposition 5.3.11, it will follow that the pre-finiteness space thus constructed is a finiteness space. The proof can be captured using the diagram below:

$$\begin{tikzcd}
(V, \mathcal{B}) \arrow[r, bend left, "\text{Our construction}"] \arrow[dr, dotted, dash, "\text{Uniform homeomorphism}"'] & (|X| := B, \mathcal{U} := \text{support of functions which converges}) \arrow[d, "\text{Ehrhard's construction}"] \\
& \mathbb{C}(X, \mathcal{U})
\end{tikzcd}$$

Now we will proceed to give our construction.

Definition 5.4.1. Let V be a reflexive Lefschetz space (so Hausdorff and complete by Corollary 5.3.18) space over $\mathbb{C}$ with a given basis $\mathcal{B} = \{v_i\}_{i \in I}$. We then define the pair $(X, \mathcal{U})$ as follows. $|X| := \mathcal{B}$ (that is web is just the basis), and

$\mathcal{U} := \{S \subseteq X : \sum_{s \in S} \alpha_s v_s$ converges in V, $\alpha_s \in \mathbb{C}$, $\alpha_s \neq 0$ for any $s \in S\}$.

Remark 5.4.2. Observe that, as V is a complete Lefschetz space, it has the scalar invariance convergence property by Proposition 5.3.15 , so we can always modify our scalars in the series, as long as we do not increase the support.

We prove that the pair defined above is a pre-finiteness space. That the null set and singletons are finitary is immediate. To prove that it is downward-closed and closed under finite unions, we use the completeness of V.

Proposition 5.4.3. The pair $(X, \mathcal{U})$ as defined above (Definition 5.4.1) is a pre-finiteness space.

Proof.

- $\emptyset \in \mathcal{U}$. This is immediate, the sum over the empty set converges to 0_V.

- For each $v_x \in X$, $\{v_x\} \in \mathcal{U}$. This is also immediate, as v_x converges to $v_x \in V$.

- If $S' \subseteq S \in \mathcal{U}$, then $S' \in \mathcal{U}$. For this as $S \in \mathcal{U}$, so $\sum_{s \in S \subseteq X} \alpha_s v_s$ is convergent in V for some choice of scalars $\alpha_s \in \mathbb{C}, \alpha_s \neq 0$ for any $s \in S$. We will now prove that $\sum_{s' \in S' \subseteq S \subseteq X} \alpha_{s'} v_{s'}$ is convergent in V. As V is complete, it is enough to prove that $\sum_{s' \in S' \subseteq S \subseteq X} \alpha_{s'} v_{s'}$ is Cauchy.

  Now, as $\sum_{s \in S \subseteq X} \alpha_s v_s$ is convergent, so it is Cauchy. So by definition, for any open linear subspace V_0 around 0_V, there exists $A_0 \in \mathrm{Fin}(S)$, such that $\sum_{a \in A \subseteq S} \alpha_a v_a - \sum_{b \in B \subseteq S} \alpha_b v_b \in V_0$, for all $A, B \in \mathrm{Fin}(S)$, so that $A, B \supseteq A_0$. In particular for any $v_x \in S - A_0$, $\sum_{a \in A_0 \subseteq S} \alpha_a v_a - \sum_{a \in A_0 \cup \{v_x\} \subseteq S} \alpha_a v_a \in V_0$, but this implies $v_x \in V_0$ for any $v_x \in S - A_0$.

Now we define $A'_0 := S' \cap A_0$, then $|A'_0| < \infty$, as $A_0 \in \mathrm{Fin}(S)$. Then we have for any $A', B' \in \mathrm{Fin}(S')$, so that $A', B' \supseteq A'_0$,$\sum_{a' \in A' \subseteq S'} \alpha_{a'} v_{a'} - \sum_{b' \in B' \subseteq S'} \alpha_{b'} v_{b'} \in V_0$, as this is a finite linear combinations of vectors in V_0. As the choice of open subspace was arbitrary, so we have $\sum_{s' \in S' \subseteq S \subseteq X} \alpha_{s'} v_{s'}$ is Cauchy and hence convergent by completeness. So we have $S' \in \mathcal{U}$.

- If $S, T \in \mathcal{U}$, then $S \cup T \in \mathcal{U}$. For this as $S \in \mathcal{U}$, so $\sum_{s \in S \subseteq X} \alpha_s v_s$ is convergent in V for some choice of scalars $\alpha_s \in \mathbb{C}, \alpha_s \neq 0$ for any $s \in S$. Similarly, as $T \in \mathcal{U}$, so $\sum_{t \in T \subseteq X} \alpha_t v_t$ is convergent in V for some choice of scalars $\alpha_t \in \mathbb{C}, \alpha_t \neq 0$ for any $t \in T$. We now define a new series $\sum_{st \in S \cup T} \alpha_{st} v_{st}$, where α_{st} is defined as follows:

$$\alpha_{st} := \begin{cases} \alpha_s, & \text{if } v_{st} \in S \text{ , and } v_{st} \notin T \\ \alpha_t, & \text{if } v_{st} \in T \text{ and } v_{st} \notin S \\ (\alpha_s + \alpha_t), & \text{if } v_{st} \in S \text{ and } v_{st} \in T \end{cases}$$

Then we will prove that $\sum_{st \in S \cup T} \alpha_{st} v_{st}$ is Cauchy. First as $\sum_{s \in S \subseteq X} \alpha_s v_s$ is convergent so Cauchy, so by definition, for any open subspace V_0 around 0_V, there exists $S_0 \in \mathrm{Fin}(S)$, such that $\sum_{a \in A \subseteq S} \alpha_a v_a - \sum_{b \in B \subseteq S} \alpha_b v_b \in V_0$, for all $A, B \in \mathrm{Fin}(S)$, so that $A, B \supseteq S_0$. In particular this implies that for any $v_s \in S - S_0, v_s \in V_0$.

Similarly, as $\sum_{t \in T \subseteq X} \alpha_t v_t$ is Cauchy, so there exists $T_0 \in \mathrm{Fin}(T)$, such that $\sum_{c \in C \subseteq T} \alpha_c v_c - \sum_{d \in D \subseteq T} \alpha_d v_d \in V_0$, for all $C, D \in \mathrm{Fin}(T)$, so that $C, D \supseteq T_0$. This implies that for any $v_t \in T - T_0, v_t \in V_0$.

Now we define $S_0T_0 := S_0 \cup T_0$, then as $S_0 \in \mathrm{Fin}(S)$, and $T_0 \in \mathrm{Fin}(T)$, so $S_0T_0 \in \mathrm{Fin}(S \cup T)$. Now for any $S_1T_1, S_2T_2 \in \mathrm{Fin}(S \cup T)$, so that $S_1T_1, S_2T_2 \supseteq S_0T_0$, $\sum_{st \in S_1T_1} \alpha_{st} v_{st} - \sum_{s't' \in S_2T_2} \alpha_{s't'} v_{s't'} \in V_0$, as this is a finite linear combination of vectors in V_0. As the choice of open subspace was arbitrary, so we have $\sum_{st\in S\cup T} \alpha_{st} v_{st}$ is Cauchy and hence convergent by completeness. Note that if $\alpha_s = -\alpha_t$ for some $s \in S \cap T$, then we can always modify our series (see Remark 5.4.2) such that $\alpha_{st} \neq 0$ for any $st \in S \cup T$.

So $S \cup T \in \mathcal{U}$.

So $(X, \mathcal{U})$ thus constructed from a reflexive Lefschetz space is a pre-finiteness space. We next show that if we construct a Lefschetz space over it using Ehrhard's construction, then it is uniformly homeomorphic to the original space we started with. Now, since V is reflexive and therefore complete, this will imply that the pre-finiteness space obtained in Definition 5.4.1 is actually a finiteness space.

Proposition 5.4.4. Let $\mathbb{C}(X, \mathcal{U})$ be the Lefschetz space constructed from the pre- finiteness space obtained above (Proposition 5.4.3). Then V (the space we started with in Definition 5.4.1) is uniformly homemorphic (that is a linear continuous map, with a continuous (linear) inverse) to $\mathbb{C}(X, \mathcal{U})$.

Proof. Let V be the Lefschetz Space as in Definition 5.4.1, with basis $\mathcal{B} = \{v_i\}_{i\in I}$. Then we define a map $F : V \to \mathbb{C}(X, \mathcal{U})$ as follows $F(v_i) := f_{v_i}$, where $f_{v_i} : X \to \mathbb{C}$ is the map which takes the value $1_{\mathbb{C}}$ at $x = v_i$, 0 elsewhere. We then extend this map linearly to obtain $F : V \to \mathbb{C}(X, \mathcal{U})$. We could see F as sending a vector to its function of scalars, as $F(\sum_{S\subseteq\mathcal{B}} \alpha_s v_s)$ will be the function $f : X \to \mathbb{C}$, where $f(v_s) = \alpha_s$. Now we claim the following:

- F is well-defined. For this, as $\{v_i\}_{i\in I}$ is a basis for V, so any vector $v \in V$ is the convergent limit of finite sums over some subset $J \subseteq I$

(see Proposition 4.4.7). Moreover, $v \in V$ is the limit of a unique linear combination over (some subset) of I. Now $F : V \to \mathbb{C}(X, \mathcal{U})$ just sends a vector to its function of scalars, and hence is well-defined.

- F is injective: For this, as V is Hausdorff, so if $\sum_{j \in J \subseteq I} \alpha_j v_j = v = w$ for any $v, w \in V$, then $v = w$, so $F : V \to \mathbb{C}(X, \mathcal{U})$ is injective.

- F is surjective: For this, let $f \in \mathbb{C}(X, \mathcal{U})$. Then $\sup(f) = S \in \mathcal{U}$, so by definition, $\sum_{s \in S} \alpha_s v_s$ converges in $\mathbb{C}(X, \mathcal{U})$ for some choice of scalars $\alpha_s \in \mathbb{C}, \alpha_s \neq 0$ for any $s \in S$. Now let $f(v_s) = \alpha'_s, \alpha'_s \neq 0$, for any $s \in S$. Then by Proposition 5.3.15, $v' = \sum_{s \in S} \alpha'_s v_s$ also converges in $\mathbb{C}(X, \mathcal{U})$, as V is complete. And we have, $F(v') = f$.

- F is open: For this, first note that as continuous functions between topological vector spaces are completely determined by their action around the origin, it is enough to prove that F is open around the origin, 0_V. Now, as V is a reflexive space, so the natural inclusion map $\mathrm{ev}_V : V \to (V^* \multimap \mathbb{C})$ is open. We also know that $v \mapsto \mathrm{ev}_V(v)$ via ev_V, where $\mathrm{ev}_V(v)(\phi) = \phi(v)$. Now by assumption, for any $S \in \mathcal{U}, \sum_{s \in S} \alpha_s v_s$, $\alpha_s \neq 0$ for any $s \in S$, converges in $\mathbb{C}(X, \mathcal{U})$. So an exact argument like in Proposition 5.3.10 will show that for any $\phi \in V^*, \sup(\phi) \in \mathcal{U}^{\perp}$. Note that here we are identifying ϕ as a function $X \to \mathbb{C}$, as the value of ϕ is fixed by the value it takes at the basis vectors f_x.

  Now let V_0 be any open linear subspace around 0_V, then $\mathrm{ev}_V : V \to (V^* \multimap \mathbb{C})$ maps V_0 to an open neighbourhood linear subspace around $0_{(V^* \multimap \mathbb{C})}$. But the neighbourhood basis around $0_{(V^* \multimap \mathbb{C})}$ is given by finite intersections of the sets of the form $\{\Phi : V^* \to \mathbb{C} : \Phi(\phi) = 0, \phi \in V^*\}$. From the previous paragraph, we have, for any $\phi \in V^*$, $\sup(\phi) \in \mathcal{U}^{\perp}$. Now the map $F : V \to \mathbb{C}(X, \mathcal{U})$ just sends a vector to its function of scalars, hence V_0 via F will get mapped to finite intersection of sets of the form $\{f : X \to \mathbb{C} : \sup(f) \in \mathcal{U}$ and $\sup(f) \cap u' = 0, u' \in \mathcal{U}^{\perp}\}$. But this is just $V_{u_1} \cap V_{u_2} \cap V_{u_3} \cap ...... \cap V_{u_n} = V_{u_1 \cup u_2 \cup u_3 \cup ..... \cup u_n}$, which is an open neighbourhood basis around $0_{\mathbb{C}(X,\mathcal{U})}$. So $F : V \to \mathbb{C}(X, \mathcal{U})$ maps open neighbouhood subspaces around 0_V to open neighbourhood subspaces around $0_{\mathbb{C}(X,\mathcal{U})}$, and hence F is open.

- F is continuous: Again, it is enough to prove that F is continuous around the origin. Now, as V is a reflexive space, so the natural inclusion map $\mathrm{ev}_V : V \to (V^* \multimap \mathbb{C})$ is continuous. We also know that $v \mapsto \mathrm{ev}_V(v)$ via ev_V, where $\mathrm{ev}_V(v)(\phi) = \phi(v)$.

  Now for any $u' \in \mathcal{U}^{\perp}$, such that there exists $\phi_{u'} \in V^*$, with $\sup(\phi_{u'}) = u'$, the set $\{\Phi : V^* \to \mathbb{C} : \Phi(\phi_{u'}) = 0\}$ will be an open neighbourhood subspace around $0_{(V^* \multimap \mathbb{C})}$. And so $i^{-1}(\{\Phi : V^* \to \mathbb{C} : \Phi(\phi_{u'}) = 0\})$ will be an open neighbourhood subspace around 0_V. Again, as the map $F : V \to \mathbb{C}(X, \mathcal{U})$ just sends a vector to its function of scalars, this implies that $F^{-1}(V_{u'})$ will be an open neighbourhood subspace around 0_V. Note that $F^{-1}(V_{u'})$ is the set of vectors whose support intersects trivially with u'.

  So we have that $F^{-1}(V_{u'})$ an open neighbourhood subspace around 0_V for some $u' \in \mathcal{U}^{\perp}$. Let there be some $v \in \mathcal{U}^{\perp}$, such that $F^{-1}(V_v)$ is not an open neighbourhood subspace around 0_V. Then $\sum_{s \in S} \alpha_s v_s$ will be Cauchy, hence convergent in V if and only if $|S \cap u'| < \infty$, for all $u' \in \mathcal{U}^\perp$, with $u' \neq v$. But then by assumption, $\sum_{s\in S} \alpha_s v_s$ is convergent in V if and only if $S \in \mathcal{U}$, since $S \subseteq X$, was arbitrary, so this implies that $F^{-1}(V_{u'})$ is an open neighbourhood subspace around 0_V for any $u' \in \mathcal{U}^\perp$.

Hence, we have that $F : V \to \mathbb{C}(X, \mathcal{U})$ is well-defined, bijective, continuous, and open, so F is a homeomorphism (of topological spaces). So $V \cong \mathbb{C}(X, \mathcal{U})$.

Moreover, note that this is a uniform homeomorphism, as F is a linear map (See Lemma 2.3.13), so $V \cong \mathbb{C}(X, \mathcal{U})$ is a uniform homeomorphism. Note that the inverse of a linear map (if it exists) is linear, so a uniform homeomorphism is a continuous linear map with a continuous inverse. □

As uniform homeomorphisms preserve completeness, we have:

Corollary 5.4.5. The pair $(X, \mathcal{U})$ as in Definition 5.4.1 is a finiteness space.

Proof. As V is uniformly homeomorphic to $\mathbb{C}(X, \mathcal{U})$, so $\mathbb{C}(X, \mathcal{U})$ inherits all the uniform properties of V, in particular completeness. But then by Proposition 5.3.11, the Lefschetz space constructed from a pre-finiteness space $(X, \mathcal{U})$ is complete if and only if $(X, \mathcal{U})$ is a finiteness space. So $(X, \mathcal{U})$ as in Definition 5.4.1 is a finiteness space. □

We next show that, on objects, the support construction is inverse to Ehrhard's construction. For this, we first note that the spaces arising out of Ehrhard's construction are indeed reflexive. Then by Proposition 5.4.4, it already follows that if we first apply support construction and then follow it by Ehrhard's construction, we get homeomorphic spaces. The other way round follows from Proposition 5.3.12.

Proposition 5.4.6. The Lefschetz spaces arising out of Ehrhard's construction, when $(X, \mathcal{U})$ is a finiteness space is reflexive.

Proof. This is immediate using Proposition 5.3.10, as we showed that if we first apply Ehrhard's construction to a finiteness space and then take its dual in RLef or if we first take the dual of a finiteness space in FinRel and then apply Ehrhard's construction, we get the same space. But then, as taking the double dual of a finiteness space gets us back to the same finiteness space, so we have that the Lefschetz spaces arising out of Ehrhard's construction, when $(X, \mathcal{U})$ is a finiteness space, are reflexive. □

Proposition 5.4.7. The construction described in this section (Definition 5.4.1) is inverse to Ehrhard's construction.

