Lecture #16: Proofs of Undecidability — Examples I

Lecture Presentation

Languages of Interest

- TM+I: The language of encodings of Turing machines

$$M = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}})$$

  and input strings $\omega \in \Sigma^\star$. This language is ***decidable***.

- A_{TM}: A subset of TM+I, including encodings of Turing machines M (as above) and strings $\omega \in \Sigma^\star$ such that M ***accepts*** ω. This language is ***recognizable***, since it is the language of the "universal Turing machine" introduced in Lecture #12 — but it is also ***undecidable***, as shown in Lecture #13.

- $\mathsf{HALT}_{\mathsf{TM}}$: Another subset of TM+I, including encodings of Turing machines M (as above) and strings $\omega \in \Sigma^\star$ such that M ***halts*** when executed on input ω.

Goal for Today

The goal for this presentation is to establish that

$$\mathsf{A}_{\mathsf{TM}} \preceq_{\mathsf{M}} \mathsf{HALT}_{\mathsf{TM}}$$

— introducing a process that can be used to establish many-one reductions, like these, along the way.

Thus ***participation in this lecture presentation will, very possibly, be useful preparation for Assignment #2***.

What Do We Want To Produce?

Useful Aspects of This Problem — and How to Use This

Using the Decidability of the Language TM+I

Thinking about Turing Machines and Input Strings

Suppose that the input string, $\mu \in \Sigma_{\mathsf{TM}}^{\star}$ encodes a Turing machine

$$M = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}})$$

and an input string

$$\omega \in \Sigma^{\star}.$$

- Either M ***accepts*** ω, M ***rejects*** ω, or M ***loops*** on ω.
- Since the membership of the string $f(\mu)$ in the language $\mathsf{HALT}_{\mathsf{TM}} \subseteq \mathsf{TM+I}$ is being considered, we can consider $f(\mu)$ to be an encoding of a Turing machine

  $$\widehat{M} = (\widehat{Q}, \widehat{\Sigma}, \widehat{\Gamma}, \widehat{\delta}, \widehat{q}_0, \widehat{q}_{\text{accept}}, \widehat{q}_{\text{reject}})$$

  and a string

  $$\widehat{\omega} \in \widehat{\Sigma}^{\star}$$

  — so that $f(\mu) \in \mathsf{TM+I}$ when $\mu \in \mathsf{TM+I}$.
- Since we will need to show that the function f, $\widehat{M}$ and $\widehat{\omega}$ should easy to describe from M and $\widehat{\omega}$.

Case: M accepts ω. What property should $\widehat{M}$ and $\widehat{\omega}$ satisfy? Why?

Case: M rejects ω. What property should $\widehat{M}$ and $\widehat{\omega}$ satisfy? Why?

Case: M loops on ω. What property should $\widehat{M}$ and $\widehat{\omega}$ satisfy? Why?

What Should $\widehat{M}$ and $\widehat{\omega}$ Be?

Describing This in More Detail

Once again,

$$M = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}})$$

and

$$\widehat{M} = (\widehat{Q}, \widehat{\Sigma}, \widehat{\Gamma}, \widehat{\delta}, \widehat{q}_0, \widehat{q}_{\text{accept}}, \widehat{q}_{\text{reject}}).$$

How is $\widehat{Q}$ Related To Q?

How is $\widehat{\Sigma}$ Related To Σ?

How is $\widehat{\Gamma}$ Related To Γ?

How is $\widehat{\delta}$ Related To δ?

How are $\widehat{q}_0$ $\widehat{q}_{\text{accept}}$ and $\widehat{q}_{\text{reject}}$ Related To q_0, q_{accept} and q_{reject}?

How is $f(\mu)$ Related To μ When $\mu \in$ TM+I?

