Modelling Computation of Functions I
Additional Examples of Primitive Recursive Functions, Predicates, and Closure Properties

Additional Examples Using "Bootstrapping"

As mentioned in the lecture notes, one way to show that a function is primitive recursive is to describe a function that can be obtained by a finite number of applications of composition and primitive recursive, using functions that have been shown to be primitive recursive already, and then prove that the described function is equal the function that you are interested in. Since you can use functions that you have already proved to be primitive recursive using this approach, this can be though of as a simple "bootstrapping" technique.

A "Monus One" Function

To begin, consider a function *monus_one* : $\mathbb{N} \to \mathbb{N}$ such that, for all $x_1 \in \mathbb{N}$,

$$\textit{monus\_one}(x_1) = x_1 \dot{-} 1 = \begin{cases} x - 1 & \text{if } x \geq 1, \\ 0 & \text{if } x = 0. \end{cases} \tag{1}$$

Consider the following sequence of functions.

(a) $f_1 : \mathbb{N}^2 \to \mathbb{N}$ such that, for all $t, y \in \mathbb{N}$, $f_1(t, y) = u_1^2(t, y) = t$. This is one of the initial functions (namely, a *projection function*), so it is primitive recursive.

(b) $f_2 : \mathbb{N} \to \mathbb{N}$ is obtained using the constant $k = 0$ and the above function f_1 using the first form of *primitive recursion*:

$$f_2(0) = 0$$

and, for every integer $t \in \mathbb{N}$,

$$\begin{aligned} f_2(t+1) &= f_1(t, f_2(t)) \\ &= t. \end{aligned}$$

Since f_1 is primitive recursive, it follows that f_2 is primitive recursive as well.

Claim 1. *The function monus_one* $: \mathbb{N} \to \mathbb{N}$ *shown at line* (1)*, above, is primitive recursive.*

Proof. It suffices to note that the function"*monus_one*" is the same as the primitive recursive function $f_2 : \mathbb{N} \to \mathbb{N}$, shown above.

- One can see by the definitions of "*monus_one*" and f_2 that *monus_one*$(0) = f_2(0) = 0$.
- Suppose that $x_1 \in \mathbb{N}$ and $x_n \geq 1$. Then $x_1 = t + 1$ for a number $t \in \mathbb{N}$, so that

$$\begin{aligned} \textit{monus\_one}(x_1) &= \textit{monus\_one}(t+1) \\ &= t && \text{(since } t = x_1 - 1\text{)} \\ &= f_2(t+1) && \text{(as shown in the derivation of } f_2\text{, above)} \\ &= f_2(x_1). \end{aligned}$$

  Thus *monus_one* $= f_2$, so that the function "*monus_one*" is primitive recursive, since f_2 is.

□

A "Monus" Function

Consider a function *monus* : $\mathbb{N}^2 \to \mathbb{N}$ such that, for all $x_1, x_2 \in \mathbb{N}$,

$$\textit{monus}(x_1, x_2) = x_1 \dot{-} x_2 = \begin{cases} x_1 - x_2 & \text{if } x_1 \geq x_2, \\ 0 & \text{if } x_1 < x_2. \end{cases} \tag{2}$$

Once again, a sequence of primitive recursive functions should also be considered:

(a) This time, let $f_1 : \mathbb{N} \to \mathbb{N}$ such that, for all $x \in \mathbb{N}$, $f_1(x) = u_1^1(x) = x$. This is primitive recursive because it is one of the initial functions, namely, a *projection* function.

(b) Let $f_2 : \mathbb{N}^3 \to \mathbb{N}$ such that, for all $t, y, x \in \mathbb{N}$, $f_2(t, y, x) = u_2^3(t, y, x) = y$. This is also primitive recursive because it is one of the initial functions, namely, a *projection* function.

(c) Let $f_3 : \mathbb{N} \to \mathbb{N}$ such that, for $x \in \mathbb{N}$, $f_3(x) =$ *monus_one*$(x) = x \dot{-} 1$. It follows by Claim 1, above, that the function f_3 is also primitive recursive.