Proof. Observe that the above arguments already proves that if we start with a reflexive Lefschetz space V, with a given basis, build a finiteness space out of it using the construction described in this section, and then follow it by Ehrhard's construction, then we get $V \cong \mathbb{C}(X, \mathcal{U})$.

Other way around, if we start with a finiteness space $(X, \mathcal{U})$, build a Lefschetz space, using Ehrhard's construction, $\mathbb{C}(X, \mathcal{U})$, and then follow it by our construction here to get $(X', \mathcal{U}')$. Then by Proposition 5.3.12, we have that $\sum_{x \in S \subseteq X} f_x$ converges in $\mathbb{C}(X, \mathcal{U})$ if and only if $S \in \mathcal{U}$. Hence, identifying points of the web $x \in X$ with the points of the web $f_x \in X'$, we see that we get the same finiteness space back.

Hence, the construction described of a finiteness space from a reflexive Lefschetz space with a basis is inverse to Ehrhard's construction. So there is a bijective correspondence between a reflexive Lefschetz space (with a given basis) and a finiteness space. Note that by Proposition 5.3.10, this correspondence also respects duals. ◻

Thus, we can classify the topologies on reflexive Lefschetz spaces as finiteness topologies, which leads to some immediate corollaries using Ehrhard's classification [19].

Corollary 5.4.8. Any reflexive Lefschetz Space has a topology coarser than the discrete topology (on $\mathbb{C}^{|X|}$ and finer than the product topology (with discrete topology on $\mathbb{C}$ and $\mathbb{C}(X, \mathcal{U})$ being considered as the product $\mathbb{C}^{|X|}$).

We can also classify the finiteness structures that give rise to the discrete topology on Lefschetz spaces constructed on it via Ehrhard's construction.

Lemma 5.4.9. Suppose $\mathbb{C}(X, \mathcal{U})$ is a reflexive Lefschetz space obtained from Ehrhard's construction on the finiteness space $(X, \mathcal{U})$. Then $\mathbb{C}(X, \mathcal{U})$ has the discrete topology if and only if $\mathcal{U} = \mathcal{P}_f(X)$, that is, the finitary sets are exactly the finite subsets of X.

Proof. Suppose $\mathbb{C}(X, \mathcal{U})$ has the discrete topology then we know that $\sum_{s\in S\subseteq X} f_x$ converges in $\mathbb{C}(X, \mathcal{U})$ if and only if $\mathrm{Lim}_{A\in \mathrm{Fin}(S)} \sum_{a\in A} f_a$ is eventually constant, which means that $|S| < \infty$. But then by Proposition 5.3.12, we know that $\sum_{s\in S\subseteq X} f_x$ converges if and only if $S \in \mathcal{U}$. So that mean that $\mathcal{U} = \mathcal{P}_f(X)$.

For the other way round, if we start with the finiteness structure $(X, \mathcal{P}_f(X), \mathcal{P}(X))$. Then in $\mathbb{C}(X, \mathcal{U})$, the constant 0 function is open as $V'_u = 0$ for $u' = X$. This implies that every singleton $\{f\}$ is open as $f + 0 \subseteq f$. Hence, the topology on $\mathbb{C}(X, \mathcal{U})$ is discrete. ◻

So in particular, since a finiteness space with a finite web can only have the trivial finiteness structure $(X, \mathcal{P}(X), \mathcal{P}(X))$, we have,

Corollary 5.4.10. Any finite-dimensional reflexive Lefschetz space has a discrete topology.

In fact, for the finite-dimensional case, demanding reflexivity is redundant.

Lemma 5.4.11. [4] Every finite-dimensional Lefschetz space is reflexive.

5.5 Functoriality of support construction

In this section, we will use the construction described in Section 5.4 and show that it extends as a functor from the category of reflexive Lefschetz Spaces, with a basis and morphisms as positive maps (See Definition 5.5.1), and the category of finiteness spaces and relations.

Category of interest: $\mathsf{pRLef_B}$

First recall from Chapter 4 that $\mathsf{RLef_B}$ is a star ($*$-) autonomous category.

Definition 5.5.1. Let $(U, I = \{u_i\}_{i\in I})$ and $(W, J = \{w_j\}_{j\in J})$ be reflexive Lefschetz spaces over $\mathbb{C}$ with a basis, then a linear continuous map $f : (U, I) \to (W, J)$ is said to be positive if for any $i \in I$, $f(u_i) = \sum_{j'\in J'\subseteq J} \alpha_{j'} w_{j'}$ such that $\alpha_{j'} \in \mathbb{R}$ and $\alpha_{j'} \geq 0$ for all $j' \in J'$.

We next want to show that when we restrict to just positive maps, the resulting category of reflexive Lefschetz Spaces with a given basis and morphisms as positive continuous linear maps, denoted by $\mathsf{pRLef_B}$ is still a star autonomous category. And so we claim:

Proposition 5.5.2. $\mathsf{pRLef_B}$ is a star autonomous category.

Proof. As identity is a positive map and composition of positive maps is positive, $\mathsf{pRLef_B}$ is clearly a category.

Next, note that the internal hom functor $V \multimap (\_) : \mathsf{RLef_B} \to \mathsf{RLef_B}$ restricts to a functor from $\mathsf{pRLef_B}$ to $\mathsf{pRLef_B}$, as it sends a positive map $f : (U, I = \{u_i\}_{i \in I}) \to (W, J = \{w_j\}_{j \in J})$, where $f(u_i) = \sum_{j' \in J' \subseteq J} \alpha_{j'} w_{j'}$ to $V \multimap f : (V \multimap U, K \times I) \to (V \multimap W, K \times J)$, which sends $\phi_{v_k \to u_i}$ to $\sum_{j' \in J' \subseteq J} \alpha_{j'} \phi_{v_k \to w'_j}$.

Next, as there is already a tensor hom adjunction in $\mathsf{RLef_B}$, to show that there is now a tensor hom adjunction in $\mathsf{pRLef_B}$, we need to show that the unit map of the adjunction is positive and that the adjoint of a positive map is positive. As this is a concrete category, so the unit map $\eta_U : U \to V \multimap (U \otimes V)$ on the basis vectors u_i is given by $\eta_U(u_i) = \sum \phi_{v_k \to u_i \otimes v_j}$, here all the scalars are 1, and hence this is a positive map. Moreover the adjunct of a positive map $f : U \to V \multimap W$, which sends u_i to $f(u_i) = \sum_{k' \in K' \subseteq K, j' \in J' \subseteq J} \alpha_{k'j'} \phi_{v_{k'} \to w_{j'}}$ is $f^\# : U \otimes V \to W$, which takes $u_i \otimes v_j$ to $f^\#(u_i \otimes v_j) = \sum_{k' \in K' \subseteq K, j' \in J' \subseteq J} \alpha_{k'j'} w_{j'}$. It is easy to check that $\eta_U V \multimap (f^\#) = f$. And hence the adjunct of a positive map is positive. Finally the evaluation map $\mathrm{ev}_V : V \to V^{**}$ is also positive as it takes v_k to $\phi_{(v_k \to *) \to *}$. □

The support functor

We want to define a functor, which we will call the support functor, motivated by our construction of a finiteness space from a reflexive Lefschetz space. The functor $\sup : \mathsf{pRLef_B} \to \mathsf{FinRel}$ is defined as follows:

- On Objects: $\sup(V, K) = (K, \mathcal{U} := \{S \subseteq K : \sum_{s \in S} \alpha_s v_s$ converges in V, $\alpha_s \in \mathbb{C}$, $\alpha_s \neq 0$ for any $s \in S\})$ as in the previous section. Similarly, $\sup(U, I) =$

$(I, \mathcal{V})$.

- On morphisms: Let $f : (V, K) \to (U, I)$ be any positive linear continuous map. Then, as a linear map is determined by its action on a basis, so let $f(v_k) = \sum_{i' \in I' \subseteq I} \alpha_{i'} u_{i'}$, where all the scalars are positive. Then $\sup(f) \subseteq K \times I$ and $v_k \sim_{\sup(f)} u_i$ (that is v_k is related to u_i) if and only if the scalar corresponding to u_i in $f(v_k)$ is non-zero.

Note that if we were just working with continuous linear maps, rather than restricting to positive continuous linear maps, then the support construction fails to be functorial, as the scalars might cancel against each other. But when we work with positive maps, the support construction is functorial.

Proposition 5.5.3. $\sup : \mathsf{pRLef_B} \to \mathsf{FinRel}$ is a functor.

Proof. We will first show that the functor is well-defined. It is well-defined on objects by the the previous section (see Corollary 5.4.5). On morphisms, we need to show that $\sup(f)$ preserves finitary and reflects co-finitary sets. First, we show that $A \rhd \sup(f) \in \mathcal{V}$ for any $A \in \mathcal{U}$. For this note that, as $A \in \mathcal{U}$, so $\sum_{a \in A, A \subseteq K, A \in \mathcal{U}} v_a$ converges in V, and as $f : V \to U$ is a continuous map, so $\sum_{a \in A, A \subseteq K, A \in \mathcal{U}} f(v_a)$ converges in W. Now note that as f is a positive map, so no basis vectors are getting canceled against each other, and hence the support of $\sum_{a \in A, A \subseteq K, A \in \mathcal{U}} f(v_a)$ is exactly $A \rhd \sup(f)$, and as this converges in U, so by definition $A \rhd \sup(f) \in \mathcal{V}$.

Next, we show that for any $B \in \mathcal{V}^\perp$, $\sup(f) \lhd B \in \mathcal{U}^\perp$. For this note that as the correspondence between relfexive Lefschetz Spaces and finiteness spaces carries over to the dual, so if we take the dual of $f : (V, K) \to (U, I)$ in

$\mathsf{pRLef_B}$, $f^* : (U^*, I) \to (V^*, K)$, then it induces a relation (which preserves finitary sets by above), $\sup(f^*) : I \to K$.

Now if we prove that $\sup(f^*) = \sup(f)^t$, where on the right hand side we are taking the transpose (dual) in FinRel, then we are done, as the transpose of a relation in FinRel preserves finitary sets if and only if the original relation reflects co-finitary sets.

First we will show that $\sup(f^*) \subseteq (\sup(f))^t$. For this let $u_i \sup(f^*) v_k$, then by definition, it means that the linear functional $\phi_{u_i} : U \to \mathbb{C}$, which takes the value 1 at u_i and 0 everywhere else, when pre-composed with f gives a linear function which from V to $\mathbb{C}$, which is non-zero on v_k. But then it means that $f(v_k)$ must have a non-zero component of u_i, so $v_k \sup(f) u_i$ and hence $u_i (\sup(f))^t v_k$. So we have $\sup(f^*) \subseteq (\sup(f))^t$.

Next we shall prove $(\sup(f))^t \subseteq \sup(f^*)$. For this let $u_i(\sup(f))^t v_k$, then $v_k \sup(f) u_i$. We want to show that $u_i \sup(f^*) v_k$. For this note that since $v_k \sup(f) u_i$, so $f(v_k)$ has a non-zero component of u_i and hence the linear functional ϕ_{u_i} when pre-composed with f will give a linear functional from V to $\mathbb{C}$ with a non-zero component of ϕ_{v_k}. And hence $u_i \sup(f^*) v_k$. So we have $(\sup(f))^t \subseteq \sup(f^*)$.

And we are done, as we already know $\sup(f^*)$ preserves finitary sets, but since it is the transpose of $\sup(f)$, so $\sup(f)$ must reflect co-finitary sets.

This shows that $\sup : \mathsf{pRLef_B} \to \mathsf{FinRel}$ is well-defined.

For functoriality, it is immediate that the support functor preserves identities. For composition, first we prove that $\sup(fg) \subseteq \sup(f)\sup(g)$. For this, let $v_k \sup(fg) w_j$. Then $g(f(v_k))$ has a non-zero component of w_j, by definition of function composition, this implies that $f(v_k)$ has a non-zero component of u_i such that $v_k \sup(f) u_i$ and $u_i \sup(g) w_j$. So by definition of relational composition, $v_k \sup(f) \sup(g) w_j$ and we are done. Finally, we show that $\sup(f) \sup(g) \subseteq \sup(fg)$. For this let $v_k \sup(f) \sup(g) w_j$, then there exists $u_i \in I$, such that $v_k \sup(f) u_i$ and $u_i \sup(g) w_j$. So we have that $f(v_k)$ has a non-zero component of u_i and $g(u_i)$ has a non-zero component of w_j. Since f, g are *positive* maps, and hence their composition, so $g(f(v_k))$ has a non-zero component of w_j and so $v_k \sup(fg) w_j$, and we are done.

So we have that $\sup : \mathsf{pRLef_B} \to \mathsf{FinRel}$ as defined above is a functor. Note that in particular, if we restrict ourselves to basis-to-basis maps(that is, those maps which take the chosen basis elements to chosen basis elements and hence are positive), then the support functor lands in FinF. $\square$

In the next chapter, we will use this support functor to give an adjunction between the category of finiteness spaces and functions and the category of dagger linear semigroups in the category of reflexive Lefschetz spaces.

Chapter 6

The semigroup story

6.1 Motivation

Lefschetz spaces and finiteness spaces, the two main mathematical objects of our study, were introduced in Chapter 4 and Chapter 5, respectively. We also saw how one can construct a reflexive Lefschetz space on a finiteness space using Ehrhard's construction. The basis of the Lefschetz space arising out of Ehrhard's construction is given by functions of point support, and the finitary sets are encoded as the support of convergent nets over the basis. Using this observation, we gave a functorial construction of a finiteness space, starting with the data of a reflexive Lefschetz space with a basis.

In this chapter, we will use Ehrhard's construction and the support construction to establish an adjunction between the category of finiteness spaces with functions and the category of dagger linear semigroups in Lefschetz spaces with semigroup homomorphisms. We first give some motivation to study such an adjunction and present the result from the finite-dimensional case in some detail. Then we will proceed to generalise the result from the core to a full structure of a mixed unitary category. We will see that outside the core, we only get a coreflection, rather than a full equivalence.

In Quantum mechanics, observables are physical quantities that can be measured in a lab, such as the position, momentum, or spin of an electron. They are represented by a self-adjoint (Hermitian) operator. Hermitian (and more generally Normal) operators have an orthonormal eigen-basis [3]. So in particular, any state has a unique decomposition in terms of its eigenvectors, and the eigenvectors corresponding to different eigenvalues are also orthogonal to each other, assuring perfect distinguishability. For example, consider the Hilbert space $\mathbb{C}^2$ (generally used in the theory of quantum computation), with the computational basis $|0\rangle$ and $|1\rangle$, then any state $|\varphi\rangle$ can be written as,

$$|\varphi\rangle \quad = \quad \alpha\,|0\rangle + \beta\,|1\rangle\,.$$

such that $\alpha, \beta \in \mathbb{C}$ and $|\alpha|^2 + |\beta|^2 = 1$. Hence, to study observables in quantum theory, it is natural to study orthonormal bases.

Another motivation to study orthonormal bases in finite-dimensional Hilbert spaces is the no-cloning theorem [42] which states that there exists no unitary operator $f : H \otimes H \to H \otimes H$ on any Hilbert space (with dimension > 1), such that for any $|\psi\rangle \in H$, $f(|s\rangle \otimes |\psi\rangle) = |\psi\rangle \otimes |\psi\rangle$, where $|s\rangle$ is some standard state. However, given an orthonormal basis of H, $\{|\phi_i\rangle\}_{i=1}^n$, we can always "freely" copy these basis elements and define the copy operator by extending it linearly. We can then define a multiplication structure by taking the dagger of this free copy map to give a dagger Frobenius algebra structure on the underlying Hilbert space. The copyables of this algebra encode the orthonormal basis that we started with. It was proven in [16] that every (commutative special) dagger Frobenius algebra on a Hilbert space arises this way. One can think of picking a basis to be cloned, as picking a "classical" fragment within the quantum theory, where arbitrary cloning is not possible. For this reason, commutative dagger special Frobenius algebras are also known as classical structures.

Categorically speaking, in the finite-dimensional case, an equivalence between the category of finite sets (interpreted as bases of the Hilbert space over them) with isomorphisms and the category of commutative dagger special Frobenius algebras and morphisms that preserve the monoid and the comonoid structure was shown [16]. If we require the Frobenius algebra homomorphisms to preserve only comultiplication, we get an equivalence between the category of finite sets with functions and commutative dagger special Frobenius algebras and morphisms that preserve the comonoid structure.

In this chapter, we will first give the outline of this equivalence, as in [16]. This will be up to Section 6.7, and then, we will see how this result can be extended from the core to the full structure of a mixed unitary category.