Since $\mu \in$ TM+I,

$$\mu = (\mu_1, \mu_2) \tag{1}$$

where μ_1 is the encoding of the Turing machine μ_1 and μ_2 is the encoding of the string ω. If $f(\mu)$ is as described above then

$$f(\mu) = (\widehat{\mu}_1, \widehat{\mu}_2) \tag{2}$$

where $\widehat{\mu}_1$ is the encoding of the Turing machine $\widehat{\mu}_1$ and $\widehat{\mu}_2$ is the encoding of the string $\widehat{\mu}_2$.

As described in the lecture on "universal Turing machines",

$$\mu_1 = (\mu_{1,1}, \mu_{1,2}, \mu_{1,3}, \mu_{1,4}) \tag{3}$$

where the following properties are satisfied.

- $\mu_{1,1}$ is the encoding of the set of states Q — the unpadded decimal representation of the integer k such that $|Q| = k + 1$.
- $\mu_{1,2}$ is the encoding of the input alphabet Σ — the unpadded decimal representation of the positive integer h such that $|\Sigma| = h$.
- $\mu_{1,3}$ is the encoding of the tape alphabet Γ — the unpadded decimal representation of the positive integer m such that $|\Gamma| = m + 1$.
- $\mu_{1,4}$ is the encoding of the transition function δ — a listing of encodings of the transitions of M, separated by commas and enclosed by brackets, given in "dictionary" order.

Similarly,

$$\widehat{\mu}_1 = (\widehat{\mu}_{1,1}, \widehat{\mu}_{1,2}, \widehat{\mu}_{1,3}, \widehat{\mu}_{1,4}) \tag{4}$$

where the following properties are satisfied.

- $\widehat{\mu}_{1,1}$ is the encoding of the set of states $\widehat{Q}$ — the unpadded decimal representation of the integer $\widehat{k}$ such that $|\widehat{Q}| = \widehat{k} + 1$.
- $\widehat{\mu}_{1,2}$ is the encoding of the input alphabet $\widehat{\Sigma}$ — the unpadded decimal representation of the positive integer $\widehat{h}$ such that $|\widehat{\Sigma}| = \widehat{h}$.
- $\widehat{\mu}_{1,3}$ is the encoding of the tape alphabet $\widehat{\Gamma}$ — the unpadded decimal representation of the positive integer $\widehat{m}$ such that $|\widehat{\Gamma}| = \widehat{m} + 1$.
- $\widehat{\mu}_{1,4}$ is the encoding of the transition function $\widehat{\delta}$ — a listing of encodings of the transitions of $\widehat{M}$, separated by commas and enclosed by brackets, given in "dictionary" order.

How is $\widehat{k}$ related to k? How is $\widehat{\mu}_{1,1}$ related to $\mu_{1,1}$?

How is $\widehat{h}$ related to h? How is $\widehat{\mu}_{1,2}$ related to $\mu_{1,2}$?

How is $\widehat{m}$ related to m? How is $\widehat{\mu}_{1,3}$ related to $\mu_{1,3}$?

How are the transitions of $\widehat{M}$ related to the transitions of M? How is $\widehat{\mu}_{1,4}$ related to $\mu_{1,4}$?

How is the encoding, $\widehat{\mu}_2$, of $\widehat{\omega}$ related to the encoding, μ_2, of ω?

How Can We Use All of This?

Suppose that an algorithm has already been used to confirm that $\mu \in$ TM+I and, furthermore, that the strings $\mu_{1,1}$, $\mu_{1,2}$, $\mu_{1,3}$, $\mu_{1,4}$ and μ_2 (as shown at lines (1) and (3)) have all been written to separate tapes of a multi-tape Turing machine — so that a "high-level" algorithm would have access to the values k, h, m and the transitions of M, as described above.

A "high-level" algorithm that completes the computation of $f(\mu)$, from μ, in this case, is as follows:

Implementation-level details that might help a reader to understand, and agree, that the function f is computable, might include the following.

What about “Stage #1” of Assignment #2?

Writing Up the Solution: A First Claim and Its Proof

Writing Up the Solution: A Third Claim and Its Proof

Writing Up the Solution: A Third Claim and Its Proof

Conclusion

A Suggested Exercise