Lecture #3: Modelling Computation of Functions II

Exercises and Review

Questions for Review

1. Describe several results involving ***iterated operators*** that can be used to give proofs that functions are primitive recursive, that are shorter and simpler than the proofs (for these functions) would be, if the definition of a "primitive recursive function" was used directly, instead.
2. Describe ***bounded quantifiers*** that can be used to give proofs that predicates are primitive recursive, that are shorter and simpler than the proofs for these predicates would be if the definition of a "primitive recursive function" was used instead.
3. Describe ***bounded minimization*** and the functions that can be defined using this. Note, once again, that the use of bounded minimization can also be used to simplify proofs that functions and predicates are primitive recursive.
4. Describe ***unbounded minimization*** and the partial or total functions that can be defined using this.
5. Say what it means for a partial or total function to be ***μ-recursive***. How is the set of "μ-recursive" partial and total functions related to the set of "primitive recursive" functions?

Practice Problems

These problems will not be discussed during the lecture, and solutions for these problems will not generally be made available. They can be used as "practice" problems that can help you to practice skill considered in the lecture presentation for Lecture #3.

1. Show that each of the following functions is primitive recursive.

   (a) $f : \mathbb{N} \to \mathbb{N}$ such that, for all $x_1 \in \mathbb{N}$,

   $$f(x_1) = \sum_{i=0}^{x_1} i = \frac{n(n-1)}{2}.$$

(b) The ***factorial function*** $F : \mathbb{N} \to \mathbb{N}$ such that $F(0) = 1$ and, for $n \geq 1$,

$$F(n) = n! = \prod_{i=1}^{n} i.$$

(c) The predicate *square* : $\mathbb{N} \to \{0, 1\}$ that is satisfied if and only if its input is a ***perfect square*** — that is, such that, for all $x_1 \in \mathbb{N}$, *square*$(x_1) = 1$ if and only if $x_1 = x_2^2$ for some number $x_2 \in \mathbb{N}$.

(d) The function $f : \mathbb{N} \to \mathbb{N}$ such that, for all $x_1 \in \mathbb{N}$, $f(x) = \lfloor \sqrt{x_1} \rfloor$, that is, the largest integer that is less than or equal to the square root of x_1.

2. Let us say that a positive integer n is a ***twin prime*** if either both $n - 2$ and n are prime numbers, or both n and $n + 2$ are prime numbers. Thus 3 is the smallest twin prime (since 1 is not a prime number, but 5 is), and 5, 7, 11 and 13 are the next twin primes after that.

   (a) Let $p_{\mathsf{tp}} : \mathbb{N} \to \{0, 1\}$ such that, for every integer n such that $n \geq 0$,

   $$p_{\mathsf{tp}}(n) = \begin{cases} 1 & \text{if } n \text{ is a twin prime,} \\ 0 & \text{if } n \text{ is not a twin prime.} \end{cases}$$

   Prove that p_{tp} is a ***primitive recursive*** predicate.

   (b) Let $f_{\mathsf{tp}} : \mathbb{N} \to \mathbb{N}$ be the partial or total function such that $f_{\mathsf{tp}}(0)$ and such that the following properties are satisfied.

      (i) If $n \geq 1$ and there are at least n twin primes then $f_{\mathsf{tp}}(n)$ is the n^{th} twin prime (when these are listed in increasing order) —so that $f_{\mathsf{tp}}(1) = 3$, $f_{\mathsf{tp}}(2) = 5$, $f_{\mathsf{tp}}(3) = 7$, $r_{\mathsf{tp}}(4) = 11$, and $f_{\mathsf{tp}}(5) = 13$.

      (ii) If $n \geq 1$ and there are *not* at least n twin primes, then $f_{\mathsf{tp}}(n)$ is undefined.

   Prove that f_{tp} is a ***μ-recursive*** partial or total function.

   (c) The ***twin prime conjecture*** is an *unproved* conjecture that there are infinitely many twin primes. This conjecture would imply that f_{tp} is a total function.

   Explain why this would *not* be sufficient to complete a proof that the function f_{tp} is primitive recursive — and then describe something else (that has something to do with the n^{th} twin prime, for each positive integer n) that would be enough to complete a proof of this.

   (Note again, though, that we do not currently know whether f_{tp} is a total function, at all.)