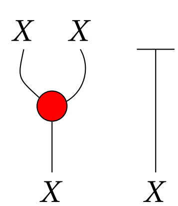
The first point of departure is that since Frobenius algebras are on self-dual objects, to move beyond the compact fragment of the theory, we consider dagger linear monoids. We construct a reflexive Lefschetz space beginning with a finiteness space, and then freely copy the basis vectors. Since we have an underlying topology, which is in general not discrete, the continuity of the proposed morphisms must be established. This restricts us to consider dagger linear semigroups in reflexive Lefschetz spaces. To come back from a dagger linear semigroup on a reflexive Lefschetz space, we first take the subspace generated by a copyable and then follow it by the support construction to land in the category of finiteness spaces and functions.

We show that this leads to an adjunction between the category of finiteness spaces with functions and the category of dagger linear semigroups with semigroup homomorphisms. This adjunction is, in fact, a coreflection, but fails to be an equivalence, as we present examples of algebra structures whose copyables do not generate the underlying space.

6.2 Frobenius algebras

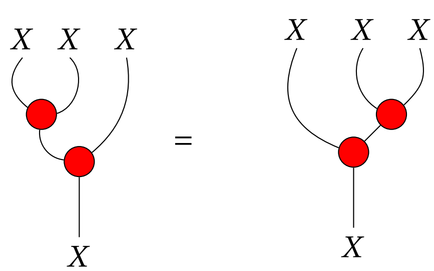
In this section, we introduce the notion of a Frobenius algebra in a monoidal category. A Frobenius algebra is an object together with a monoid and a comonoid structure that interacts via the Frobenius law.

Definition 6.2.1. Let $(\mathbb{X}, \otimes, I, a_\otimes, l_\otimes, r_\otimes)$ be a monoidal category. Then a **monoid** in $\mathbb{X}$ is an object $X \in \mathbb{X}$ together with maps $m : X \otimes X \to X$, called the multiplication map and the unit map $u : I \to X$



(6.1)

such that the multiplication is associative and unital.



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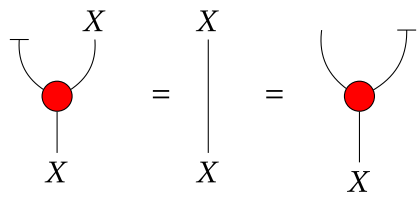

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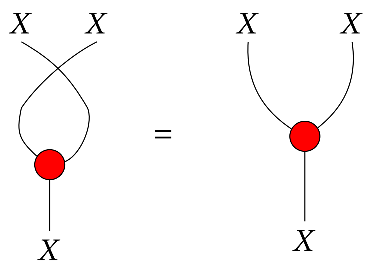
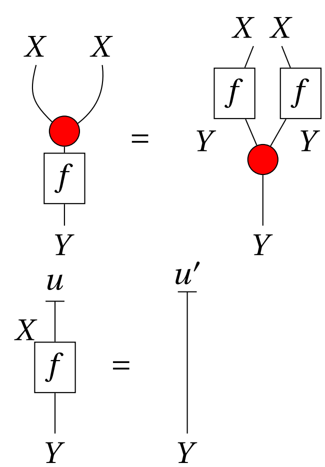
The monoid is said to be **commutative** if the underlying monoidal category is symmetric and the symmetry map followed by multiplication is the same as the multiplication map.



(6.4)

A **morphism of monoids** between $(X, m, u) \to (Y, m', u')$ is a morphism $f : X \to Y$ in the underlying monoidal category such that $mf = (f \otimes f)m'$ and $uf = u'$



(6.5)

Lemma 6.2.2. Monoid morphisms compose.

Proof. Let $f : X \to Y$, $g : Y \to Z$ be monoid morphisms. Then we want to prove that $m_X fg = (fg \otimes fg)m_Z$. For that note,

$$\begin{aligned} m_X fg &= (f \otimes f)m_Y g \\ &= (f \otimes f)(g \otimes g)m_Z \\ &= (fg \otimes fg)m_Z, \end{aligned}$$

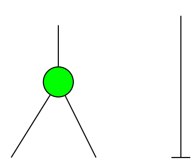
where the first and second equality use the fact that f, g are monoid morphisms, respectively. And for $u_X fg = u_Z$, note,

$$\begin{aligned} u_X fg &= u_Y g \\ &= u_Z. \end{aligned}$$

So we have that $fg : X \to Z$ is a monoid morphism. □

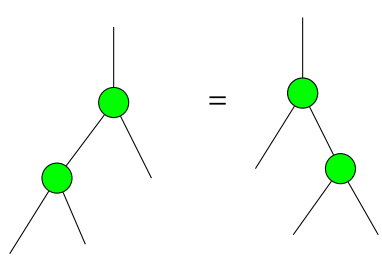
Definition 6.2.3. A **co-monoid** in a monoidal category is the dual notion of a monoid; so a co-monoid in $\mathbb{X}$ is an object $X \in \mathbb{X}$ together with a co-multiplication map $\delta : X \to X \otimes X$ and a co-unit map $\varepsilon : X \to I$ such that the co-multiplication is co-associative and co-unital. The axioms of a co-monoid can be obtained by flipping the axioms for the monoid horizontally.

We will denote the comultiplication and the counit map as:



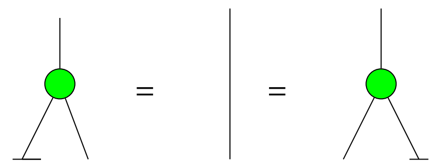
(6.6)

And the string diagram representation of the coassociativity axiom will be,



(6.7)

And for the counitality axiom, we have,



(6.8)

Definition 6.2.4. A **comonoid morphism** $f : (X, \delta, \varepsilon) \to (Y, \delta', \varepsilon')$ is a morphism $f : X \to Y$ such that $\delta(f \otimes f) = f\delta'$ and $f\varepsilon' = \varepsilon$. The diagrams can be obtained again by horizontally flipping the diagrams above for a monoid morphism.

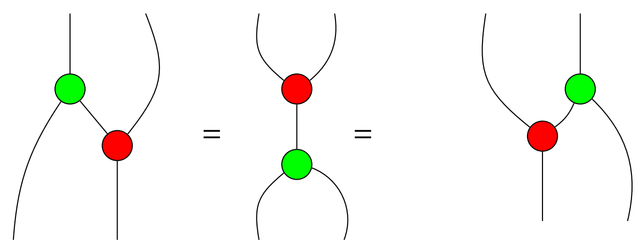
We similarly have that,

Lemma 6.2.5. Comonoid homomorphisms compose.

Definition 6.2.6. Let $(\mathbb{X}, \otimes, I)$ be a monoidal category. A **Frobenius algebra** in $\mathbb{X}$ is an object $X \in \mathbb{X}$ together with

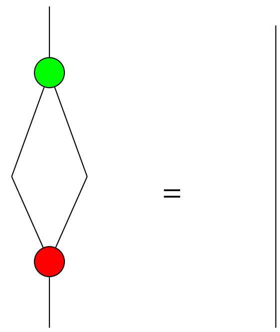
- a monoid $(m : X \otimes X \to X, u : I \to X)$;
- a co-monoid $(\delta : X \to X \otimes X, \varepsilon : X \to I)$;

such that the **Frobenius law** holds; $(\delta \otimes \mathrm{id})(\mathrm{id} \otimes m) = m\delta = (\mathrm{id} \otimes \delta)(m \otimes \mathrm{id})$.



(6.9)

The Frobenius algebra is said to be **commutative** if the monoid and the comonoid structures are commutative, and it is said to be **special** if $\delta m = \mathrm{id}$.



(6.10)

Definition 6.2.7. A Frobenius algebra (FA) in a †- monoidal category is said to be a **dagger Frobenius algebra** (†- FA) if $\delta = m^\dagger$ and $\varepsilon = u^\dagger$.

Remark 6.2.8. The monoidal unit I can be equipped with a commutative dagger special Frobenius algebra (CDSFA) structure $(I, l_\otimes : I \otimes I \to I, \mathrm{id}_I)$.

6.3 Dagger Frobenius algebras

We first consider the category of commutative dagger special Frobenius algebras in a †- monoidal category $\mathbb{X}$, which we denote by $\mathsf{CDSFA}(\mathbb{X})$ where:

Objects: Commutative dagger Frobenius algebras in a †- monoidal category $\mathbb{X}$;

Morphisms: $f : (X, m, u, \delta, \varepsilon) \to (Y, m', u', \delta', \varepsilon')$ is a morphism $f : X \to Y$ such that f is both a monoid and a comonoid homomorphism;

Composition: composition of morphisms in the underlying monoidal category;

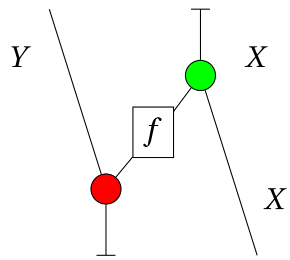
Identity: the identity morphism in the underlying category.

It is clear that the identity morphism preserves both the monoid and the comonoid structure, and by Lemma 6.2.2 and 6.2.5 we have that $\mathsf{CDSFA}(\mathbb{X})$

is a category. We also have that if we demand our algebra morphisms to preserve both the monoid and comonoid structure, then it is an isomorphism:

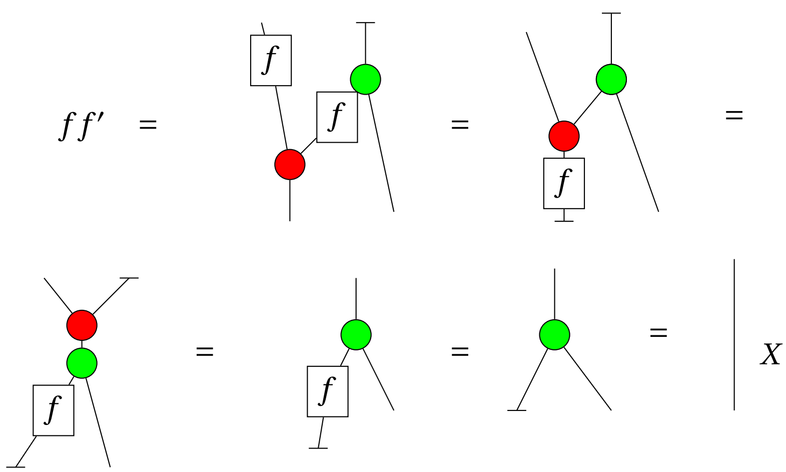
Lemma 6.3.1. Let $f : (X, m, u, \delta, \varepsilon) \to (Y, m', u', \delta', \varepsilon')$ be a morphism between two Frobenius algebras. If f is both a monoid morphism and a comonoid morphism, then f is an isomorphism.

Proof. We will give a string diagrammatic proof for this. First we define the inverse morphism $f' : Y \to X$ as $(\mathrm{id} \otimes u\delta)(\mathrm{id} \otimes f \otimes \mathrm{id})(m'\varepsilon' \otimes \mathrm{id})$.



(6.11)

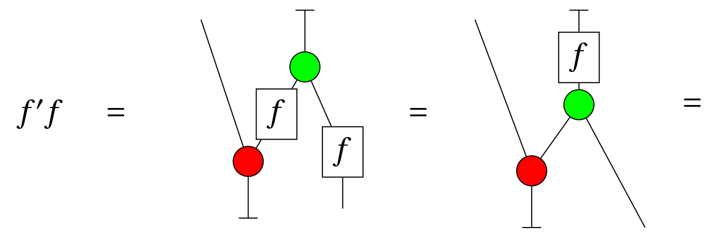
We then must prove that $ff' = \mathrm{id}_X$ and $f'f = \mathrm{id}_Y$. For $ff' = \mathrm{id}_X$, note the following sequence of string diagrammatic equalities:



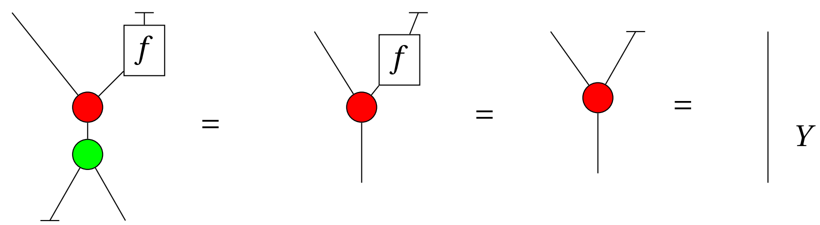
(6.12)

where the first equality is the definition, the second equality comes from the fact that f is a monoid morphism, the third equality is using the Frobenius law, the fourth equality is using the unitality of multiplication, the fifth equality is using the fact that f is a comonoid homomorphism, and finally, the last equality is using the fact that comultiplication is unital.

And for $f'f = \mathrm{id}_Y$, note the following sequence of diagrammatic equalities:



(6.13)



And hence $ff' = \mathrm{id}_X$ and $f'f = \mathrm{id}_Y$, so $f : X \to Y$ is an isomorphism in the underlying monoidal category. □

Our second category of interest will be the category of finite-dimensional Hilbert spaces FdHilb, which is the standard setting for finite-dimensional quantum theory. FdHilb has:

Objects: Finite-dimensional Hilbert spaces;

Morphisms: linear maps;

Composition: usual composition of linear transformations;

Identity: the identity linear transformation.

The final category of interest here will be the category of finite sets with isomorphisms, GFinSet:

Objects: Finite sets;

Morphisms: isomorphisms of finite sets, or equivalently, bijections;

Composition: usual composition of functions;

Identity: the identity function.

Now we will proceed to first define a functor from $F : \mathsf{GFinSet} \to \mathsf{CDSFA}(\mathsf{FdHilb})$.

6.4 From orthonormal bases to Frobenius algebras

We want to define a functor from $F : \mathsf{GFinset} \to \mathsf{CDSFA}(\mathsf{FdHilb})$, the elements of a finite set, say S, should be interpreted as the orthonormal basis for the free Hilbert space on it. We shall make this notion precise. Let $S = \{|\phi_i\rangle\}_{i=1}^{n}$. Then we define $F(S)$ by taking the free vector space over $\mathbb{C}$ on the elements of S. We must give it an inner product structure to make it a Hilbert space. We declare $\langle |\phi_i\rangle, |\phi_j\rangle\rangle := \delta_{ij}$ and then extend it linearly in the first component and conjugate linearly in the second component. This is what we meant by interpreting elements of the set as the orthonormal basis for our Hilbert Space. The space is automatically complete under the norm induced by this inner product, as it is a finite-dimensional space. Then to give it a CDSFA structure we define $\delta : H \to H \otimes H$ by freely copying the basis

$$\delta(|\phi_i\rangle) = |\phi_i\rangle \otimes |\phi_i\rangle,$$

and we define $\varepsilon : H \to \mathbb{C}$ by freely deleting the basis vectors

$$\varepsilon(|\phi_i\rangle) = 1.$$

We then define $m := \delta^\dagger$ and $u := \varepsilon^\dagger$. More explicitly, these maps are

$$m(|\phi_i\rangle \otimes |\phi_j\rangle) = |\phi_{\delta_{ij}}\rangle,$$

and,

$$u(1) = \sum_{i=1}^{n} |\phi_i\rangle \, ,$$

where by $|\phi_0\rangle$, we mean the zero vector (that is, the additive identity of the vector space structure). We denote the zero vector by 0.

We then extend these maps linearly. Now, for this to be well-defined, we must show that this defines a Frobenius algebra structure that is commutative and special. We first check that the associativity and unitality hold for the monoid structure. For this, we have,

$$\begin{aligned} m((\mathrm{id} \otimes m)(|\phi_i\rangle \otimes (|\phi_j\rangle \otimes |\phi_k\rangle))) &= m(|\phi_i\rangle \otimes |\phi_{\delta_{jk}}\rangle) \\ &= |\phi_{\delta_{ijk}}\rangle \, , \end{aligned}$$

which is

$$m((\mathrm{id} \otimes m)(|\phi_i\rangle \otimes (|\phi_j\rangle \otimes |\phi_k\rangle))) = \begin{cases} |\phi_i\rangle \, , & \text{if } i = j = k, \\ 0, & \text{otherwise} \, , \end{cases}$$

and

$$\begin{aligned} m((m \otimes \mathrm{id})((|\phi_i\rangle \otimes |\phi_j\rangle) \otimes |\phi_k\rangle)) &= m(|\phi_{\delta_{ij}}\rangle \otimes |\phi_k\rangle) \\ &= |\phi_{\delta_{ijk}}\rangle \, , \end{aligned}$$

which is

$$m((m \otimes \mathrm{id})((|\phi_i\rangle \otimes |\phi_j\rangle) \otimes |\phi_k\rangle)) = \begin{cases} |\phi_i\rangle \, , & \text{if } i = j = k, \\ 0, & \text{otherwise} \, . \end{cases}$$

So the associativity law holds. For unitality, note that,

$$\begin{aligned} m((u \otimes \mathrm{id})(1 \otimes |\phi_i\rangle)) &= m((\sum_{j=1}^{n} |\phi_j\rangle) \otimes |\phi_i\rangle) \\ &= \sum_{j=1}^{n} m(|\phi_j\rangle \otimes |\phi_i\rangle) \\ &= |\phi_i\rangle\,, \end{aligned}$$

where in the second equality, we have used the fact that m is (bi)linear by construction. Similarly, the right unitality holds. For comonoid laws, the proof for associativity is similar. For counitality, note that,

$$\begin{aligned} (\varepsilon \otimes \mathrm{id})(\delta(|\phi_i\rangle)) &= (\varepsilon \otimes \mathrm{id})((|\phi_i\rangle \otimes |\phi_i\rangle)) \\ &= 1 \otimes |\phi_i\rangle \\ &= |\phi_i\rangle\,. \end{aligned}$$

And similarly, the right unital law holds.