(d) Let $f_4 : \mathbb{N}^3 \to \mathbb{N}$ be obtained from f_3 and f_2 by *composition:* For all $t, y, x \in \mathbb{N}$,

$$f_4(t, y, x) = f_3(f_2(t, y, x)) = f_3(y) = y \dot{-} 1.$$

Since f_4 was obtained from primitive recursive functions using *composition*, this function is also primitive recursive.

(e) Let $f_5 : \mathbb{N}^2 \to \mathbb{N}$ be obtained (with $n = 1$) using the functions $f_1 : \mathbb{N} \to \mathbb{N}$ and $f_4 : \mathbb{N}^3 \to \mathbb{N}$ using the second form of *primitive recursion:* For all $x_1 \in \mathbb{N}$,

$$f_5(x_1, 0) = f_1(x_1) = x_1$$

and, for all $x_1, t \in \mathbb{N}$,

$$\begin{aligned} f_5(x_1, t+1) &= f_4(t, f_5(x_1, t), x_1) \\ &= f_5(x_1, t) \dot{-} 1. \end{aligned}$$

Since this was obtained from primitive recursive functions f_1 and f_4 using the second form of *primitive recursion*, it is a primitive recursive function as well.

Claim 2. *The function monus* : $\mathbb{N}^2 \to \mathbb{N}$ *shown at line* (2)*, above, is primitive recursive.*

Proof. Since the above function f_5 is primitive recursive, t is sufficient to show that the "*monus*" function is equal to the function f_5 that is given above, that is, *monus*$(x_1, y_1) = f_5(x_1, y_1)$ for all $x_1, y_1 \in \mathbb{N}$. This will be proved by induction on y_1, using the standard form of mathematical induction.

Basis: If $x_2 = 0$ then, for all $x_1 \in \mathbb{N}$,

$$\begin{aligned} f_5(x_1, x_2) &= f_5(x_1, 0) \\ &= x_1 && \text{(by the above definition of } f_5\text{)} \\ &= x_1 \dot{-} 0 \\ &= x_1 \dot{-} x_2 \\ &= \textit{monus}(x_1, x_2) \end{aligned}$$

as required in this case.

Inductive Step: Let $k \in \mathbb{N}$ such that $k \geq 0$. It is necessary and sufficient to use the following

<u>Inductive Hypothesis:</u> $f_5(x_1, k) = x_1 \dot{-} k =$ *monus*(x_1, k) for all $x_1 \in \mathbb{N}$.

to prove the following

<u>Inductive Claim:</u> $f_5(x_1, k+1) = x_1 \dot{-} (k+1) =$ *monus*$(x_1, k+1)$ for all $x_1 \in \mathbb{N}$.

With that noted, let $x_1 \in \mathbb{N}$. The cases $x_1 < k$, $x_1 = k$ and $x_1 \geq k+1$ will be considered separately, below.

- *Case:* $x_1 < k$. In this case

$$\begin{aligned}
f_5(x_1, k+1) &= f_5(x_1, k) \dot{-} 1 && \text{(by the definition of } f_5\text{, above)} \\
&= (x_1 \dot{-} k) \dot{-} 1 && \text{(by the inductive hypothesis)} \\
&= 0 \dot{-} 1 && \text{(since } x_1 < k\text{)} \\
&= 0 \\
&= x_1 \dot{-} (k+1) \\
&= \textit{monus}(x_1, k+1) && \text{(since } x_1 < k\text{, so that } x_1 < k+1\text{).}
\end{aligned}$$

- *Case:* $x_1 = k$. In this case

$$\begin{aligned}
f_5(x_1, k+1) &= f_5(x_1, k) \dot{-} 1 && \text{(by the definition of } f_5\text{, above)} \\
&= (x_1 \dot{-} k) \dot{-} 1 && \text{(by the inductive hypothesis)} \\
&= 0 \dot{-} 1 && \text{(since } x_1 \geq k \text{ and } x_1 - k = k - k = 0\text{)} \\
&= 0 \\
&= x_1 \dot{-} (k+1) \\
&= \textit{monus}(x_1, k+1)) && \text{(since } x_1 = k\text{, so that } x_1 < k+1\text{).}
\end{aligned}$$

