Lecture #15: Many-One Reductions
What Will Happen During the Lecture

Remember... You Had Homework!

Students were asked to work through the following set of lecture notes before this lecture.

- Lecture Notes — "Many-One Reductions".

As always, you may attend the lecture presentation if you have not worked through this material ahead of time — but it will not be repeated for you, and you might get a little bit lost, during the presentation, if you haven't worked through this.

Problems To Be Solved

1. Consider the alphabet
$$\Sigma_D = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}.$$
Let $\sigma_0 =$ 0, $\sigma_1 =$ 1, $\sigma_2 =$ 2, $\sigma_3 =$ 3, and so on (so that, for each integer i such that $0 \leq i \leq 9$, σ_i is the symbol in Σ_D that we generally use to represent i). Recall that, for every ***positive*** integer n, there is an ***unpadded decimal expansion*** of n, consisting of an integer $k \geq 0$ and digits $d_0, d_1, \dots, d_k \in \{d \in \mathbb{N} \mid 0 \leq d \leq 9\}$ such that $d_k \geq 1$ and
$$n = \sum_{\ell=0}^{k} d_\ell \times 10^\ell.$$
The ***unpadded decimal representation*** of n is the string $\sigma_{d_k}\sigma_{d_{k-1}} \dots \sigma_{d_1}\sigma_{d_0}$, a string in $\Sigma_D^\star$ with length $k+1$.
For example, if $n = 312$ then the ***unpadded decimal expansion*** of n consists of the integer $k = 2$ and the digits $d_0 = 2$, $d_1 = 1$ and $d_2 = 3$ — since $d_k = d_2 \geq 1$ — and
$$312 = 2 \times 10^0 + 1 \times 10^1 + 3 \times 10^2$$

— and the ***unpadded decimal representation*** of n is the string 312 — a string in $\Sigma_D^\star$ with length $k+1=3$.[1]

In order to have "unpadded decimal representations" of all natural numbers — including zero — we generally define the ***unpadded decimal representation*** of zero to be the string 0 — a string in $\Sigma_D^\star$ with length one.

Let Unpad $\subseteq \Sigma_D^\star$ be the set of unpadded decimal representations of natural numbers. This is certainly a decidable language — indeed, it is even a ***regular*** language — because this is just the union of the set of all non-empty strings in $\Sigma_D^\star$ that do not begin with 0, and the set $\{0\}$.

A (possibly) ***padded decimal representation*** of a natural number n is any string in $\Sigma_D^\star$ that is the concatenation of a string of zero more 0's and the unpadded decimal representation of n. Thus the ***padded decimal representations*** of zero include the strings 0^i for every positive integer i, and the ***padded decimal representations*** of the natural number 312 are the strings of the form 0^j312 for every non-negative integer j.

Let Pad $\subseteq \Sigma_D^\star$ be the set of (possibly) padded decimal representations of natural numbers. This is also certainly a decidable language — indeed, it is even a ***regular*** language — because this is just $\Sigma_D^+ = \Sigma_D^\star \setminus \{\lambda\}$, that is, the set of all ***non-empty*** strings in $\Sigma_D^\star$.

Now let $S \subseteq \mathbb{N}$. This can be used to define ***two*** languages, that are each subsets of $\Sigma_D^\star$:

- $\mathcal{U}_S \subseteq$ Unpad is the set of all ***unpadded decimal representations*** of numbers $n \in S$.
- $\mathcal{P}_S \subseteq$ Pad is the set of all ***padded decimal representations*** of numbers $n \in S$.

For example, if $S_1 = \emptyset \subseteq \mathbb{N}$ then $\mathcal{U}_{S_1} = \mathcal{P}_{S_1} = \emptyset \subseteq \Sigma_D^\star$. On the other hand, if $S_2 = \{312\} \subseteq \mathbb{N}$ then $\mathcal{U}_{S_2} = \{312\} \subseteq$ Unpad and $\mathcal{P}_{S_2} = \{0^j312 \mid j \in \mathbb{N}\} \subseteq$ Pad.

We will prove that $\mathcal{U}_S \preceq_{\mathsf{M}} \mathcal{P}_S$ (for *every* subset S of $\mathbb{N}$).

2. The lecture notes introduced a language $\mathsf{HALT}_{\mathsf{TM}} \subseteq$ TM+I and a proof that

$$\mathsf{HALT}_{\mathsf{TM}} \preceq_{\mathsf{M}} \mathsf{A}_{\mathsf{TM}}.$$

We will consider at least one attempt to prove that A_{TM} is many-one reducible to $\mathsf{HALT}_{\mathsf{TM}}$ as well — in order to discuss at least one ***mistake that many students make***, when they try to give many-one reductions, and which you should watch for and avoid.

[1] Describing this can be tricky, because we are so used to working with the "unpadded decimal representations" of positive integers that it might not be easy to see the difference between the unpadded decimal representation of the positive integer n — which is a non-empty string in $\Sigma_D^\star$ — and the integer n, itself.