Modelling Computation of Functions II
Proofs of Claims

This document includes sketches of proofs of claims that were given in the lecture notes. The completion of a proof is often left as an exercise in the use of techniques that were introduced in the previous lecture.

Iterated Sums

Claim 1. *Let $n \in \mathbb{N}$ and let $\varphi : \mathbb{N}^{n+1} \to \mathbb{N}$ be primitive recursive. Then the function $S_\varphi : \mathbb{N}^{n+1} \to \mathbb{N}$ such that, for all $x_1, x_2, \dots, x_{n+1}$,*

$$S_\varphi(x_1, x_2, \dots, x_n, x_{n+1}) = \sum_{i=0}^{x_{n+1}} \varphi(x_1, x_2, \dots, x_n, i) \tag{1}$$

is primitive recursive as well.

Sketch of Proof. Suppose, first, that $n \geq 1$.

- Let $f : \mathbb{N}^n \to \mathbb{N}$ such that, for all $x_1, x_2, \dots, x_n \in \mathbb{N}$,

  $$f(x_1, x_2, \dots, x_n) = \varphi(x_1, x_2, \dots, x_n, 0).$$

  Then f can be shown to be a primitive recursive function.

- Let $g : \mathbb{N}^{n+2} \to \mathbb{N}$ such that, for all $t, y, x_1, x_2, \dots, x_n \in \mathbb{N}$,

  $$g(t, y, x_1, x_2, \dots, x_n) = y + \varphi(x_1, x_2, \dots, x_n, t+1).$$

  Then g can also be shown to be primitive recursive.

- Let $h : \mathbb{N}^n + 1 \to \mathbb{N}$ be the function obtained from f and g using the second form of primitive recursion.

- Then, it is easily shown by induction on x_{n+1} that, for all $x_1, x_2, \dots, x_{n+1} \in \mathbb{N}$,

$$h(x_1, x_2, \dots, x_{n+1}) = S_\varphi(x_1, x_2, \dots, x_{n+1}) = \sum_{i=0}^{x_{n+1}} \varphi(x_1, x_2, \dots, x_n, i)$$

—- so that the function S_φ is primitive recursive, as claimed. ☐

Exercises:

(a) Use techniques from the previous lecture to prove that the function $f : \mathbb{N}^n \to \mathbb{N}$, introduced in the above proof, is primitive recursive if the function φ is.

(b) Use techniques from the previous lecture to prove that the function $g : \mathbb{N}^{n+2} \to \mathbb{N}$, introduced in the above proof, is primitive recursive if the function φ is.

(c) Prove, by induction on x_{n+1}, that if the function $h : \mathbb{N}^{n+1} \to \mathbb{N}$ is as defined (using primitive recursion) as above, then

$$h(x_1, x_2, \dots, x_{n+1}) = S_\varphi(x_1, x_2, \dots, x_{n+1}) = \sum_{i=0}^{x_{n+1}} \varphi(x_1, x_2, \dots, x_n, i)$$

for all $x_1, x_2, \dots, x_{n+1} \in \mathbb{N}$, as claimed above.

(d) Modify the proof of Claim 1, above, to prove that if the function $\varphi : \mathbb{N} \to \mathbb{N}$ is primitive recursive then the function $S_\varphi : \mathbb{N} \to \mathbb{N}$ such that

$$S_\varphi(x_1) = \sum_{i=0}^{x_1} \varphi(i),$$

for all $x_1 \in \mathbb{N}$, is primitive recursive as well.

(e) Modify the proof of Claim 1, above, to prove that if $n \in \mathbb{N}$ and the function $\varphi : \mathbb{N}^{n+1} \to \mathbb{N}$ is primitive recursive then the function $S_\varphi : \mathbb{N}^{n+1} \to \mathbb{N}$ such that

$$S_\varphi(x_1, x_2, \dots, x_{n+1}) = \sum_{i=1}^{x_{n+1}} \varphi(x_1, x_2, \dots, x_n, i),$$

for all $x_1, x_2, \dots, x_{n+1} \in \mathbb{N}$, is primitive recursive as well.

Iterated Products

Claim 2. *Let $n \in \mathbb{N}$ and let the function $\varphi : \mathbb{N}^{n+1} \to \mathbb{N}$ be primitive recursive.*

(a) The iterated product $P_\varphi : \mathbb{N}^{n+1} \to \mathbb{N}$ such that, for all $x_1, x_2, \ldots, x_{n+1} \in \mathbb{N}$,

$$P_\varphi(x_1, x_2. \ldots, x_n, x_{n+1}) = \prod_{i=0}^{x_{n+1}} \varphi(x_1, x_2, \ldots, x_n, i)$$

is primitive recursive.

(b) The (modified) inner product $\widehat{P}_\varphi : \mathbb{N}^{n+1} \to \mathbb{N}$ such that, for all $x_1, x_2, \ldots, x_{n+1} \in \mathbb{N}$,

$$\widehat{P}_\varphi(x_1, x_2, \ldots, x_n, x_{n+1}) \prod_{i=1}^{x_{n+1}} \varphi(x_1, x_2, \ldots, x_n, i)$$

is primitive recursive as well.

Exercise: Modify the proof of Claim 1, to prove Claim 2.

Construction: Bounded "Universal" Quantifiers

See the lecture notes for the definitions of the "bounded quantifiers" that are mentioned in the following claim.