- *Case:* $x_1 \geq k+1$. In this case

$$\begin{aligned}
f_5(x_1, k+1) &= f_5(x_1, k) \dot{-} 1 && \text{(by the definition of } f_5\text{, above)} \\
&= (x_1 \dot{-} k) \dot{-} 1 && \text{(by the inductive hypothesis)} \\
&= (x_1 - k) \dot{-} 1 && \text{(since } x_1 \geq k\text{)} \\
&= (x_1 - k) - 1 && \text{(since } x_1 \geq k+1\text{, so that } x_1 - k \geq 1\text{)} \\
&= x_1 - (k+1) \\
&= x_1 \dot{-} (k+1) && \text{(since } x_1 \geq k+1\text{)} \\
&= \textit{monus}(x_1, k+1).
\end{aligned}$$

Thus the inductive claim is established in every case, as needed to complete the inductive step.

It follows that $f_5(x_1, x_2) = x_1 \dot{-} x_2 = \textit{monus}(x_1, x_2)$ for all $x_1, x_2 \in \mathbb{N}$. Thus the "*monus*" function is primitive recursive, as claimed. $\square$

Multiplication

Consider the function *mult* : $\mathbb{N}^2 \to \mathbb{N}$ such that

$$\textit{mult}(x_1, x_2) = x_1 \times x_2 \tag{3}$$

for all $x_1, x_2 \in \mathbb{N}$.

The following sequence of functions should now be considered.

(a) The function $f_1 : \mathbb{N} \to \mathbb{N}$ such that $f_1(x) = n(x) = 0$ is primitive recursive because it is one of the initial functions, namely, the *nullify function*.

(b) As proved during this lecture, the function $f_2 : \mathbb{N}^2 \to \mathbb{N}$ such that, for all $y, x \in \mathbb{N}$, $f_2(y, x) = \textit{plus}(y, x) = y + x$ (that is, the *addition* function) is primitive recursive.

(c) The function $f_3 : \mathbb{N}^3 \to \mathbb{N}$ such that, for all $t, y, x \in \mathbb{N}$, $f_3(t, y, x) = u_2^3(t, y, x) = y$, is primitive recursive because it is one of the initial functions, namely, a *projection* function.

(d) The function $f_4 : \mathbb{N}^3 \to \mathbb{N}$ such that, for all $t, y, x \in \mathbb{N}$, $f_3(t, y, x) = u_3^3(t, y, x) = x$, is primitive recursive because it is one of the initial functions, namely, a *projection* function.

(e) The function $f_5 : \mathbb{N}^3 \to \mathbb{N}$ such that, for all $t, y, z \in \mathbb{N}$,

$$\begin{aligned} f_5(t, y, x) &= f_2(f_3(t, y, x), f_4(t, y, x)) \\ &= f_2(y, x) \\ &= y + x \end{aligned}$$

is primitive recursive because it has been obtained from the primitive recursive functions f_2, f_3 and f_4 using *composition*.

(f) The function $f_6 : \mathbb{N}^2 \to \mathbb{N}$ obtained from the primitive recursive functions f_1 and f_5 using the second form of *primitive recursion* (with $n = 1$) is a primitive recursive function: For all $x_1 \in \mathbb{N}$,

$$f_6(x_1, 0) = f_1(x) = 0$$

and, for all $x_1, t \in \mathbb{N}$,

$$\begin{aligned} f_6(x_1, t+1) &= f_5(t, f_6(x_1, t), x_1) \\ &= f_6(x_1, t) + x_1. \end{aligned}$$

Claim 3. *The function mult* : $\mathbb{N}^2 \to \mathbb{N}$ *shown at line* (3)*, above, is primitive recursive.*

Since the above function f_6 is primitive recursive, t is sufficient to show that the "*mult*" function is equal to the function f_6 that is given above, that is, *mult*$(x_1, y_1) = f_6(x_1, y_1)$ for all $x_1, y_1 \in \mathbb{N}$. This can be proved by induction on y_1, using the standard form of mathematical induction — and a completion of this proof (whose structure should resemble that of the proof of Claim 2, above) is left as an ***exercise***.