To check the Frobenius law, we must check first check $(\delta \otimes \mathrm{id})(\mathrm{id} \otimes m) = m\delta$. It suffices to check it on the basis vectors, as any linear map is uniquely determined by its action on this basis. For this, we have,

$$\begin{aligned} (\mathrm{id} \otimes m)[(\delta \otimes \mathrm{id})(|\phi_i\rangle \otimes |\phi_j\rangle)] &= (\mathrm{id} \otimes m)(|\phi_i\rangle \otimes (|\phi_i\rangle \otimes |\phi_j\rangle)) \\ &= |\phi_i\rangle \otimes |\phi_{\delta ij}\rangle\,, \end{aligned}$$

which is

$$(\mathrm{id} \otimes m)[(\delta \otimes \mathrm{id})(|\phi_i\rangle \otimes |\phi_j\rangle)] = \begin{cases} |\phi_i\rangle \otimes |\phi_i\rangle\,, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases}$$

and

$$\delta(m(|\phi_i\rangle \otimes |\phi_j\rangle)) = \delta(|\phi_{\delta_{ij}}\rangle),$$

which is

$$\delta(m(|\phi_i\rangle \otimes |\phi_j\rangle)) = \begin{cases} |\phi_i\rangle \otimes |\phi_i\rangle, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

And for the second equality, we must check, $(\mathrm{id} \otimes \delta)(m \otimes \mathrm{id}) = m\delta$, again it suffices to check it on the basis vectors. For this, note that,

$$\begin{aligned} (m \otimes \mathrm{id})[(\mathrm{id} \otimes \delta)(|\phi_i\rangle \otimes |\phi_j\rangle)] &= (m \otimes \mathrm{id})((|\phi_i\rangle \otimes |\phi_j\rangle) \otimes |\phi_j\rangle) \\ &= |\phi_{\delta_{ij}}\rangle \otimes |\phi_j\rangle, \end{aligned}$$

which is

$$(m \otimes \mathrm{id})[(\mathrm{id} \otimes \delta)(|\phi_i\rangle \otimes |\phi_j\rangle)] = \begin{cases} |\phi_i\rangle \otimes |\phi_i\rangle, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

So we have that the Frobenius law holds; $(\delta \otimes \mathrm{id})(\mathrm{id} \otimes m) = m\delta = (\mathrm{id} \otimes \delta)(m \otimes \mathrm{id})$. It is also a †- Frobenius algebra by construction. Next, we must check that the algebra structure is commutative and special. For commutative, note that in FdHilb, the braiding map is given by $c_\otimes(|\phi_i\rangle \otimes |\phi_j\rangle) = |\phi_j\rangle \otimes |\phi_i\rangle$. Now, note that,

$$\begin{aligned} m(c_\otimes(|\phi_i\rangle \otimes |\phi_j\rangle)) &= m(|\phi_j\rangle \otimes |\phi_i\rangle) \\ &= |\phi_{\delta_{ij}}\rangle \\ &= m(|\phi_i\rangle \otimes |\phi_j\rangle), \end{aligned}$$

and similarly,

$$\begin{aligned} c_\otimes(\delta(|\phi_i\rangle)) &= c_\otimes(|\phi_i\rangle \otimes |\phi_i\rangle) \\ &= |\phi_i\rangle \otimes |\phi_i\rangle \\ &= \delta(|\phi_i\rangle). \end{aligned}$$

So the algebra is commutative, and finally, the algebra is also special, as

$$\begin{aligned} m(\delta(|\phi_i\rangle)) &= m(|\phi_i\rangle \otimes |\phi_i\rangle) \\ &= |\phi_i\rangle \,. \end{aligned}$$

So $\delta m = \mathrm{id}$ and hence the algebra is special. So, on objects, we have that the functor $F : \mathsf{GFinset} \to \mathsf{CDSFA}(\mathsf{FdHilb})$ is well-defined on objects.

Now for any morphism $f : S \to T$ in $\mathsf{GFinset}$, we define a morphism in FdHilb by $F(f) = f'$, where $f'(|\phi_i\rangle) := f(|\phi_i\rangle)$ and then extending it linearly. Now to show that this is a morphism in $\mathsf{CDSFA}(\mathsf{FdHilb})$, we must show that $f' : F(S) \to F(T)$ preserves both monoid and comonoid structure. We call $H_1 := F(S)$ and $H_2 := F(T)$. For the monoid-structure, we check it on the basis elements as the map is linear by construction. For this, we shall show that $f'(m_{H_1}(|\phi_i\rangle \otimes |\phi_j\rangle)) = m_{H_2}[(f' \otimes f')(|\phi_i\rangle \otimes |\phi_j\rangle)]$. For this, note that,

$$f'(m_{H_1}(|\phi_i\rangle \otimes |\phi_j\rangle)) = f'(|\phi_{\delta_{ij}}\rangle),$$

which is

$$f'(m_{H_1}(|\phi_i\rangle \otimes |\phi_j\rangle)) = \begin{cases} f'(|\phi_i\rangle), & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases}$$

and

$$m_{H_2}[f' \otimes f'](|\phi_i\rangle \otimes |\phi_j\rangle) = m_{H_2}(f'(|\phi_i\rangle) \otimes f'(|\phi_j\rangle)),$$

which is

$$m_{H_2}[f' \otimes f'](|\phi_i\rangle \otimes |\phi_j\rangle) = \begin{cases} f'(|\phi_i\rangle), & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

Thus $f' : H_1 \longrightarrow H_2$ preserves multiplication. To show that it preserves the monoid structure, we must also show that $u_{H_1} f' = u_{H_2}$. For this, note that

$$\begin{aligned} f'(u_{H_1}(1)) &= f'(\sum_{i=1}^{n} |\phi_i\rangle) \\ &= \sum_{i=1}^{n} f'(|\phi_i\rangle) \\ &= u_{H_2}(1), \end{aligned}$$

where in the last equality, we have used the fact that $f : S \longrightarrow T$ is an isomorphism, in particular a surjective map. Now we shall show that f' also preserves the comonoid structure. For the comultiplication, note that

$$\begin{aligned} f' \otimes f'(\delta_{H_1}(|\phi_i\rangle)) &= f' \otimes f'(|\phi_i\rangle \otimes |\phi_i\rangle) \\ &= f'(|\phi_i\rangle) \otimes f'(|\phi_i\rangle), \end{aligned}$$

and

$$\delta_{H_2}(f'(|\phi_i\rangle)) = f'(|\phi_i\rangle) \otimes f'(|\phi_i\rangle).$$

And finally for the counit,

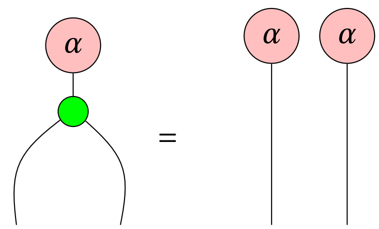
$$\varepsilon_{H_2}(f(|\phi_i\rangle)) = 1 = \varepsilon_{H_1} f(|\phi_i\rangle).$$

So we have that $F : \mathsf{GFinset} \longrightarrow \mathsf{CDSFA}(\mathsf{FdHilb})$ is well-defined. The functorality is immediate as the composition structure is the same in both $\mathsf{GFinset}$ and $\mathsf{CDSFA}(\mathsf{FdHilb})$.

Remark 6.4.1. We also observe that the $f' := G(f)$ in $\mathsf{CDSFA}(\mathsf{FdHilb})$ for any $f : S \longrightarrow T$ in $\mathsf{GFinSet}$ is not only an isomorphism between the underlying space by Lemma 6.3.1, but in fact a unitary isomorphism, as $\langle |\phi_i\rangle \mid |\phi_j\rangle\rangle = \delta_{ij} = \langle f'(|\phi_i\rangle) | f'(|\phi_j\rangle)\rangle$, as f is in particular injective.

We shall now prove the fullness and faithfulness of this functor. But before that, we note some interesting observations.

Definition 6.4.2. A copyable element of a co-monoid (X, δ, ε) is a co-monoid morphism from I, the identity for the monoidal tensor on X.

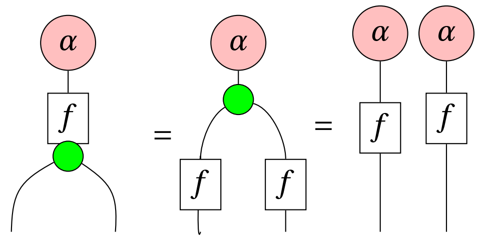


(6.14)

Remark 6.4.3. Now, in the co-monoid structure defined on the free Hilbert space by the functor S, the only copyable elements are the basis elements. To see that, note that the basis elements are copyable by construction, and for any vector that involves a sum of any two basis elements, say $|\psi\rangle = |\phi_1\rangle + |\phi_2\rangle$ and $\delta(|\psi\rangle) = (|\phi_1\rangle \otimes |\phi_1\rangle) + (|\phi_2\rangle \otimes |\phi_2\rangle)$. And this is not equal to $|\psi\rangle \otimes |\psi\rangle$, as $(|\phi_1\rangle \otimes |\phi_1\rangle) + (|\phi_2\rangle \otimes |\phi_2\rangle)$ is an entangled state (not factorisable as a tensor product of two states), whereas $|\psi\rangle \otimes |\psi\rangle$ is not. So F faithfully encodes the basis as the copyable elements of our co-monoid.

Lemma 6.4.4. Let $f : (X, \delta, \varepsilon) \to (Y, \delta', \varepsilon')$ be a co-monoid homomorphism. Then f takes a copyable element of X to a copyable element of Y.

Proof. Let $\alpha : I \to X$ be a copyable element of X. We then want to show that $f(\alpha)$ is a copyable element of Y. For this note, the following string diagrammatic proof,



(6.15)

where the first equality is using the fact that f is a comonoid homomorphism and the second equality comes from the fact that α is copyable.

Remark 6.4.5. Note that the above Remark 6.4.3 and Lemma 6.4.4 apply also in the case where the algebra structure does not have a unit. This will be crucial in the case of arbitrary dimensions.

Lemma 6.4.6. $F : \mathsf{GFinset} \to \mathsf{CDSFA}(\mathsf{FdHilb})$ is a fully faithful functor .

Proof. Faithfullness of F is immediate. For fullness, let $f' : F(S) \to F(T)$ be a map between the free Hilbert space on S and T. Then, since f' preserves both monoid and comonoid structure, by Lemma 6.3.1, it is an isomorphism between $F(S)$ and $F(T)$, moreover by Lemma 6.4.4, it takes copyable elements of $F(S)$ to copyable elements of $F(T)$. Now by construction, copyable elements of $F(S)$ and $F(T)$ are elements of S and T respectively, so we get a map $f : S \to T$ in $\mathsf{GFinset}$ define by $f(|\phi_i\rangle) := f'(|\phi_i\rangle)$. It is clear that $F(f) = f'$ and hence $F : \mathsf{GFinset} \to \mathsf{CDSFA}(\mathsf{FdHilb})$ is full.

6.5 Interlude: C^*- algebras

Now we shall proceed to define a functor $G : \mathsf{CDSFA}(\mathsf{Fdhilb}) \to \mathsf{GFinset}$. At objects, this will be achieved by taking the copyable elements of the Frobenius algebra. Essential to the proof of the equivalence will be the fact that the copyable elements of a CDSFA in FdHilb will form a basis for the underlying Hilbert space. The main idea is that we will embed the elements of our space into FdHilb(H, H); this monoid embedding will be involution preserving, where the FdHilb$(\mathbb{C}, H)$ has the monoid structure coming from the Frobenius algebra. And thus FdHilb$(\mathbb{C}, H)$ will have the C^* algebra structure, and a structure theorem for C^* algebras will give us the desired result. To see this more concretely, we will first need to build some mathematical preliminaries. Our first definition is of a C^* algebra, which is an abstraction of the complex numbers.

Definition 6.5.1. [33] A C^***- algebra** is a non-empty set A together with the following operations:

- addition; +, which is commutative and associative;
- multiplication; $\times$, which is associative;
- multiplication by complex scalars; $\times_c$;
- an involution $a \longrightarrow a^*$, which is conjugate linear;

such that for any $a, b \in A$, $\lambda, \mu \in \mathbb{C}$, $\lambda(ab) = (\lambda a)b = a(\lambda b)$ and $(\lambda\mu)a = \lambda(\mu a)$.

Crucially, there is a norm $||.||$ on A such that for all $a, b \in A$ and $\lambda \in \mathbb{C}$ such that

- $||\lambda a|| = |\lambda|\,||a||$,
- $||a + b|| \leq ||a|| + ||b||$,
- $||ab|| \leq ||a||\,||b||$, and

- $||a^*a|| = ||a||^2$,

Finally, the space A is complete under this norm.

Example 6.5.2. The space of all bounded operators between a Hilbert space and itself, denoted by $\mathcal{B}(H)$ is a C^* algebra. The linear structure is just the addition of operators. Multiplication is given by composition, and the involution is given by taking the adjoint of an operator (dagger). And the norm is given by the supremum over the unit ball,

$$||f|| := \text{supp}_{||x||\leq 1}\{f(x) : x \in H\}$$

Note that in the finite-dimensional case, all norms are equivalent, and if we fix a basis, then we have that $M_n(\mathbb{C})$ is a C^* algebra.

Example 6.5.3. For any compact Hausdorff topological space X, $C(X) = \{f : X \to \mathbb{C} : f \text{ is continuous}\}$ is a C^* algebra. Here, the addition, multiplication, and scalar multiplication are all pointwise, and the involution is given by taking the pointwise complex conjugate. The norm again is given by the supremum norm,

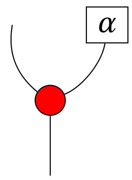
$$||f|| = \text{supp}_x\{f(x) : x \in X\}$$

Theorem 6.5.4. [41] Every finite-dimensional commutative C^* algebra over $\mathbb{C}$ is a product of one-dimensional complex algebras. And the co-algebra morphisms form the basis for the underlying space.

We will be using this structure theorem in the next section to show that the copyable elements of a CDSFA in FdHilb forms a basis for the underlying space.

6.6 From Frobenius algebras to orthonormal bases

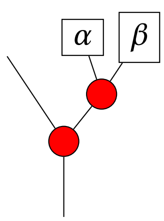
We begin by defining a map $R : \mathrm{FdHilb}(\mathbb{C}, H) \to \mathrm{FdHilb}(H, H)$ by multiplication on the right, that is $R(\alpha)(|\phi\rangle) = (|\phi\rangle \otimes \alpha)m$. Diagrammatically, $R(\alpha)$ is

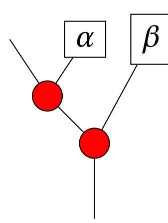


(6.16)

Lemma 6.6.1. $R : \mathrm{FdHilb}(\mathbb{C}, H) \to \mathrm{FdHilb}(H, H)$ is a monoid morphism, where the monoid structure on $\mathrm{FdHilb}(\mathbb{C}, H)$ is induced by the monoid (H, m, u) of the Frobenius algebra, and the monoid structure on $\mathrm{FdHilb}(H, H)$ is of composition.

