Lecture #15: Many-One Reductions

Lecture Presentation

1. Recall that an alphabet
$$\Sigma_D = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$
was introduced. Following the definitions of the ***unpadded decimal expansion*** of a positive integer n, the ***unpadded decimal representation*** of a natural number n as a string in $\Sigma_D^\star$, the set Unpad $\subseteq \Sigma_D^\star$ of unpadded decimal representations of natural numbers, a (possibly) ***padded decimal representation*** of a natural number n, and the set Pad $\subseteq \Sigma_D^\star$ of (possibly) padded decimal representations of natural numbers, a set $S \subseteq \mathbb{N}$ was introduced and used to define ***two*** languages, $\mathcal{U}_S \subseteq$ Unpad $\subseteq \Sigma_D^\star$ and $\mathcal{P}_S \subseteq$ Pad $\subseteq \Sigma_D^\star$:

   - $\mathcal{U}_S \subseteq$ Unpad is the set of all ***unpadded decimal representations*** of numbers $n \in S$.
   - $\mathcal{P}_S \subseteq$ Pad is the set of all ***padded decimal representations*** of numbers $n \in S$.

   Our goal is to prove that $\mathcal{U}_S \preceq_{\mathsf{M}} \mathcal{P}_S$ for *every* subset $S \subseteq \mathbb{N}$.

   What We Need To Provide — and the Properties It Must Satisfy:

Let $\omega \in \Sigma_D^\star$. ***What Can We Set $f(\omega)$ To Be, When $\omega \notin$ Unpad? Why?***

What Can We Set $f(\omega)$ To Be, When $\omega \in$ Unpad Why?

The Function f:

A First Claim about f and Its Proof:

A Second Claim About f and Its Proof:

A Third Claim About f and Its Proof:

2. The lecture notes introduced a language $\mathsf{HALT_{TM}} \subseteq$ TM+I and a proof that

$$\mathsf{HALT_{TM}} \preceq_{\mathsf{M}} \mathsf{A_{TM}}.$$

Recall that $\mathsf{HALT_{TM}}$ was the set of encodings of Turing machines

$$M = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}})$$

and input string $\omega \in \Sigma^\star$ such that M's execution on the input string ω *halts*, while $\mathsf{A_{TM}}$ is another subset of TM+I, namely the set of encodings of Turing machines M (as above) and input strings $\omega \in \Sigma^\star$ such that M *accepts* ω.

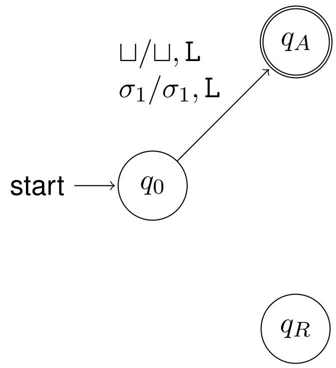
Suppose we wish to prove that $\mathsf{A_{TM}} \preceq_{\mathsf{M}} \mathsf{HALT_{TM}}$. ***What do we need to provide — and what properties must it satisfy?***

Let $f_1 : \Sigma^{\star}_{\mathsf{TM}} \rightarrow \Sigma^{\star}_{\mathsf{TM}}$ such that $f_1(\mu) = \mu$ for every string $\mu \in \Sigma^{\star}_{\mathsf{TM}}$.

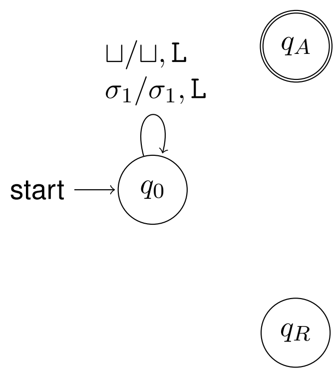
Is f_1 a Many-One Reduction from $\mathsf{A_{TM}}$ to $\mathsf{HALT_{TM}}$?

Why — or Why Not?

Next let us consider a pair of Turing machines with input alphabet $\Sigma = \{\sigma_1\}$ and tape alphabet $\Gamma = \{\sigma_1, \sqcup\}$. The first of these Turing machines, M_Y, is as follows:



The second of these Turing machines, M_N, is as follows:



In both of these pictures the accept state is shown as "q_A" instead of q_{accept}, and the reject state is shown as "q_R" instead of "q_{reject}", to simplify the picture.

Now note that if $x_{\text{Yes}} \in \Sigma_{\text{TM}}^{\star}$ is the encoding of the above Turing machine M_Y, and the empty string λ, then $x_{\text{Yes}} \in \text{HALT}_{\text{TM}}$. On the other hand, if $x_{\text{No}} \in \Sigma_{\text{TM}}^{\star}$ is the encoding of the Turing machine M_N, and the empty string λ, then $x_{\text{no}} \notin \text{HALT}_{\text{TM}}$.

Let $f_2 : \Sigma_{\mathsf{TM}}^{\star} \to \Sigma_{\mathsf{TM}}^{\star}$ such that, for every string $\mu \in \Sigma_{\mathsf{TM}}^{\star}$,

$$f_2(\mu) = \begin{cases} x_{\mathsf{Yes}} & \text{if } \mu \in \mathsf{A}_{\mathsf{TM}}, \\ x_{\mathsf{No}} & \text{if } \mu \notin \mathsf{A}_{\mathsf{TM}}. \end{cases}$$

Is f_2 a Many-One Reduction from A_{TM} to $\mathsf{HALT}_{\mathsf{TM}}$?

Why — or Why Not?