Predicate: Checking Whether $x_1 > 0$

Consider the predicate *nonzero* : $\mathbb{N} \to \{0, 1\}$ such that, for all $x_1 \in \mathbb{N}$,

$$\textit{nonzero}(x_1) = \begin{cases} 0 & \text{if } x_1 = 0, \\ 1 & \text{if } x_1 > 0. \end{cases} \tag{4}$$

Consider, as well, the following sequence of functions.

(a) The function $f_1 : \mathbb{N} \to \mathbb{N}$ such that, for all $x_1 \in \mathbb{N}$, $f_1(x_1) = c_1(x_1) = 1$, is primitive recursive — because this is one of the *constant functions* that were proved to be primitive recursive in the notes for this lecture.

(b) The projection function $f_2 : \mathbb{N}^2 \to \mathbb{N}$, such that $f_2(x_1, x_2) = u_1^2(x_1, x_2) = x_1$ for all $x_1, x_2 \in \mathbb{N}$, is primitive recursive.

(c) The function $f_3 : \mathbb{N}^2 \to \mathbb{N}$, obtained from f_1 and f_2 using *composition* is primitive recursive, and $f_3(x_1, x_2) = f_1(f_2(x_1, x_2)) = f_1(x_1) = 1$ for all $x_1, x_2 \in \mathbb{N}$.

(d) Let $f_4 : \mathbb{N} \to \mathbb{N}$ be the function obtained using the first form of *primitive recursion* using the constant $k = 0$ and the function f_3. Then f_4 is primitive recursive,

$$f_4(0) = 0,$$

and, for all $t \in \mathbb{N}$,

$$f_4(t + 1) = f_3(t, f_4(t)) = 1.$$

Claim 4. *The predicate nonzero* : $\mathbb{N} \to \{0, 1\}$ *shown at line* (4)*, above, is primitive recursive.*

Proof. It suffices to note that if $f_4 : \mathbb{N} \to \mathbb{N}$ is as given above then $f_4(x_1) \in \{0, 1\}$ for all $x_1 \in \mathbb{N}$ — so that $f_4 : \mathbb{N} \to \{0, 1\}$ and f_4 is a *predicate*. Furthermore one can see by an examination of the above that *nonzero*$(x_1) = f_4(x_1)$ for all $x_1 \in \mathbb{N}$. Thus “*nonzero*” and f_4 are the same predicates — and the predicate “*nonzero*” is primitive recursive, as claimed. ☐

Useful Closure Properties

It can be shown that the set of primitive recursive functions — and the set of primitive recursive predicates — are closed under a variety of operations. This can sometimes be used to prove that functions, and predicates, are primitive recursive — more quickly, or simply than if the “bootstrapping” technique was the only thing being used.

Closure of the Primitive Recursive Predicates Under Logical Operations

The following result is stated as "Claim #3" in the lecture notes.

Claim 5. *Let n be a positive integer and let $p, q : \mathbb{N}^n \to \{0, 1\}$ be primitive recursive predicates.*

(a) $\neg p$ is a primitive recursive predicate.

(b) $p \wedge q$ is a primitive recursive predicate.

(c) $p \vee q$ is a primitive recursive predicate.

Proof. Let n be a positive integer and suppose that $p, q : \mathbb{N}^n \to \{0, 1\}$ are primitive recursive predicates.

(a) Note, first, that — for all $x_1, x_2, \ldots, x_n \in \mathbb{N}$ —

$$\neg p(x_1, x_2, \ldots, x_n) = 1 \dot{-} p(x_1, x_2, \ldots, x_n).$$

Consider, now, the following sequence of functions.

(i) The function $f_1 = u_1^n : \mathbb{N}^n \to \mathbb{N}$ is primitive recursive because it is one of the *projection* functions. Recall that, for all $x_1, x_2, \ldots, x_n$,

$$f_1(x_1, x_2, \ldots, x_n) = u_1^n(x_1, x_2, \ldots, x_n) = x_1.$$

(ii) As shown in the lecture notes, the *constant* function $f_2 = c_1 : \mathbb{N} \to \mathbb{N}$ such that, for all $x_1 \in \mathbb{N}$, $c_1(x_1) = 1$, is primitive recursive.