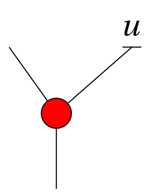
Proof. First note that R preserves multiplication. This is just by associativity of the multiplication. Note the following string diagrammatic proof,

$R(\alpha\beta) =$ 

$=$ 

$= R(\alpha)R(\beta)$ (6.17)

And for unitality, we have,

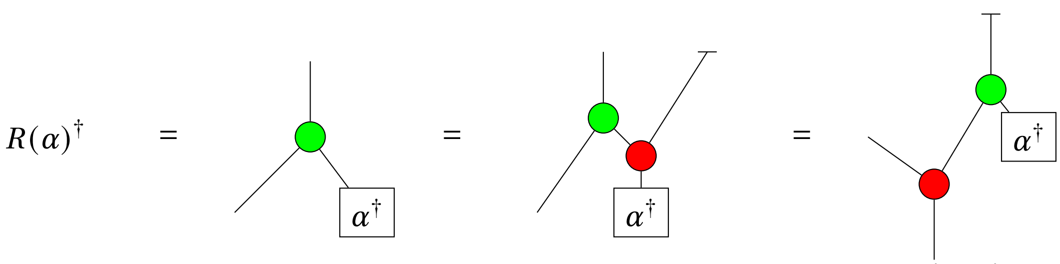
$R(u) =$ 

$=$
(6.18)

□

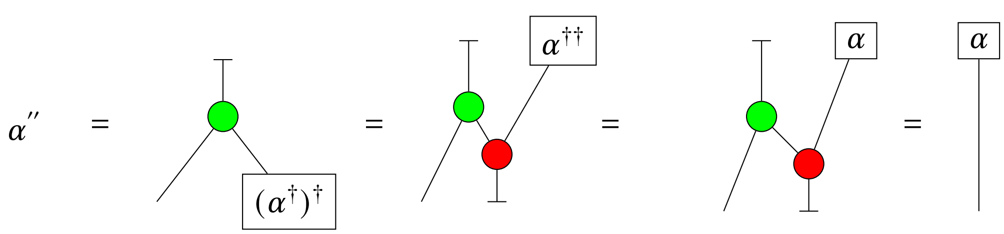
Now also note that R is indeed an embedding as if for α, β, $R(\alpha) = R(\beta)$, then $uR(\alpha) = uR(\beta)$, but then using unitality, this says $\alpha = \beta$. R also preserves the vector space structure, this is just by linearity of m.

Now since there is a $\dagger$ on $\mathrm{FdHilb}(H,H)$, we can take $R(\alpha)^\dagger$, then the question is does there exists $(\_)' : \mathrm{FdHilb}(\mathbb{C},H) \to \mathrm{FdHilb}(\mathbb{C},H)$ such that $R(\alpha') = R(\alpha)^\dagger$. The answer is positive, we will construct such an α'. For that, note the following string diagrammatic proof,



(6.19)

where the second equality is using the unitality of multiplication and the third equality comes from the Frobenius law. So equationally, we have $\alpha' = u\delta(\mathrm{id} \otimes \alpha^\dagger)$. Moreover, $(\_)' : \mathrm{FdHilb}(\mathbb{C},H) \to \mathrm{FdHilb}(\mathbb{C},H)$ is involutive as,



(6.20)

where the first two equalities are just by definition, the third equality uses the fact that $\dagger$ is involutive, and the final equality is by unitality.

So from the above discussion, we have that $R : \mathrm{FdHilb}(\mathbb{C},H) \to \mathrm{FdHilb}(H,H)$ is an involution-preserving monoid embedding.

Remark 6.6.2. The proof that there exists an operation $(\_)' : \mathrm{FdHilb}(\mathbb{C},H) \to \mathrm{FdHilb}(\mathbb{C},H)$ such that $R(\alpha') = R(\alpha)^\dagger$ uses the unitality axiom. Abramsky and Heunen in [2] considered non-unital Frobenius algebras (motivated by the fact that a Frobenius algebra in Hilb is unital if and only if it is finite dimensional [27, 25]), where they take the existence of such an involutive operation in a general dagger category to be an axiom. They then consider a different kind of algebra called the H^* algebra to derive their results. We will see that when we consider the Frobenius structure in RLef , the free copy map is continuous, but the free deletion map is not.

Theorem 6.6.3. [33] Any finite-dimensional involution closed subalgebra of a C^* algebra is a C^* algebra.

So by the above discussion and theorem, we have that $\mathrm{FdHilb}(\mathbb{C}, H)$ is a finite-dimensional C^* algebra. Moreover, this is commutative, as the multiplication here is commutative. So by the structure theorem 6.5.4, we have that,

$$\mathrm{FdHilb}(\mathbb{C}, H) \cong H \cong \bigoplus_{i=1}^{n} (\mathbb{C}, \delta_{\mathbb{C}})$$

And hence, the co-algebra morphisms $|\phi_i\rangle : \mathbb{C} \to H$ form a basis for H, but the co-algebra morphisms are exactly the copyable elements of the co-monoid (H, δ, ε), so we have that the copyable elements of a CDSFA in FdHilb form a basis for the underlying space.

That this basis is orthogonal with norm 1 follows from structural reasons.

Lemma 6.6.4. [2] The copyable elements of a Frobenius algebra in FdHilb are pairwise orthogonal.

Proof. Let a, b be two distinct non-zero copyable elements of (H, δ, ε) such that $\langle a|b\rangle \neq 0$, then

$$\begin{aligned}
\langle a|a\rangle\langle a|a\rangle\langle b|a\rangle &= \langle a\otimes a\otimes b|a\otimes a\otimes a\rangle \\
&= \langle(\delta\otimes id)(a\otimes b)|(id\otimes\delta)(a\otimes a)\rangle \\
&= \langle a\otimes b|(\delta^\dagger\otimes id)\circ(id\otimes\delta)(a\otimes a)\rangle \\
&= \langle a\otimes b|(id\otimes\delta^\dagger)\circ(\delta\otimes id)(a\otimes a)\rangle, \text{ by Frobenius law} \\
&= \langle(id\otimes\delta)(a\otimes b)|(\delta\otimes id)(a\otimes a)\rangle \\
&= \langle a\otimes b\otimes b|a\otimes a\otimes a\rangle \\
&= \langle a|a\rangle\langle b|a\rangle\langle b|a\rangle.
\end{aligned}$$

So we have $\langle a|a\rangle\langle a|a\rangle\langle b|a\rangle = \langle a|a\rangle\langle b|a\rangle\langle b|a\rangle$, canceling $\langle a|a\rangle$, and $\langle b|a\rangle$, we get $\langle a|a\rangle = \langle b|a\rangle$. Similar computation will give us $\langle b|b\rangle\langle b|b\rangle\langle a|b\rangle = \langle b|b\rangle\langle a|b\rangle\langle a|b\rangle$, and canceling $\langle b|b\rangle$, and $\langle a|b\rangle$ will give us $\langle b|b\rangle = \langle a|b\rangle$. Then observe,

$$\begin{aligned}
\langle a|a\rangle &= \langle b|a\rangle \\
&= \overline{\langle a|b\rangle} \\
&= \langle b|b\rangle \\
&= \langle b|b\rangle.
\end{aligned}$$

So it follows that all inner products here are real and equal. But then, $\langle a-b|a-b\rangle = \langle a|a\rangle - \langle a|b\rangle - \langle b|a\rangle + \langle b|b\rangle = 0$, contradicting the fact that a, b were distinct. □

So we have that non-zero copyable elements of a $\mathsf{CDSFA}(\mathsf{FdHilb})$ form an orthogonal basis for the underlying space. Moreover, they have norm 1 as,

Lemma 6.6.5. Non-zero copyable elements of a $\mathsf{CDSFA}(\mathsf{FdHilb})$ have norm 1.

Proof. Note,

$$\begin{aligned}\langle a|a\rangle &= \langle \delta a|\delta a\rangle,\text{ as the algebra is special}\\ &= \langle a\otimes a|a\otimes a\rangle\\ &= \langle a|a\rangle\langle a|a\rangle.\end{aligned}$$

Since the element is non-zero by assumption, we have that $||a|| = 1$. □

Hence, we have that the copyable elements of a $\mathsf{CDSFA}(\mathsf{FdHilb})$ form an orthonormal basis for the underlying space. We will now proceed to define a functor $G : \mathsf{CDSFA}(\mathsf{FdHilb}) \to \mathsf{GFinSet}$. On objects, we take the copyable elements of a Frobenius algebra, which we have shown above to form a basis for the underlying Hilbert space. On morphisms, we define G as follows. Let $f : (H, \delta, \varepsilon) \to (H', \delta', \varepsilon')$ be a morphism of Frobenius algebras in $\mathsf{CDSFA}(\mathsf{FdHilb})$. We then define $G(f) = f'$, where $f'(|\phi_i\rangle) := f(|\phi_i\rangle)$. Note that this is well-defined as f is a co-algebra morphism so it takes copyable elements to copyable elements by Lemma 6.4.4, moreover as f is both algebra and co-algebra morphism so f is an isomorphism of the underlying spaces, so $f' : G(H, \delta, \varepsilon) \to G(H', \delta', \varepsilon')$ is a morphism in $\mathsf{GFinSet}$. Functoriality of G is then immediate.

We also have that $G : \mathsf{CDSFA}(\mathsf{FdHilb}) \to \mathsf{GFinSet}$ is fully faithful.

Lemma 6.6.6. The functor $G : \mathsf{CDSFA}(\mathsf{FdHilb}) \to \mathsf{GFinSet}$ is fully faithful.

Proof. Faithfullness of G is immediate. For fullness, let $f : G(H, \delta, \varepsilon) \to G(H', \delta', \varepsilon')$ be a morphism in $\mathsf{GFinset}$, we then define $f' : (H, \delta, \varepsilon) \to$

$(H', \delta', \varepsilon')$ by setting $f'(|\phi_i\rangle) = f(|\phi_i\rangle)$ and then extending it linearly. We must show that f' preserves the co-monoid and monoid structure for this to be well-defined. For the co-monoid structure, note that as $G(H, \delta, \varepsilon)$, and $G(H', \delta', \varepsilon')$ are the copyable elements of (H, δ, ε), and $(H', \delta', \varepsilon')$ respectively. Hence, it fixes the co-multiplication structure on these algebras where the co-multiplication map copies these basis elements. Then we have,

$$\begin{aligned} f' \otimes f'(\delta(|\phi_i\rangle)) &= f' \otimes f'(|\phi_i\rangle \otimes |\phi_i\rangle) \\ &= f'(|\phi_i\rangle) \otimes f'(|\phi_i\rangle), \end{aligned}$$

and

$$\delta'(f'(|\phi_i\rangle)) = f'(|\phi_i\rangle) \otimes f'(|\phi_i\rangle).$$

Fixing the co-multiplication also fixes the counit and, by dagger, the multiplication and the unit map. The fact that f' also preserves the counit and the monoid structure is exactly analogous to Section 6.4. It is then clear that $G(f') = f$. Hence $G : \mathsf{CDSFA}(\mathsf{FdHilb}) \to \mathsf{GFinSet}$ is fully faithful.

6.7 Equivalence between bases and coalgebras

We will now prove that $\mathsf{GFinSet}$ and $\mathsf{CDSFA}(\mathsf{FdHilb})$ are equivalent as categories. For this, we will first set up an adjunction between them. We already have a pair of functors $F : \mathsf{GFinSet} \to \mathsf{CDSFA}(\mathsf{FdHilb})$ and $G : \mathsf{CDSFA}(\mathsf{FdHilb}) \to \mathsf{GFinSet}$. We will show that this pair of functors forms part of an adjunction. We will give the adjunction by proving the triangle identities. For this, first note that the constructions in Section 6.4 and 6.6

are inverses to each other. If we first construct a Hilbert space on a finite set and define the comultiplication by copying these elements, and then take the copyable elements of that Frobenius algebra, we end up getting the same set back. And if we first take copyable elements of a Frobenius algebra, we know that they form a basis of the underlying space, and thus, constructing a Frobenius algebra on this set will get us back to the algebra we started with.

So the unit and counit map both are both identity maps, which are clearly well-defined and natural. The triangle identities thus reduce to simply equality of identity maps in both categories. So $F : \mathsf{GFinSet} \to \mathsf{CDSFA}(\mathsf{FdHilb})$ and $G : \mathsf{CDSFA}(\mathsf{FdHilb}) \to \mathsf{GFinSet}$ form part of an adjunction. Moreover, since both the unit and counit maps are isomorphisms, this gives us an equivalence of categories, so we have $\mathsf{GFinSet} \cong \mathsf{CDSFA}(\mathsf{FdHilb})$.

Lemma 6.3.1 tells us that if we require morphisms of Frobenius algebras to preserve both the monoid and the comonoid structures, then they are necessarily isomorphisms. So if we define a new category $\mathsf{CDSFA}'(\mathsf{FdHilb})$ where we require morphisms only to preserve the comultiplication structure, then we get that the category of finite sets and functions FinSet is equivalent to the category $\mathsf{CDSFA}'(\mathsf{FdHilb})$, that is $\mathsf{FinSet} \cong \mathsf{CDSFA}'(\mathsf{FdHilb})$.

So we have,

Proposition 6.7.1. [16] $(\eta, \varepsilon) : F \dashv G : \mathsf{FinSet} \to \mathsf{CDSFA}'(\mathsf{FdHilb})$ is an equivalence of categories.

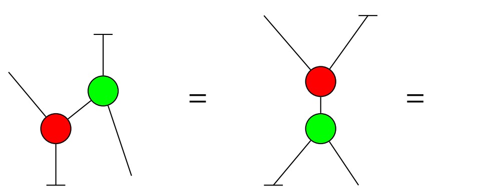
Now we proceed to see how this story about the correspondence between bases and algebras pans out when we go to arbitrary dimensions. As explained at the beginning of this chapter, Frobenius algebras are on self-dual objects; therefore, to move beyond the compact fragment of the theory, we consider linear semigroups. The absence of units from our algebras is due to continuity considerations, which we have to take into account as we have an underlying (Lefschetz) topology, so the continuity of the proposed maps must be established.

6.8 Dagger linear semigroups

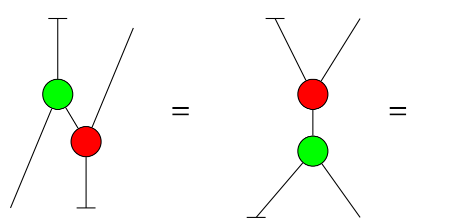
The main motivation to look beyond Frobenius algebras is the following theorem

Proposition 6.8.1. [27] A Frobenius algebra $(X, m, u, \delta, \varepsilon)$ in a monoidal category $\mathbb{X}$ is on a self-dual object.

Proof. We define the unit and the counit map as follows: $\eta : I \to X \otimes X := u\delta$ and $\varepsilon' : X \otimes X \to I := m\varepsilon$. We must prove that they obey the triangle identities, for that note the following string diagrammatic proof for $(\mathrm{id}_X \otimes \eta)(\varepsilon \otimes \mathrm{id}_X) = \mathrm{id}_X$,

 (6.21)

where the first equality is using the Frobenius law and the second equality is by right unitality of multiplication and left unitality of comultiplication. And for the second triangle identity, $(\eta \otimes \mathrm{id})(\mathrm{id} \otimes \varepsilon)$we have,

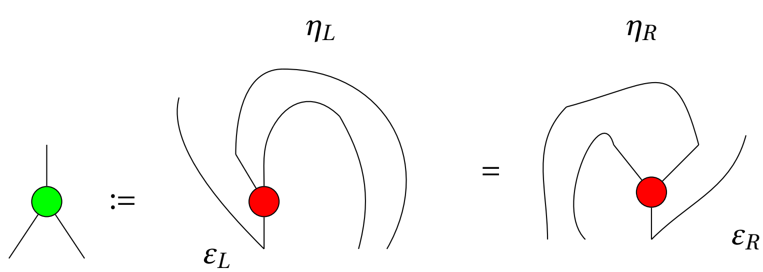
 (6.22)

where we have used the symmetric version of the same laws as for the first triangle equality. □

So this says that if we have a Frobenius algebra structure in any monoidal category, then it automatically degenerates to a compact closed setting. Since compact structures can only account for the finite-dimensional fragment of the theory [22], we consider linear monoids, which makes sense in any LDC with a specified dual for that object; in particular, the notion of a linear monoid makes sense for any $*$- autonomous category, allowing us to move beyond the core.

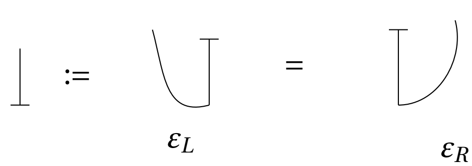
A linear monoid is simply a monoid object in a LDC with a specified dual, such that the comonoid induced by left and right duals agrees.

Definition 6.8.2. [12, 40] A **linear monoid** $A \overset{\circ}{-\!\!\|}\, B$ in an LDC consists of a monoid $(A, m : A \otimes A \to A, u : I \to A)$, a left dual $(\eta_L, \varepsilon_L) : A -\!\!\| \, B$, a right dual $(\eta_R, \varepsilon_R) : B -\!\!\| \, A$ such that the comonoid structure induced by both duals are same, that is,



(6.23)

and,



(6.24)

A linear monoid is said to be commutative if the underlying category is symmetric and $c_\otimes m = m$.

Thus linear monoid is simply a monoid in an LDC on an object for which the right and left dual is specified.

In particular, in a $*$- autonomous category, the definition of a linear monoid simply reduces to having a monoid structure (or equivalently a comonoid structure, as there is symmetry in the definition) on a object, the comonoid structure (respectively a monoid structure) on the dual of that object is then automatically specified by the action of the $(\_)^*$ functor (See Lemma 3.1.16).

There is an equivalent definition of a linear monoid where the resemblance to a Frobenius algebra is clearer.

