Lecture #12: Classical Reductions

Exercises and Review

Additional Exercises

1. A Boolean formula $\mathcal{F}$ is in 4***-conjunctive normal form*** if $\mathcal{F}$ and every clause in $\mathcal{F}$ includes exactly ***four*** literals. If the same encoding scheme for Boolean formulas (as strings in $\Sigma_F^\star$) is used as in recent lectures, then one can define

   - a language $L_{\text{4CNF}} \subseteq \Sigma_F^\star$ of encodings of Boolean formulas in 4-conjunctive normal form, and
   - a language $L_{\text{4CNF-SAT}} \subseteq L_{\text{4CNF}}$ of encodings of *satisfiable* Boolean formulas in 4-conjunctive normal form.

   Prove that $L_{\text{4CNF}} \in \mathcal{P}$ and that $L_{\text{4CNF-SAT}}$ is $\mathcal{NP}$-complete.

2. If $G = (V, E)$ is an undirected graph then a graph $\widehat{G} = (\widehat{V}, \widehat{E})$ is a ***subgraph*** of G if $\widehat{V} \subseteq V$ and $\widehat{E} \subseteq E$.

   A pair of undirected graphs, $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, are ***isomorphic*** if there is a bijective function $f : V_1 \to V_2$ such that, for all vertices $u, v \in V_1$, $(u, v) \in E_1$ if and only if $(f(u), f(v)) \in E_2$.

   Suppose that undirected graphs are encoded, as strings over the alphabet Σ_G, as described in the lecture notes for Lecture #12. Then a pair of undirected graphs, G_1 and G_2, can be encoded as a string over the alphabet Σ_G consisting of the encodings of G_1 and G_2, separated by a comma.

   Consider the following languages:

   - $L_{\text{2Graphs}} \subseteq \Sigma_G^\star$ is the set of encodings of pairs of undirected graphs.
   - $L_{\text{SubgraphIso}} \subseteq L_{\text{2Graphs}}$ is the set of encodings of ordered pairs of subgraphs, G_1 and G_2, such that G_2 is ***isomorphic to a subgraph of*** G_1.

   Prove that $L_{\text{2Graphs}} \in \mathcal{P}$ and that $L_{\text{SubgraphIso}}$ is $\mathcal{NP}$-complete.

Questions for Review

1. Describe a recommended ***process*** that can now be followed to prove that a given language $L \subseteq \Sigma^\star$ is $\mathcal{NP}$-complete.

   How do you use this when you are a given a description of a ***decision problem*** to start with, instead of a language?

2. Describe Boolean formulas that are in ***conjunctive normal form***. What is the ***CNF Satisfiability Problem*** and what is known about the complexity of this problem?

3. Describe Boolean formulas that are in ***3-conjunctive normal form***. What is the ***$\mathbf{3}$-CNF Satisfiability Problem*** and what is known about the complexity of this problem?

4. What is the ***k-Clique Problem***, and what is known about the complexity of this problem?

5. How does the process of "proving $\mathcal{NP}$-completeness" change as one discovers more, and more, languages that are provably $\mathcal{NP}$-complete? What part of this process can become more complicated? What part of the process can become easier?