(iii) Let $f_3 : \mathbb{N}^n \to \mathbb{N}$ be obtained from f_2 and f_1 by *composition* — so that, for all $x_1, x_2, \ldots, x_n$,

$$\begin{aligned} f_3(x_1, x_2, \ldots, x_n) &= f_2(f_1(x_1, x_2, \ldots, x_n)) \\ &= f_2(x_1) \\ &= 1 \end{aligned}$$

— is a primitive recursive function, since f_2 and f_1 are.

(iv) It is given that the function $f_4 = p : \mathbb{N}^n \to \{0, 1\}$ is a primitive recursive predicate (so that it is also a primitive recursive function).

(v) It follows by Claim 2 that the function $f_5 =$ *monus* $: \mathbb{N}^2 \to \mathbb{N}$ — such that $f_5(x_1, x_2) =$ *monus*$(x_1, x_2) = x_1 \dot{-} x_2$, for all $x_1, x_2 \in \mathbb{N}$ — is also a primitive recursive function.

(vi) Let $f_6 : \mathbb{N}^2 \to \mathbb{N}$ be the function obtained from f_5, f_3 and f_4 by *composition*. Then f_6 is also a primitive recursive function and, for all $x_1, x_2, \dots, x_n \in \mathbb{N}$,

$$\begin{aligned} f_6(x_1, x_2, \dots, x_n) &= f_5(f_3(x_1, x_2, \dots, x_n), f_4(x_1, x_2, \dots, x_n)) \\ &= f_3(x_1, x_2, \dots, x_n) \mathbin{\dot{-}} f_4(x_1, x_2, \dots, x_n) \\ &= 1 \mathbin{\dot{-}} p(x_1, x_2, \dots, x_n) \\ &= \neg p(x_1, x_2, \dots, x_n). \end{aligned}$$

Thus $\neg p$ is a primitive recursive function and — since it is also a predicate — it is a primitive recursive predicate, as claimed.

(b) Note that — since p_1 and p_2 are both *predicates* (so that $p_1(x_1, x_2, \dots, x_n), p_2(x_1, x_2, \dots, x_n) \in \{0, 1\}$)

$$\begin{aligned} (p_1 \wedge p_2)(x_1, x_2, \dots, x_n) &= p_1(x_1, x_2, \dots, x_n) \wedge p_2(x_1, x_2, \dots, x_n) \\ &= p_1(x_1, x_2, \dots, x_n) \times p_2(x_1, x_2, \dots, x_n) \\ &= \textit{mult}(p_1(x_1, x_2, \dots, x_n), p_2(x_1, x_2, \dots, x_n)). \end{aligned}$$

Since the function "*mult*" is primitive recursive, by Claim 3, and $p_1 \wedge p_1$ has been obtained from "*mult*" and the primitive recursive predicates p_1 and p_2 by *composition*, it follows that $p_1 \wedge p_2$. This function is also a *predicate*, $p_1 \wedge p_2$ is a primitive recursive predicate, as claimed.

(c) Recall that $y_1 \vee y_2 = \neg(\neg y_1 \wedge \neg y_2)$ for all Boolean values y_1 and y_2.

- Since the predicates $p_1 : \mathbb{N}^n \to \{0, 1\}$ and $p_2 : \mathbb{N}^n \to \{0, 1\}$ are primitive recursive it follows by part (a) that the predicates $\neg p_1$ and $\neg p_2$ are primitive recursive predicates too.
- It now follows by part (b) that the function $\neg p_1 \wedge \neg p_2$ is a primitive recursive predicate as well.
- It follows by another application of the result of part (a) that the function $\neg(\neg p_1 \wedge \not p_2)$ is also a primitive recursive predicate. However, as noted above, this is the same as the predicate $p_1 \vee p_2$. Thus this is also a primitive recursive predicate, as claimed.

All parts of the claim have now been established. $\square$

Another Closure Property: Definition by Cases

The following result is stated as "Claim #4" in the lecture notes.