Claim 3. *Let $n \in \mathbb{N}$ and let $p : \mathbb{N}^{n+1} \to \{0, 1\}$. If p is primitive recursive then each of the predicates*

$$\underset{0 \leq i \leq x_{n+1}}{\forall} p, \quad \underset{1 \leq i \leq x_{n+1}}{\forall} p, \quad \underset{0 \leq i \leq x_{n+1}}{\exists} p \quad \textit{and} \quad \underset{1 \leq i \leq x_{n+1}}{\exists} p$$

is a primitive recursive predicate as well.

Proof. Note, first, that, for all $x_1, x_2, \ldots, x_{n+1} \in \mathbb{N}$,

$$\underset{0 \leq i \leq x_{n_1}}{\forall} p(x_1, x_2, \ldots, x_n, i) = \prod_{i=0}^{x_{n+1}} p(x_1, x_2, \ldots, x_n, i)$$

and

$$\underset{1 \leq i \leq x_{n_1}}{\forall} p(x_1, x_2, \ldots, x_n, i) = \prod_{i=1}^{x_{n+1}} p(x_1, x_2, \ldots, x_n, i).$$

It therefore follows by Claim 2 that the predicates $\underset{0 \leq i \leq x_{n+1}}{\forall} p$ and $\underset{1 \leq i \leq x_{n+1}}{\forall} p$ are primitive recursive.

For all $x_1, x_2, \ldots, x_{n+1} \in \mathbb{N}$, it follows by DeMorgan's Law that

$$\underset{0 \leq i \leq x_{n+1}}{\exists} p(x_1, x_2, \ldots, x_n, i) = \neg \left(\underset{0 \leq i \leq x_{n+1}}{\forall} (\neg p(x_1, x_2, \ldots, x_n, i)) \right).$$

Since the set of primitive recursive predicates is closed under negation, it follows that the predicate $\underset{0 \leq i \leq x_{n+1}}{\exists} p$ is primitive recursive.

A similar argument establishes the the predicate $\underset{1 \leq i \leq x_{n+1}}{\exists} p$ is also primitive recursive, establishing the claim. ☐

Construction: Bounded Minimization

See the lecture notes for an introduction to "bounded minimization" and the definition of the function that is mentioned in the following claim.

Claim 4. *Let $n \in \mathbb{N}$ and let $p : \mathbb{N}^{n+1} \to \{0, 1\}$. If the predicate p is primitive recursive then the function*

$$\underset{0 \leq i \leq x_{n+1}}{\text{first}} p : \mathbb{N}^{n+1} \to \mathbb{N}$$

is primitive recursive as well.

Proof. Let $n \in \mathbb{N}$ and let $p : \mathbb{N}^{n+1} \to \{0, 1\}$ be a primitive recursive predicate.

Since the set of primitive recursive predicates is closed under negation it follows by Claim 2 that the iterated product $q : \mathbb{N}^{n+1} \to \mathbb{N}$ such that, for $x_1, x_2, \ldots, x_{n+1} \in \mathbb{N}$,

$$q(x_1, x_2, \ldots, x_{n+1}) = \prod_{j=0}^{x_{n+1}} (\neg p(x_1, x_2, \ldots, x_n, j))$$

is primitive recursive as well. Since the value of q is either 0 or 1, q is also a primitive recursive predicate.

Since it is an iterated sum obtained from q, it now follows by Claim 1 that the function $f : \mathbb{N}^{n+1} \to \mathbb{N}$

such that, for all $x_1, x_2, \ldots, x_{n+1} \in \mathbb{N}$,

$$\begin{aligned} f(x_1,&x_2, \ldots, x_n, x_{n+1}) \\ &= \sum_{i=0}^{n+1} q(x_1, x_2, \ldots, x_n, i) \\ &= \sum_{i=0}^{n+1} \prod_{j=0}^{i} (\neg p(x_1, x_2, \ldots, x_n, j)) \end{aligned}$$

is a primitive recursive function. It can be shown by induction on k that if there exists an integer k such that $0 \leq k \leq x_{n+1}$ and $p(x_1, x_2, \ldots, x_n, k) = 1$, then the value of this function is the *smallest* integer for which this is true.

Since p is primitive recursive it follows by Claim 3 that the predicate $\underset{0 \leq i \leq x_{n+1}}{\exists} p$ is primitive recursive as well.

This can be used, with the above, to prove that the function $g : \mathbb{N}^{n+1} \to \mathbb{N}$ such that, for $x_1, x_2, \ldots, x_{n+1} \in \mathbb{N}$,

$$\begin{aligned} &g(x_1 x_2, \ldots, x_{n+1}) \\ &\quad = \begin{cases} f(x_1, x_2, \ldots, x_{n+1}) & \text{if } \underset{0 \leq i \leq x_{n+1}}{\exists} p(x_1, x_2, \ldots, x_{n+1}) = 1, \\ 0 & \text{if } \underset{0 \leq i \leq x_{n+1}}{\exists} p(x_1, x_2, \ldots, x_{n+1}) = 0 \end{cases} \end{aligned}$$

is primitive recursive. It now suffices to note that this is the same as the function $\underset{0 \leq i \leq x_{n+1}}{\textit{first}} p$ in order to complete the proof. $\square$