Proposition 6.8.3. [40] A linear monoid $A \overset{\circ}{-\!\!\|} B$ in a LDC is equivalent to the following data:

- a monoid $(A, m : A \otimes A \to A, u : \top \to A)$;
- a comonoid $(B, \delta : B \to B \oplus B, \varepsilon : B \to \bot)$;
- and action maps $\rho_L : A \otimes B \to B$, $\phi_L : A \to B \oplus A$, and their symmetrical counterparts, such that,

(i) $(u_A \otimes \mathrm{id}_B)\rho_L = \mathrm{id}_B$,

(ii) $(m_A \otimes \mathrm{id}_B)\rho_L = (\mathrm{id}_A \otimes \rho_L)\rho_L$,

(iii) $(\mathrm{id}_A \otimes \rho_R)\rho = (\rho_L \otimes \mathrm{id}_A)\rho_R$, and

(iv) $(\mathrm{id}_A \otimes \phi_L)(\rho_L \otimes \mathrm{id}_A) = m_A \rho_L = (\phi_L \otimes \mathrm{id}_A)(\mathrm{id}_B \otimes m_A)$.

The morphism of linear monoids is defined as follows.

Definition 6.8.4. [40] A **morphism of linear monoids** is a pair of maps $(f, g) : (A \overset{\circ}{-\!\!\|} B) \to (A' \overset{\circ}{-\!\!\|} B')$ such that $f : A \to A'$ is a monoid morphism (or equivalently $g : B' \to B$ is a comonoid morphim) such that $(f, g), (g, f)$ preserves the left dual and right dual respectively.

We will be interested in †- linear monoids, which are linear monoids where the induced comonoid structure is on the object which is dagger of the object on which the monoid structure is given, and the comonoid structure induced by the dual map of LDC and the dagger functor coincide. More precisely, we have,

Definition 6.8.5. [40] A linear monoid $A \overset{\circ}{-\!\!\|} B$ in a $\dagger - LDC$ is said to be a **dagger linear monoid** if and only if $B = A^\dagger$ and moreover $\varepsilon_{A^\dagger} = (u_A)^\dagger$, $\delta_{A^\dagger} = (m_A)^\dagger$. That is to say, the induced coalgebra map from the dual matches the induced map via daggering.

Again, when working in the case of $*$- autonomous categories, where the dual map is coming from the $(\_)^*$ functor (possibly composed with the identity functor or the conjugation functor which leaves the underlying map unchanged), this just reduces to having a monoid structure on one of the objects, the comonoid structure on its dagger is then determined. (See Proposition 4.3.6).

Also note that the morphisms of a dagger linear monoid are of the form $(f, f^\dagger)$. Now, for reasons which we will see ahead, we will have to do with just having a semigroup structure (that is monoid without the unit), but the rest remains the same. And we will be interested in the category of dagger linear monoids in RLef.

At the end of this section, we note the following categories of interest for our discussion ahead. Our first category of interest will be commutative dagger linear semigroups in RLef, $\mathsf{CDLS}(\mathsf{RLef})$, where

Objects: Commutative †- linear semigroups in RLef;

Morphisms: mrphisms of linear semigroups (that is, morphisms of linear monoids without the requirement of preserving units);

Composition: usual composition of functions in RLef;

Identity: the identity continuous linear function in RLef.

Our other two categories of interest will be $\mathsf{RLef}_{\mathsf{B}}$ and FinF, which we have already discussed before, but we recall them here for convenience. $\mathsf{RLef}_{\mathsf{B}}$ has

Objects: Reflexive Lefschetz spaces with a chosen basis;

Morphisms: continuous linear maps independent of the basis;

Composition: composition of (continuous linear) functions;

Identity: the identity continuous linear function.

And FinF has

Objects: Finiteness spaces $(X, \mathcal{U}, \mathcal{U}^{\perp})$, $(Y, \mathcal{V}, \mathcal{V}^{\perp})$;

Morphisms: $f : X \to Y$ such that for any $u \in \mathcal{U}$, $f(u) \in \mathcal{V}$ and for any $v' \in \mathcal{V}^{\perp}$, $f^{-1}(v) \in \mathcal{U}^{\perp}$;

Composition: usual composition of functions;

Identity: the identity function.

6.9 From bases to dagger linear semigroups

We first want to define a functor $F : \mathsf{FinF} \to \mathsf{CDLS}(\mathsf{RLef})$. For this, we first construct a Lefschetz space on a given finiteness space. Recall that the web of the finiteness space acts as the *basis* for the Lefschetz space constructed on it via Ehrhard's construction. Now we freely copy these basis elements to give them a cosemigroup structure. But we must prove that the copy map is continuous in RLef for it to be well-defined. First, recall that the dagger of V, where V is a reflexive Lefschetz space, is $\overline{V^*}$, where V^* is the dual of V, and the conjugation functor changes the scalar action to $c \bar{\bullet} v := \overline{c} \bullet v$. Now we will prove that the copy map is exactly the dagger of the multiplication map, which we first show to be continuous.

Lemma 6.9.1. Let V be any reflexive Lefschetz space with a basis $\{f_x\}_{x\in X}$.

(i) The map $m : \overline{V^*} \otimes \overline{V^*} \to \overline{V^*}$ given by $f_x^* \otimes f_y^* \to f_x^*$ if $x = y$, 0 otherwise is continuous.

(ii) The map $\delta : V \to V \oplus V := (\overline{V^*} \otimes \overline{V^*})^*$ given by $f_x \to f_x \oplus f_x$ is the dagger of the above map and is thereby continuous.

Proof.

(i) We know that $m : \overline{V^*} \otimes \overline{V^*} \to \bar{V}^*$ is continuous if and only if the induced linear maps $m_1(-, \varphi) : \overline{V^*} \to \overline{V^*}$ and $m_2(\varphi, -) : \overline{V^*} \to \overline{V^*}$ are continuous for any choice of $\varphi \in \overline{V^*}$. Now m_1 pulls back an open subspace U, around $0_{\overline{V^*}}$ to $\overline{V^*} - \mathrm{Span}[\{\varphi\}]$, which is open (See Lemma 6.10.2), if $f_x^* \notin U$. And if $f_x^* \in U$, then m pulls back U to whole of $\overline{V^*}$, which again is open. So m_1 is continuous. The proof of m_2 being continuous is exactly identical. Hence $m : \overline{V^*} \otimes \overline{V^*} \to \overline{V^*}$ is continuous.

(ii) We will now prove that $\delta : V \to V \oplus V := (\overline{V^*} \otimes \overline{V^*})^*$ is actually $(m)^\dagger$, and is hence continuous (as $\dagger$ is a functor). For this first note that δ sends f_x to $f_x \oplus f_x$, which is the map that takes the value 1 at $f_x^* \otimes f_x^* \in \overline{V^*} \otimes \overline{V^*}$ and 0 elsewhere. Now, identifying $f_x \in V$ with $\mathrm{ev}_V(f_x) \in \overline{V^{**}}$, we see that $\mathrm{ev}_V(f_x)$ goes to $m\, \mathrm{ev}_V(f_x)$ via $m^\dagger$. And $\mathrm{ev}_V(f_x)(m(f_x^* \otimes f_y^*)) = 1$ if $x = y$ and 0 elsewhere, which is exactly the map δ and hence δ is continuous.

$\square$

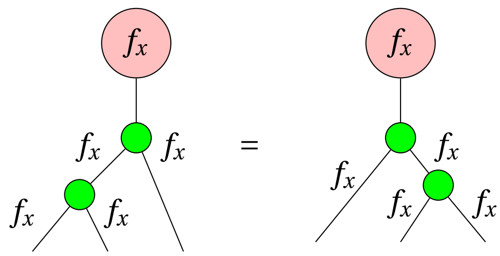
Thus, the copy map, which copies every basis vector of a reflexive Lefschetz space, is indeed continuous.

Now we will define a functor $F : \mathsf{FinF} \to \mathsf{CDLS}(\mathsf{RLef})$. On objects we define the functor $F : \mathsf{FinF} \to \mathsf{CDLS}(\mathsf{RLef})$ as $F(X, \mathcal{U}, \mathcal{U}^\perp) := \mathbb{C}(X, \mathcal{U}, \mathcal{U}^\perp)$, with the basis of maps of point support $\{f_x\}_{x \in X}$, as in Ehrhard's construction. Then we define the cosemigroup structure by linear extension of copying all the basis elements, $\delta(f_x) := f_x \oplus f_x$. This map is continuous by Lemma 6.9.1. We first show that this defines a cosemigroup. We check this on the basis elements, as a linear map is determined by its action on the basis. For this, note that

$$
\begin{aligned}
(\delta \oplus \mathrm{id})(\delta(f_x)) &= \delta \oplus \mathrm{id}(f_x \oplus f_x) \\
&= f_x \oplus f_x,
\end{aligned}
$$

and

$$
\begin{aligned}
(\mathrm{id} \oplus \delta)(\delta(f_x)) &= \mathrm{id} \oplus \delta(f_x \oplus f_x) \\
&= f_x \oplus f_x.
\end{aligned}
$$



(6.25)

Moreover, this cosemigroup structure is clearly commutative. Note that in RLef, the symmetry map is given by $c_\oplus(f_x \oplus f_y) := f_y \oplus f_x$, so,

$$\begin{aligned} c_\oplus(\delta(f_x)) &= c_\oplus(f_x \oplus f_x) \\ &= f_x \oplus f_x \\ &= \delta(f_x). \end{aligned}$$

This induces a semigroup structure on the dagger linear dual of $\mathbb{C}(X, \mathcal{U}, \mathcal{U}^\perp)$, which is $\overline{\mathbb{C}(X, \mathcal{U}, \mathcal{U}^\perp)}^*$. Note that this is by construction a dagger linear semigroup, as the dual map comes from the action of the dualizing functor, and moreover, the conjugation functor leaves the underlying map unchanged.

Next, we define the functor $F : \mathsf{FinF} \to \mathsf{CDLS}(\mathsf{RLef})$ on maps. For a $\alpha : (X, \mathcal{U}, \mathcal{U}^\perp) \to (Y, \mathcal{V}, \mathcal{V}^\perp)$ in FinF, we define

$$F(\alpha)(f)(y) := \sum_{x \in \alpha^{-1}(y)} f(x).$$

Note that this sum is finite as α is a morphism of finiteness space, so $\alpha^{-1}(y) \in \mathcal{U}^\perp$ and $\sup(f) \in \mathcal{U}$. For this functor to be well-defined, we must show that the $F(\alpha) : \mathbb{C}(X, \mathcal{U}, \mathcal{U}^\perp) \to \mathbb{C}(Y, \mathcal{V}, \mathcal{V}^\perp)$ is a continuous linear map, which in addition is a cosemigroup homomorphism that also preserves the dual. Linearity is immediate as

$$\begin{aligned} F(\alpha)(\beta f_{x_1} + f_{x_2})(y) &:= \sum_{x\in\alpha^{-1}(y)} \beta f_{x_1} + f_{x_2}(x) \\ &= \sum_{x\in\alpha^{-1}(y)} \beta f_{x_1}(x) + \sum_{x\in\alpha^{-1}(y)} f_{x_2}(x) \\ &= \beta \sum_{x\in\alpha^{-1}(y)} f_{x_1}(x) + \sum_{x\in\alpha^{-1}(y)} f_{x_2}(x) \\ &= (\beta F(\alpha))(f_{x_1}) + (F(\alpha)(f_{x_2}))(y). \end{aligned}$$

Note that the basis vectors f_x gets sends via $F(\alpha)$ to $f_{\alpha(x)}$.

For continuity, observe that for any $h \in \mathbb{C}(Y, \mathcal{V}, \mathcal{V}^\perp), v' \in \mathcal{V}^\perp$ and $f \in F(\alpha)^{-1}(h + V_{v'})$, we have, $f + V_{\alpha^{-1}(v')} \subseteq F(\alpha)^{-1}(h + V_{v'})$ and so $F(\alpha)$ pulls back any open set in $\mathbb{C}(Y, \mathcal{V}, \mathcal{V}^\perp)$ to an open set in $\mathbb{C}(X, \mathcal{U}, \mathcal{U}^\perp)$ and is hence continuous.

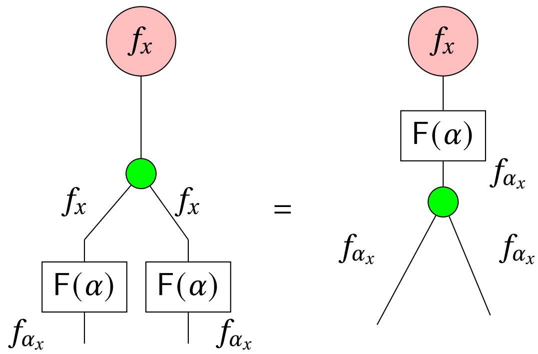
Now we must show that $F(\alpha)$ is a cosemigroup homomorphism. We show that $F(\alpha)$ is a cosemigroup homomorphism on the basis elements, then it follows by linearity that $F(\alpha)$ is a cosemigroup homomorphism on the whole space. For this, note that,

$$\begin{aligned} F(\alpha) \oplus F(\alpha)(\delta_{\mathbb{C}\langle X,\mathcal{U},\mathcal{U}^\perp\rangle}(f_x)) &= F(\alpha) \oplus F(\alpha)(f_x \oplus f_x) \\ &= f_{\alpha(x)} \oplus f_{\alpha(x)}, \end{aligned}$$

and

$$\begin{aligned} \delta_{\mathbb{C}(Y,\mathcal{V},\mathcal{V}^\perp)}(F(\alpha)(f_x)) &= \delta_{\mathbb{C}(Y,\mathcal{V},\mathcal{V}^\perp)}(f_{\alpha(x)}) \\ &= f_{\alpha(x)} \oplus f_{\alpha(x)}. \end{aligned}$$

And thus $F(\alpha) : \mathbb{C}(X, \mathcal{U}, \mathcal{U}^\perp) \to \mathbb{C}(Y, \mathcal{V}, \mathcal{V}^\perp)$ is a cosemigroup homomorphism.



(6.26)

Finally, for well-definedness, we need to show that $F(\alpha)$ preserves duals. We will only show that it preserves the counit, the case for unit, and their symmetrical counterpart follows similarly. Also note that it is equivalent to prove that $F(\alpha)$ preserves duals when the dual is coming from the dualizing functor, as the conjugation functor leaves the underlying map unchanged. For this, we have,

$$\begin{aligned}
\varepsilon_{\mathbb{C}(X,\mathcal{U},\mathcal{U}^{\perp})}((F(\alpha)^* \otimes \mathrm{id})(f_y^* \otimes f_x)) &= \varepsilon_{\mathbb{C}(X,\mathcal{U},\mathcal{U}^{\perp})}(F(\alpha)f_y^* \otimes f_x) \\
&= f_y^*(F(\alpha)f_x) \\
&= f_y^*(f_{\alpha(x)}),
\end{aligned}$$

which is

$$\varepsilon_{\mathbb{C}(X,\mathcal{U},\mathcal{U}^{\perp})}((F(\alpha)^* \otimes id)(f_y^* \otimes f_x)) = \begin{cases} 1, & \text{if } \alpha(x) = y, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$\varepsilon_{\mathbb{C}(Y,\mathcal{V},\mathcal{V}^{\perp})}((\mathrm{id} \otimes F(\alpha))(f_y^* \otimes f_x)) = f_y^* \otimes f_{\alpha(x)},$$

which is

$$\varepsilon_{\mathbb{C}(Y,\mathcal{V},\mathcal{V}^{\perp})}((\mathrm{id} \otimes F(\alpha))(f_y^* \otimes f_x)) = \begin{cases} 1, & \text{if } \alpha(x) = y, \\ 0, & \text{otherwise}. \end{cases}$$

And thus $F(\alpha) : \mathbb{C}(X, \mathcal{U}, \mathcal{U}^{\perp}) \to \mathbb{C}(Y, \mathcal{V}, \mathcal{V}^{\perp})$ is a valid map in the category of $\mathsf{CDLS}(\mathsf{RLef})$. For functoriality, it is immediate that $F(\mathrm{id}_X) = \mathrm{id}_{\mathbb{C}(X,\mathcal{U},\mathcal{U}^{\perp})}$. For composition, note that,

$$F(\alpha\gamma)(f)(z) = \sum_{x \in (\gamma\circ\alpha)^{-1}(z)} f(x).$$

But then $x \in (\gamma \circ \alpha)^{-1}(z)$ if and only if there exists $y \in Y : y \in \gamma^{-1}(z)$, and $x \in \alpha^{-1}(y)$, so we have,

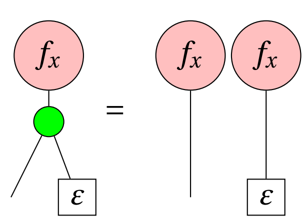
$$\begin{aligned} F(\alpha\gamma)(f)(z) &= \sum_{x \in (\gamma\circ\alpha)^{-1}(z)} f(x) \\ &= \sum_{y \in \gamma^{-1}(z)} \sum_{x \in \alpha^{-1}(y)} f(x), \end{aligned}$$

and

$$\begin{aligned} F(\gamma)(E(\alpha)(y)) &= F(\gamma)\Big(\sum_{x \in \alpha^{-1}(y)} f(x)\Big) \\ &= \sum_{y \in \gamma^{-1}(z)} \sum_{x \in \alpha^{-1}(y)} f(x). \end{aligned}$$

And thus $F(\alpha\gamma) = F(\alpha)F(\gamma)$. So we have a functor $F : \mathsf{FinF} \to \mathsf{CDLS}(\mathsf{RLef})$.