Claim 6. *Suppose that $f, g : \mathbb{N}^n \to \mathbb{N}$ are primitive recursive functions and suppose that $p : \mathbb{N}^n \to \{0,1\}$ is a primitive recursive predicate. Consider the function $h : \mathbb{N}^n \to \mathbb{N}$ such that, for all $x_1, x_2, \ldots, x_n \in \mathbb{N}$,*

$$h(x_1, x_2, \ldots, x_n) = \begin{cases} f(x_1, x_2, \ldots, x_n) & \textit{if } p(x_1, x_2, \ldots, x_n) = 1, \\ g(x_1, x_2, \ldots, x_n) & \textit{if } p(x_1, x_2, \ldots, x_n) = 0. \end{cases} \tag{5}$$

This function is also primitive recursive.

Proof. This can be proved using the fact that if the function h is as given at line (5), above, then

$$h = p \times f + (\neg p) \times f.$$

With that noted, consider the following sequence of functions.

(a) Let $f_1 : \mathbb{N}^n \to \mathbb{N}$ such that

$$\begin{aligned} f_1(x_1, x_2, \ldots, x_n) &= p(x_1, x_2, \ldots, x_n) \times f(x_1, x_2, \ldots, x_n) \\ &= \textit{mult}(p(x_1, x_2, \ldots, x_n), f(x_1, x_2, \ldots, x_n)) \end{aligned}$$

for all $x_1, x_2, \ldots, x_n \in \mathbb{N}$. It follows by Claim 3 that the function *mult* : $\mathbb{N}^2 \to \mathbb{N}$ is primitive recursive. The function f_1 is therefore primitive recursive since it has been obtained from the primitive recursive functions *mult*, p and f by *composition*.

(b) Since p is a primitive recursive predicate it follows by part (a) of Claim 5 that the predicate $(\neg p) : \mathbb{N}^n \to \{0,1\}$ is a primitive recursive predicate too.

(c) Let $f_2 : \mathbb{N}^n \to \mathbb{N}$ such that

$$\begin{aligned} f_2(x_1, x_2, \ldots, x)n) &= \\ &= (\neg p)(x_1, x_2, \ldots, x_n) \times g(x_1, x_2, \ldots, x_n) \\ &= \textit{mult}((\neg p)(x_1, x_2, \ldots, x_n), g(x_1, x_2, \ldots, x_n)) \end{aligned}$$

for all $x_1, x_2, \ldots, x_n$. This function is primitive recursive since it has been obtained from the primitive recursive functions *mult*, $\neg p$ and g by *composition*.

(d) Let $f_3 : \mathbb{N}^n \to \mathbb{N}$ such that

$$\begin{aligned} f_3(x_1, x_2, \ldots, x_n) &= f_1(x_1, x_2, \ldots, x_n) + f_2(x_1, x_2, \ldots, x_n) \\ &= \textit{plus}(f_1(x_1, x_2, \ldots, x_n), f_2(x_1, x_2, \ldots, x_n)) \end{aligned}$$

for all $x_1, x_2, \ldots, x_n \in \mathbb{N}$. As shown in the lecture note, the addition function, *plus*, is primitive recursive function. The function f_3 is therefore primitive recursive because it has been obtained from the primitive recursive functions *plus*, f_1 and f_2 by *composition*.

As noted above,

$$h(x_1, x_2, \ldots, x_n) = f_3(x_1, x_2, \ldots, x_n)$$

for all $x_1, x_2, \ldots, x_n \in \mathbb{N}$ — so that f_3 and h are the same functions. The function h is therefore primitive recursive, as claimed. $\square$

Application of Closure Properties: A Remainder Function

Let *rem* : $\mathbb{N}^2 \to \mathbb{N}$ such that, for all $x_1, x_2 \in \mathbb{N}$,

$$\textit{rem}(x_1, x_2) = \begin{cases} x_1 \bmod x_2 & \text{if } x_2 \geq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Claim 7. *The function rem* : $\mathbb{N}^2 \to \mathbb{N}$*, given at line* (6)*, above, is primitive recursive.*

Proof. Consider the following sequence of functions.[1]

(a) The function $f_1 : \mathbb{N}^3 \to \mathbb{N}$ such that, for all $t, y, x \in \mathbb{N}$,

$$f_1(t, y, x) = s(u_2^3(t, y, x)) = s(y) = y + 1,$$

is primitive recursive because it has been obtained from the initial functions s and u_2^3 by *composition*.

(b) The function $f_2 : \mathbb{N}^3 \to \mathbb{N}$ such that, for all $t, y, x \in \mathbb{N}$,

$$f_2(t, y, x) = n(u_2^3(t, y, x)) = n(0) = 0,$$

is primitive recursive because it has been obtained from the initial functions n and u_2^3 by *composition*.

(c) It is argued, in the lecture notes, that the predicate *gt* : $\mathbb{N}^2 \to \{0, 1\}$ such that, for all $x_1, x_2 \in \mathbb{N}$,

$$\textit{gt}(x_1, x_2) = \begin{cases} 1 & \text{if } x_1 > x_2, \\ 0 & \text{if } x_1 \leq x_2 \end{cases}$$

is a primitive recursive predicate.

[1] This list is slightly less complete than the lists in previous example proofs — functions that are primitive recursive because they are *initial functions* are not listed separately.

(d) The predicate $p_1 : \mathbb{N}^3 \to \{0,1\}$ such that, for all $t, y, x \in \mathbb{N}$,

$$p_1(t,y,x) = \textit{gt}(u_3^3(t,y,x), f_1(t,y,x)) = \begin{cases} 1 & \text{if } x > y+1, \\ 0 & \text{if } x \le y+1 \end{cases}$$

is primitive recursive because it has been obtained from the primitive recursive predicate *gt*, the initial function u_3^3, and the primitive recursive function f_1 by *composition*.

(e) Since the functions $f_1, f_2 : \mathbb{N}^3 \to \mathbb{N}$ and the predicate $p_1 : \mathbb{N}^3 \to \{0,1\}$ are primitive recursive, it follows by Claim 6, above, that the function $f_3 : \mathbb{N}^3 \to \mathbb{N}$ such that, for all $t, y, x \in \mathbb{N}$,

$$\begin{aligned} f_3(t,y,x) &= \begin{cases} f_1(t,y,x) & \text{if } p_1(t,y,x) = 1, \\ f_2(t,y,x) & \text{if } p_1(t,y,x) = 0 \end{cases} \\ &= \begin{cases} y+1 & \text{if } x > y+1, \\ 0 & \text{if } x \le y+1. \end{cases} \end{aligned}$$

is primitive recursive.

(f) Let $f_4 : \mathbb{N}^2 \to \mathbb{N}$ be the function obtained from the nullify function n and the function f_3 using the second form of primitive recursion (with $n = 1$). Then

$$f_4(x_1, 0) = n(x_1) = 0$$

for all $x_1 \in \mathbb{N}$ and

$$\begin{aligned} f_4(x_1, t+1) &= f_3(t, f_4(x_1,t), x_1) \\ &= \begin{cases} f_4(x_1,t)+1 & \text{if } x_1 > f_4(x_1,t)+1, \\ 0 & \text{if } x_1 \le f_4(x_1,t)+1. \end{cases} \end{aligned}$$

(g) Finally, let $f_5 : \mathbb{N}^2 \to \mathbb{N}$ be the function obtained from above function, f_4, and the initial functions $u_2^2, u_1^2 : \mathbb{N}^2 \to \mathbb{N}$ by *composition*: For all $x_1, x_2 \in \mathbb{N}$,

$$\begin{aligned} f_5(x_1,x_2) &= f_4(u_2^2(x_1,x_2), u_1^2(x_1,x_2)) \\ &= f_4(x_2, x_1). \end{aligned}$$

Then f_5 is also a primitive recursive function.

It can be shown (by induction on x_1) that *rem*$(x_1, x_2) = f_5(x_1, x_2)$ for all $x_1, x_2 \in \mathbb{N}$ — and this part of the proof is left as an *exercise*. It now follows that the function *rem* is primitive recursive, as claimed. $\square$