Remark 6.9.2. The reason we are working with a coalgebra structure, without a counit, is that, suppose we have a counit map for the coalgebra structure defined on spaces arising out of Ehrhard's construction by $\delta : \mathbb{C}(X, \mathcal{U}) \to \mathbb{C}(X, \mathcal{U}) \oplus \mathbb{C}(X, \mathcal{U}); f_x \to f_x \oplus f_x$. Then, as every basis element f_x is copyable by construction, the counit map $\varepsilon : \mathbb{C}(X, \mathcal{U}) \to \mathbb{C}$ must satisfy the following diagrammatic equality,



(6.27)

But that means that the counit map $\varepsilon : \mathbb{C}(X, \mathcal{U}) \to \mathbb{C}$ must send every basis vector f_x to $1_{\mathbb{C}}$. But such a map is not continuous as it pulls back the open set $0_{\mathbb{C}}$ to 0_V, which, if open, gives the discrete topology on $\mathbb{C}(X, \mathcal{U})$, as every translation of an open set is open.

Now, by Lemma 5.4.9, spaces arising out of Ehrhard's construction on finiteness spaces are discrete if and only if we started with the finiteness structure $(X, \mathcal{P}_f(X), P(X))$. So we know that spaces arising out of Ehrhard's construction are not, in general, discrete.

Thus, to capture the correspondence in reflexive Lefschetz spaces, it is necessary that we work with the broader category of cosemigroup structures.

We next prove that the functor $F : \mathsf{FinF} \to \mathsf{CDLS}(\mathsf{RLef})$ is fully faithful.

Proposition 6.9.3. The functor $F : \mathsf{FinF} \to \mathsf{CDLS}(\mathsf{RLef})$ is fully faithful.

Proof. It is immediate that F is faithful. For fullness, let $h : (\mathbb{C}(X, \mathcal{U}, \mathcal{U}^\perp), f_x \to f_x \oplus f_x) \to (\mathbb{C}(Y, \mathcal{V}, \mathcal{V}^\perp), f_y \to f_y \oplus f_y)$ be a map in $\mathsf{CLDS}(\mathsf{RLef})$, then we need to produce a $g : \mathbb{C}(X, \mathcal{U}, \mathcal{U}^\perp) \to \mathbb{C}(Y, \mathcal{V}, \mathcal{V}^\perp)$ in FinF such that $F(g) = h$.

Since h is a cosemigroup homomorphism, it takes a copyable element to a copyable element; then identifying x with f_x, we see that the underlying map of h is a set map $|h| : X \to Y$. We then define $g : X \to Y$ as $g(x) := h(f_x)$. It is then clear that $F(g) = h$.

But for this to be well-defined, we need g to be a morphism in the category of finiteness spaces and functions. So we must show that g preserves finitaries and reflects co-finitaries. Now for any $S \subseteq X, S \in \mathcal{U}$, by Proposition 5.3.12 we have that $\sum_{s \in S \subseteq X} f_s$ converges in $\mathbb{C}(X, \mathcal{U}, \mathcal{U}^\perp)$, and since h is a continuous map, so $\sum_{s \in S \subseteq X} h(f_s)$ converges in $\mathbb{C}(Y, \mathcal{V}, \mathcal{V}^\perp)$.

But then the support of this sum is exactly $g(S)$ and again by Proposition 5.3.12, $\sum_{s\in S\subseteq X} h(f_s)$ converges in $\mathbb{C}(Y, \mathcal{V}, \mathcal{V}^{\perp})$ if and only if $g(S) \in \mathcal{V}$. So $g : \mathbb{C}(X, \mathcal{U}, \mathcal{U}^{\perp}) \to \mathbb{C}(Y, \mathcal{V}, \mathcal{V}^{\perp})$ does preserves finitary sets.

To show that it reflects co-finitary sets, by Lemma 5.2.8 , it is enough to prove that for any $y \in Y$, $g^{-1}(y) \in \mathcal{U}^{\perp}$. Now by our Proposition 5.3.10, it is equivalent to prove that there exists $\varphi \in \mathbb{C}(X, \mathcal{U}, \mathcal{U}^{\perp})^*$ such that $|\sup(\varphi)| = g^{-1}(y)$. Now since $h : \mathbb{C}(X, \mathcal{U}, \mathcal{U}^{\perp}) \to \mathbb{C}(Y, \mathcal{V}, \mathcal{V}^{\perp})$ is a continuous map, we can take its dual to get $h^* : \mathbb{C}(Y, \mathcal{V}, \mathcal{V}^{\perp})^* \to \mathbb{C}(X, \mathcal{U}, \mathcal{U}^{\perp})^*$, where $f_y^* \in \mathbb{C}(Y, \mathcal{V}, \mathcal{V}^{\perp})^*$ goes to $hf_y^* \in \mathbb{C}(X, \mathcal{U}, \mathcal{U}^{\perp})^*$. Then observe that support of hf_y^* is exactly $g^{-1}(y)$, hence $g^{-1}(y) \in \mathcal{U}^*$. So $g : \mathbb{C}(X, \mathcal{U}, \mathcal{U}^{\perp}) \to \mathbb{C}(Y, \mathcal{V}, \mathcal{V}^{\perp})$ does reflects co-finitary sets.

And hence g is a morphism in FinF and $F(g) = h$, so F is full, and we are done. □

6.10 From dagger linear semigroups to finiteness spaces

Next, we want to get a functor $G : \mathsf{CDLS}(\mathsf{RLef}) \to \mathsf{FinF}$, but before that, we need a couple of important lemmas:

Proposition 6.10.1. Suppose U and V are reflexive Lefschetz space with bases given by $\{u_i\}_{i\in I}$, $\{v_j\}_{j\in J}$ respectively. Then $\{u_i \otimes v_j\}_{i\in I, j\in J}$ is a basis for the reflexive Lefschetz space $U \otimes V$.

Before we prove this, we need a lemma:

Lemma 6.10.2. Suppose V is a reflexive Lefschetz space. Then for any $v \in V$, $\mathrm{Span}[\{v\}]$ is closed.

Proof. We first prove that $\{v\}$ is closed. For that, we must prove that $V - \{v\}$ is open. Take $w \in V - \{v\}$, then as every reflexive Lefschetz space is Hausdorff (Corollary 5.3.18), so there exists open sets U_w, V_w such that $v \in U_w, w \in V_w$ and $U_w \cap V_w = \emptyset$. So $w \in V_w \subseteq V - \{v\}$. Since the choice of $w \in V - \{v\}$ was arbitrary, we have that $V - \{v\}$ is open, or equivalently, $\{v\}$ is closed. Now observe that the same choices of open sets U_w, V_w also work when we pick $w \in V - \mathrm{Span}[\{v\}]$. □

Now we are ready to give a proof of Proposition 6.10.1. It is clear that $\{u_i \otimes v_j\}_{i\in I, j\in J}$ spans $U \otimes V$ as every vector in $U \otimes V$ is of the form $u \otimes v$ for some $u \in U$ and $v \in V$ and since $\{u_i\}_{i\in I}$, $\{v_j\}_{j\in J}$ are bases for U and V respectively, we have that $\{u_i \otimes v_j\}_{i\in I, j\in J}$ spans $U \otimes V$. We next show linear independence of the set $\{u_i \otimes v_j\}_{i\in I, j\in J}$. Suppose $\sum_{(i,j)\in I\times J} \gamma_{ij} u_i \otimes v_j = 0$. Then we define a map $F : U \otimes V \longrightarrow \mathbb{C}$, which takes (u_{i_0}, v_{j_0}) for some fixed $i_0 \in I, j_0 \in J$ to 1 and everything else to $0_{\mathbb{C}}$. We must show that the bilinear map $F : U \otimes V \longrightarrow \mathbb{C}$ thus defined is continuous. For that, note that the linear map $F(u_{i_0}, -) : V \longrightarrow \mathbb{C}$ is continuous as it pulls back $0_{\mathbb{C}}$ to $V - \mathrm{Span}[\{v_{j_0}\}]$, which is open by Lemma 6.10.2. Similarly $F(-, v_{j_0}) : U \longrightarrow \mathbb{C}$ is continuous. And hence $F : U \otimes V \longrightarrow \mathbb{C}$ is continuous. Now observe that if we apply F on the sum $\sum_{(i,j)\in I\times J} \gamma_{ij} u_i \otimes v_j = 0$, we get $\gamma_{i_0 j_0} = 0_{\mathbb{C}}$. Since the choice of (u_{i_o}, v_{j_0}) was arbitrary, this implies that $\gamma_{ij} = 0$ for all $(i, j) \in I \times J$. And therefore $\{u_i \otimes v_j\}_{i\in I, j\in J}$ is a linearly independent set in $U \otimes V$. And thus $\{u_i \otimes v_j\}_{i\in I, j\in J}$ is a basis for $U \otimes V$.

Note that Lemma 6.10.2 and Proposition 6.10.1 also justify our choice of bases for various constructs on reflexive Lefschetz spaces in Section 4.5. □

We can now prove the following:

Lemma 6.10.3. In $\mathsf{CDLS}(\mathsf{RLef})$, the non-zero copyable elements of any cosemigroup form a linearly independent set.

Proof. Let $\{c_i\}_{i\in I}$ be a linearly independent set of non-zero copyable elements. And let $c_0 \neq c_i$ for any $i \in I$ be a non-zero copyable element such that

$$c_0 = \sum_{i\in I} \alpha_i c_i \text{ for some } \alpha_i \in \mathbb{C}.$$

This makes $\{c_0, c_i\}_{i\in I}$ a linearly dependent set (See Proposition 4.4.8). Then, on applying the comultiplication map, we get:

$$\begin{aligned}
\delta(c_0) &= \sum_{i\in I} \alpha_i \delta(c_i) \\
&= \sum_{i\in I} \alpha_i c_i \otimes c_i.
\end{aligned}$$

Since c_0 is itself copyable, we have,

$$\begin{aligned}
\delta(c_0) &= \sum_{i\in I} \alpha_i c_i \otimes \sum_{i\in I} \alpha_i c_i \\
&= \sum_{(i,i)\in I\times I} \alpha_i^2 c_i \otimes c_i + \sum_{(i,j)\in I\times I, i\neq j} \alpha_i \alpha_j c_i \otimes c_j.
\end{aligned}$$

Since addition and scalar multiplication are continuous operations in RLef, this implies that

$$\sum_{(i,i)\in I\times I} (\alpha_i^2 - \alpha_i) c_i \otimes c_i + \sum_{(i,j)\in I\times I, i\neq j} \alpha_i \alpha_j c_i \otimes c_j = 0.$$

Since $\{c_i\}_{i\in I}$ is a linearly independent set and the tensor product of linearly independent sets is linearly independent (in general, the tensor product of a basis is a basis by Proposition 6.10.1), so we have,

$$\begin{aligned}
\alpha_i^2 &= \alpha_i, \text{ for all } i \in I, \text{and} \\
\alpha_i \alpha_j &= 0, \text{ for all } i, j \in I.
\end{aligned}$$

So we have $\alpha_i = 0$ or 1, for all $i \in I$. And the second equation implies that if $\alpha_{i_0} = 1$ for some fixed $i_0 \in I$ then $\alpha_j = 0$ for all $j \neq i_0$. And so $c_0 = c_{i_0}$, but this contradicts the assumption that $c_0 \neq c_i$ for any $i \in I$. So we must have that $\alpha_i = 0$ for all $i \in I$, but then $c_0 = 0$, which is again a contradiction. □

Remark 6.10.4. Note that by the previous lemma (Lemma 6.10.3), we have that $F : \mathsf{FinF} \to \mathsf{CDLS}(\mathsf{RLef})$ faithfully encodes the basis elements as the *only* copyable elements, since copyable elements are linearly independent, so any sum over basis elements consisting of at least two different basis elements is not copyable.

Remark 6.10.5. Lemma 6.10.3 seems not to have much to do with (reflexive) Lefschetz spaces and has been proved in other settings, such as Hilbert spaces [2] and in C^* algebras [23], where they are called primitive elements. Though the general context in which this holds is unclear, the relevant structure seems to be of an associative co-algebra in any additively enriched category.

We now want to define a functor $G : \mathsf{CDLS}(\mathsf{RLef}) \to \mathsf{FinF}$, for this we first define a functor $U : \mathsf{CDLS}(\mathsf{RLef}) \to \mathsf{RLef}_{\mathsf{B}}$.

On objects, we take the copyable elements of the cosemigroup, and take the sub-Lefschetz space generated by that, with the basis of these copyable elements, that is, $U(V, v_i \to v_i \oplus v_i) := (\hat{V}, \{v_i\}_{i \in I})$. Now note that in general, we do not know if the copyables form a basis of the underlying space, but they are independent by Lemma 6.10.3, so we can take the sub-Lefschetz space generated by that, with the subspace topology.

For this to be well-defined, we must show that $\hat{V}$ is also a reflexive Lefschetz space. That it is a Lefschetz space is immediate, as the local basis around $0_{\hat{V}}$ is given by $W_O \cap \hat{V}$, where W_0 is an open linear subspace around 0_V. Thus, $W_O \cap \hat{V}$ is a linear subspace of $\hat{V}$, which is open by definition of subspace topology. To show that it is reflexive, we know that the natural evaluation map $\mathrm{ev}_{\hat{V}} : \hat{V} \to \hat{V}^{**}$ is always a continuous bijection by [4]. So we only need to show that it is open to prove that $\hat{V}$ is reflexive. For that, let W be any open set in $\hat{V}$, then W is of the form $O \cap \hat{V}$ for some open O in V. Then,

$$\begin{aligned} \mathrm{ev}_{\hat{V}}(O \cap \hat{V}) &= \mathrm{ev}_{\hat{V}}(O) \cap \mathrm{ev}_{\hat{V}}(\hat{V}), \text{ as evaluation is injective} \\ &= \mathrm{ev}_{V}(O) \cap V^{**} \cap \mathrm{ev}_{\hat{V}}(\hat{V}) \\ &= \mathrm{ev}_{V}(O) \cap \hat{V}^{**} \cap \hat{V}^{**} \\ &= \mathrm{ev}_{V}(O) \cap \hat{V}^{**}. \end{aligned}$$

Now $\mathrm{ev}_V(O)$ is open as V is reflexive, so $\mathrm{ev}_V(O) \cap \hat{V}^{**}$ is open in $\hat{V}^{**}$ under the subspace topology, so $\mathrm{ev}_{\hat{V}} : \hat{V} \to \hat{V}^{**}$ is open and hence $\hat{V}$ is reflexive.

Hence, U is well-defined on objects. Now for maps, let $f : (V, v_i \to v_i \oplus v_i) \to (W, w_j \to w_j \oplus v_i)$ be a morphism in $\mathsf{CDLS}(\mathsf{RLef})$ we then define $U(f) := \hat{f}$, where $\hat{f}(v_i) = f(v_i)$, and then extending it linearly. Note that as f is a cosemigroup homomorphism, it takes a copyable element to a copyable element (by Lemma 6.4.4), and hence $\hat{f}$ is well-defined. But we also must prove that $\hat{f} : (\hat{V}, \{v_i\}_{i \in I}) \to (\hat{W}, \{w_j\}_{j \in J})$ is also a continuous linear map. Linearity is just by construction. For continuity, let $\hat{O}$ be any open set in $\hat{W}$, then it is of the form $O \cap \hat{W}$, where O is an open set in W.

Then,

$$\begin{aligned}
\hat{f}^{-1}(\hat{O}) &= \hat{f}^{-1}(O \cap \hat{W}) \\
&= \hat{f}^{-1}(O) \cap \hat{f}^{-1}(\hat{W}) \\
&= \hat{f}^{-1}(O) \cap \hat{V} \\
&= f^{-1}(O) \cap \hat{V} \cap \hat{V} \\
&= f^{-1}(O) \cap \hat{V}.
\end{aligned}$$

Now $f^{-1}(O)$ is open in V as f is continuous, so $f^{-1}(O) \cap \hat{V}$ is open in $\hat{V}$ under the subspace topology, and hence $\hat{f} : (\hat{V}, \{v_i\}_{i\in I}) \to (\hat{W}, \{w_j\}_{j\in J})$ is continuous, so $U : \mathsf{CDLS}(\mathsf{RLef}) \to \mathsf{RLef}_\mathsf{B}$ is well-defined. Functorality of U is immediate.

We finally compose U with the support functor, to get a functor from $G := U\,\mathrm{sup} : \mathsf{CDLS}(\mathsf{RLef}) \to \mathsf{FinF}$. Note that the support functor when restricted to basis-to-basis maps in RLef_B gives functions in FinF. So we have a pair of functors $F : \mathsf{FinF} \to \mathsf{CDLS}(\mathsf{RLef})$ and $G : \mathsf{CDLS}(\mathsf{RLef}) \to \mathsf{FinF}$. We will now proceed to show that this pair of functors forms part of an adjunction.

6.11 Adjunction between bases and coalgebras

We will give the adjunction by proving the triangle identities. For this, we first give the unit and the counit map. Now note that by Proposition 5.4.7, we know that Ehrhard's construction and our construction are inverses to each other (up to identification of x with f_x), so the unit map is simply the identity map in FinF.

$$\eta_{(X,\mathcal{U},\mathcal{U}^\perp)} = \mathrm{id}_{(X,\mathcal{U},\mathcal{U}^\perp)}\,.$$

The identity map certainly preserves finitary sets and reflects co-finitary sets, so it is a well-defined morphism in FinF. It is also immediate that it is a natural transformation, and in fact a natural isomorphism.

Now for any $(V, v_i \to v_i \oplus v_i)$, the counit map $\varepsilon : (\hat{V}, v_i \to v_i \oplus v_i) \to (V, v_i \to v_i \oplus v_i)$ map is given by $\varepsilon_{(V,v_i \to v_i \oplus v_i)}(v_i) := v_i$.

We want to show the adjunction $(\eta, \varepsilon) : F \dashv G : \mathsf{FinF} \to \mathsf{CDLS}(\mathsf{RLef})$. For that, first note that the counit map is linear by construction. For continuity, note that for any open set O in V, $\varepsilon^{-1}_{(V,v_i \to v_i \oplus v_i)}(O) = O \cap \hat{V}$, so it is open in the subspace topology of $\hat{V}$ and hence the counit map is continuous. We shall also show that the counit map is a cosemigroup homomorphism. For that, note that,

$$\begin{aligned} \varepsilon_{(V,v_i \to v_i \oplus v_i)} \oplus \varepsilon_{(V,v_i \to v_i \oplus v_i)}(\delta_{\hat{V}}(v_i)) &= \varepsilon_{(V,v_i \to v_i \oplus v_i)} \oplus \varepsilon_{(V,v_i \to v_i \oplus v_i)}(v_i \oplus v_i) \\ &= v_i \oplus v_i, \end{aligned}$$

and

$$\begin{aligned} \delta_V(\varepsilon_{(\hat{V},v_i \to v_i \oplus v_i)}(v_i)) &= \delta_V(v_i) \\ &= v_i \oplus v_i. \end{aligned}$$

That the counit is a semigroup homomorphism on the whole space follows by linearity.

Finally, for well-definedness, all that is left to show is that it preserves duals. Again, we will only show that the counit preserves the evaluation map; the case for the co-evaluation map (and its symmetrical counterparts) follows similarly. Now by construction $\{v_i\}_{i \in I}$ forms a basis for $\hat{V}$, and for V, though $\{v_i\}_{i \in I}$ are linearly independent, they might not form a basis, but we can always extend (possibly non-uniquely) this linearly independent set to form a basis for V, we denote the elements of this basis, which are not copyable by v'_j. Now,

$$\begin{aligned} \mathrm{ev}_{\hat{V}}((\varepsilon^*_{(V,v_i\to v_i\oplus v_i)}\otimes \mathrm{id})(v^*_j\otimes v_i)) &= \varepsilon v^*_j \otimes v_i \\ &= v^*_j(\varepsilon_{(V,v_i\to v_i\oplus v_i)}(v_i)), \end{aligned}$$

which is

$$\mathrm{ev}_{\hat{V}}((\varepsilon^*_{(V,v_i\to v_i\oplus v_i)}\otimes \mathrm{id})(v^*_j\otimes v_i)) = \begin{cases} 1, & \text{if } i=j, \\ 0, & \text{otherwise }, \end{cases}$$

and

$$\mathrm{ev}_V((\mathrm{id}\otimes\varepsilon_{(V,v_i\to v_i\oplus v_i)})(v^*_j\otimes v_i)) = \mathrm{ev}_V(v^*_j\otimes v_i),$$

which is

$$\mathrm{ev}_V((\mathrm{id}\otimes\varepsilon_{(V,v_i\to v_i\oplus v_i)})(v^*_j\otimes v_i)) = \begin{cases} 1, & \text{if } i=j, \\ 0, & \text{otherwise.} \end{cases}$$

And for any of the basis elements v'_j of V which are not copyable elements, $v'^*_j \otimes v_i$ evaluates to 0 by both the evaluation maps. So the counit map preserves duals.

Hence the counit $\varepsilon : (\hat{V}, v_i \to v_i \oplus v_i) \to (V, v_i \to v_i \oplus v_i)$ is a well-defined morphism in $\mathsf{CDLS}(\mathsf{RLef})$.

The counit map is clearly a section, and that it is a natural transformation is immediate. So we have the required natural transformations, now we must show the triangle identities of the adjunction. To fix notation, let $X = (X, \mathcal{U}, \mathcal{U}^{\perp})$ and $Y = (V, v_i \to v_i \oplus v_i)$. Then we must show

$$\eta_{G(Y)}G(\varepsilon_Y) = \mathrm{id}_{G(Y)} \ \text{ and } \ F(\eta_X)\varepsilon_{F(X)} = \mathrm{id}_{F(X)}\,.$$

But by Proposition 5.4.7, as Ehrhard's construction and our construction are inverses to each other, the triangle identities simply reduce to the composition of identities in the respective category, that is,

$$\begin{aligned} \eta_{G(Y)}G(\varepsilon_Y) &= \mathrm{id}_Z\,\mathrm{id}_Z \\ &= \mathrm{id}_Z, \end{aligned}$$

where Z is the set of copyables of $(V, v_i \to v_i \oplus v_i)$, and we are using id_Z as a shorthand for the identity on the finiteness space whose web is Z. Similarly, the second triangle identity reduces to,

$$\begin{aligned} F(\eta_X)\varepsilon_{F(X)} &= \mathrm{id}_{(\mathbb{C}(X,\mathcal{U},\mathcal{U}^\perp), f_x \to f_x \oplus f_x)}\,\mathrm{id}_{(\mathbb{C}(X,\mathcal{U},\mathcal{U}^\perp), f_x \to f_x \oplus f_x)} \\ &= \mathrm{id}_{(\mathbb{C}(X,\mathcal{U},\mathcal{U}^\perp), f_x \to f_x \oplus f_x)}\,. \end{aligned}$$

Thus we have $(\eta, \varepsilon) : F \dashv G : \mathsf{FinF} \to \mathsf{CDLS}(\mathsf{RLef})$. The situation where the left adjoint functor is fully faithful is sometimes referred to as coreflection. To conclude, we have,

Proposition 6.11.1. $(\eta, \varepsilon) : F \dashv G : \mathsf{FinF} \to \mathsf{CDLS}(\mathsf{RLef})$ is a coreflection.

Remark 6.11.2. Observe that the unit map $\eta : (X, \mathcal{U}, \mathcal{U}^\perp) \to G(F(X, \mathcal{U}, \mathcal{U}^\perp))$ is an isomorphism, or equivalently, the left adjoint functor $F : \mathsf{FinF} \to \mathsf{CDLS}(\mathsf{RLef})$ is fully faithful (see Proposition 6.9.3. But the counit map $\varepsilon : (\hat{V}, v_i \to v_i \oplus v_i) \to (V, v_i \to v_i \oplus v_i)$ is just a section (or equivalently, the right adjoint functor G is full).

But G is clearly not faithful, as recall that we do not know if the copyable elements of $(V, v_i \to v_i \oplus v_i)$ forms a basis for the underlying space V, so for two maps $f \neq g : (V, v_i \to v_i \oplus v_i) \to (W, w_j \to w_j \oplus w_j)$ such that they are equal on the copyable elements, $f(v_i) = g(v_i)$, for all $i \in I$, but say for some

v_i', which is part of the basis, but not a copyable element, $f(v_i') \neq g(v_i')$, then f and g induce the same map in FinF between the sets of their copyable elements. So G is not a faithful functor.

But if all the linear cosemigroups in $\mathsf{CDLS}(\mathsf{RLef})$ were well-generated, in the sense that their copyable elements would have formed the basis of the underlying space, then the right adjoint functor G would also be faithful (and hence an isomorphism), as the map between copyable elements would completely determine the map between the underlying space. This would mean that the adjunction $(\eta, \varepsilon) : F \dashv G : \mathsf{FinF} \to \mathsf{CDLS}(\mathsf{RLef})$ is actually an equivalence.

This begs the questions that does there exists a (co) semigroup structure on any reflexive Lefschetz space whose copyables do not span the underlying space. To this end, we provide a very simple counterexample.

Example 6.11.3. Let V be the Lefschetz space constructed on a finiteness space $(X = \{a_0, a_1\}, \mathcal{U} = \mathcal{P}(X))$. For concreteness we can take $V = \mathbb{R}^2$ and denote its basis $(0,1)$ by a_0 and $(1,0)$ by a_1. As $\mathbb{R}^2$ is finite-dimensional, it is discrete (Corollary 5.4.10) and reflexive (Lemma 5.4.11). We define a cosemigroup structure on it by defining $\delta : V \to V \oplus V \cong V \otimes V$ by sending $\delta(a_0) = a_1 \oplus a_1 = \delta(a_1)$ and extending it linearly. δ is continuous as spaces here have a discrete topology. The only thing to check is that δ indeed defines a cosemigroup structure on V. For this, note that,

$$\begin{aligned} \mathrm{id} \oplus \delta((\delta(a_0))) &= \mathrm{id} \oplus \delta(a_1 \oplus a_1) \\ &= a_1 \oplus (a_1 \oplus a_1), \end{aligned}$$

and

$$\begin{aligned}\delta \oplus \mathrm{id}(\delta(a_0)) &= \delta \oplus \mathrm{id}(a_1 \oplus a_1) \\ &= (a_1 \oplus a_1) \oplus a_1.\end{aligned}$$

And similarly $\mathrm{id} \oplus \delta((\delta(a_1))) = \delta \oplus \mathrm{id}(\delta(a_1))$. So $\delta : V \to V \oplus V$ is indeed a cosemigroup structure on V. Now it is clear that a_1 is copyable, as $\delta(a_1) = a_1 \oplus a_1$. It is also clear that a_0 is not copyable as $\delta(a_0) = a_1 \oplus a_1 \neq a_0 \oplus a_0$. We then claim that any non-trivial linear combination of a_0 and a_1 is not copyable. For that let $v \in V, v = \alpha \bullet a_0 + \beta \bullet a_1$, where $\alpha, \beta \neq 0$. Then we have,

$$\begin{aligned}\delta(v) &= \delta(\alpha \bullet a_0 + \beta \bullet a_1) \\ &= \alpha \bullet \delta(a_0) + \beta \bullet \delta(a_1) \\ &= \alpha \bullet (a_1 \oplus a_1) + \beta \bullet (a_1 \oplus a_1) \\ &= (\alpha + \beta) \bullet (a_1 \oplus a_1).\end{aligned}$$

But then, $v \oplus v = \alpha^2 \bullet (a_0 \oplus a_0) + \beta^2 \bullet (a_1 \oplus a_1) + \alpha\beta \bullet (a_0 \oplus a_1) + \beta\alpha(a_1 \oplus a_0)$. So we have that the only copyable element of (V, δ) is a_0. But then by assumption V was two-dimensional (with basis as a_0 and a_1), so the copyables of (V, δ) do not generate the whole space.

To see how this breaks faithfulness of $G : \mathsf{CDLS}(\mathsf{RLef}) \to \mathsf{FinF}$. Let W be again a finite-dimensional Lefschetz space with basis b_0, b_1. We similarly define a cosemigroup structure on W by $\delta_W(b_0) = b_1 \oplus b_1 = \delta(b_1)$. Then we define two maps $f, g : V \to W$ by taking $f(a_1) = b_1 = g(a_1)$ and $f(a_0) = b_1$ and $g(a_0) = b_0$. It is easy to check that both f and g are cosemigroup homomorphisms, as

$$\begin{aligned}\delta_W(f(a_i)) &= f \oplus f(\delta_V(a_i)) \\ &= a_i \oplus a_i,\end{aligned}$$

for $i = 1, 2$. And similarly,

$$\begin{aligned}\delta_W(g(a_i)) &= g \oplus g(\delta_V(a_i)) \\ &= a_i \oplus a_i,\end{aligned}$$

for $i = 1, 2$. So both $f, g : V \to W$ are cosemigroup homorphisms. But they induce the same map between the space generated by their copyables and hence the same map between the finiteness spaces constructed from them. And hence $G : \mathsf{CDLS}(\mathsf{RLef}) \to \mathsf{FinF}$ is not faithful, and hence the coreflection $(\eta, \varepsilon) : F \dashv G : \mathsf{FinF} \to \mathsf{CDLS}(\mathsf{RLef})$ as described in Proposition 6.11.1 is not an equivalence.

We can now give a class of examples where copyables do not span the underlying space. For any finite-dimensional space $V, \{v_i\}_{i=1}^{n}$, we define the map $\delta(v_i) = v_j$ for some fixed $j \in J$, for all $i \in I$. Furthermore, we can always tensor a finite-dimensional space with an infinite-dimensional space W, where copyables do span the space and define the cosemigroup structure component-wise composed with the symmetry map

$$V \oplus W \xrightarrow{\delta_V \oplus \delta_W} V \oplus V \oplus W \oplus W \xrightarrow{\mathrm{id} \oplus c_{\oplus} \oplus \mathrm{id}} (V \oplus W) \oplus (V \oplus W)$$

to get examples of infinite-dimensional Lefschetz spaces where copyables do not span the space.

Chapter 7

Conclusion and outlook

In this thesis, we examined a possible extension of the correspondence between orthonormal bases of finite-dimensional Hilbert spaces and copyable elements of Frobenius algebras [16] in a model of a mixed unitary category [40]- namely, the reflexive Lefschetz spaces [4].

We began by showing that a full subcategory of Lefschetz spaces, namely the reflexive ones, forms a mixed unitary category, whose unitary core consists of finite-dimensional spaces, and where the dagger structure is given by the composition of the dualising functor with the conjugation functor.

We showed how the web of a finiteness space can be seen as a basis for a Lefschetz space constructed on the finiteness space by Ehrhard's construction. And the convergent nets over this basis are exactly those that have finitary support. Conversely, the finitary sets are encoded in the basis of a Lefschetz space as the support of convergent nets of finite sums over the basis. We use this encoding to provide a functor from coalgebras in reflexive Lefschetz spaces to finiteness spaces (with functions).

The basis vectors of reflexive Lefschetz spaces arising out of Ehrhard's construction are copyable, and give a coalgebra structure on the space. However, if one has a counit for this coalgebra, then the underlying space must be discrete. But we know that the reflexive Lefschetz spaces arising out of Ehrhard's construction are not, in general, discrete. Thus, to obtain a correspondence between bases and copyable elements of coalgebras necessitates dropping the counit and working with the broader category of dagger linear semigroups.

We finally construct a coreflection between the category of finiteness spaces with functions and the category of cosemigroups with homomorphisms – in reflexive Lefschetz spaces. However, this coreflection fails to be an equivalence, as we exhibit a class of examples of cosemigroups whose copyables do not span the underlying space.

We leave it for future work to completely characterise the fixed points of this coreflection. To this end, we conjecture that the adjunction is an equivalence precisely for those cosemigroup structures whose comultiplication map is a section. The comultiplication map being a section is clearly a necessary condition to obtain an equivalence. However, the more interesting case of it being a sufficient condition remains open. Taking motivation from previous works [2, 16], where structure theorems from the theory of C^*- algebras and H^*- algebras plays a crucial role in proving that the copyables generate the underlying space, it will be fruitful to investigate what sort of algebra structure we can give to the class of endomorphisms for a given object in a $*$-autonomous category, or more generally, in a linearly distributive category.

The correspondence between bases and copyable elements of algebras in finite-dimensional Hilbert spaces shows that Frobenius algebras are the right categorical notion to capture observables in finite-dimensional quantum mechanics. However, when one moves to infinite dimensions, we see that, though one can freely copy bases of Lefschetz spaces to get a cosemigroup structure, not every cosemigroup structure on reflexive Lefschetz spaces arises in this way. Thus, one needs to be cautious in concluding that Frobenius algebras (or their linear extension in $*$-autonomous categories) are, by themselves, the right categorical notion to capture bases, and, thus, observables in arbitrary-dimensional quantum mechanics.

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