Lecture #5: Encodings — Sequences and Strings
Exercises and Review

Questions for Review

1. Give ***definitions*** of each of the following functions.

   (a) The ***pairing function*** $\langle \cdot, \cdot \rangle : \mathbb{N}^2 \to \mathbb{N}$
   (b) The *first* inverse $\ell : \mathbb{N} \to \mathbb{N}$ of the pairing function
   (c) The *second* inverse $r : \mathbb{N} \to \mathbb{N}$ of the pairing function
   (d) The encoding function $\langle \cdot, \dots, \cdot \rangle_k : \mathbb{N}^k \to \mathbb{N}$ for k-tuples
   (e) The inverse function $\rho_i^k : \mathbb{N} \to \mathbb{N}$ for the above encoding function, where $1 \le i \le k$
   (f) The *Gödel encoding function* $[\cdot, \dots, \cdot] : \mathbb{N}^\star \to \mathbb{N}$
   (g) The *inverse* function $()_i : \mathbb{N} \to \mathbb{N}$ — mapping x to $(x)_i$, for $x, i \in \mathbb{N}$
   (h) The encoding of a string $\omega \in \Sigma^\star$ where Σ is an alphabet with size n

2. Let Σ be an alphabet with size n, for a positive integer n. For each of the following functions of ***strings*** over the alphabet Σ,

   - describe, in English, the computation performed by this function, and
   - describe, as precisely as you can, a corresponding function $\varphi : \mathbb{N} \to \mathbb{N}$ (or a function $\varphi : \mathbb{N}^2 \to \mathbb{N}$ if the original function is a function of a pair of strings).

   (a) The function $f : \Sigma^\star \to \mathbb{N}$ such that $f(\omega) = |\omega|$ for all $\omega \in \Sigma^\star$.
   (b) The function concat : $\Sigma^\star \times \Sigma^\star \to \Sigma^\star$
   (c) The function rtend : $\Sigma^\star \to \Sigma^\star$
   (d) The function rtrunc : $\Sigma^\star \to \Sigma^\star$
   (e) The function ltend : $\Sigma^\star \to \Sigma^\star$
   (f) The function ltrunc : $\Sigma^\star \to \Sigma^\star$

   Are these functions computable? Are they primitive recursive?

3. Let n_1 and n_2 be positive integers such that $n_1 < n_2$. Describe, as precisely as you can, the functions $\text{upchange}_{n_1,n_2} : \mathbb{N} \to \mathbb{N}$ and $\text{downchange}_{n_2,n_1} : \mathbb{N} \to \mathbb{N}$. What computations, concerning strings over alphabets $\Sigma_1^\star$ and $\Sigma_2^\star$, with sizes n_1 and n_2 respectively, do these functions correspond to? Are these functions computable? Are they primitive recursive?

Practice Problems

These problems will not be discussed during the lecture, and solutions for these problems will not generally be made available. They can be used as "practice" problems that can help you to practice skill considered in the lecture presentation for Lecture #5.

Ice Breakers

These problems can be used to check that you understand the definitions of the encoding functions (and other related functions) that were introduced in this lecture.

1. Compute each of the following values.
   (a) $\langle 0, 0\rangle$
   (b) $\langle 1, 0\rangle$
   (c) $\langle 0, 1\rangle$
   (d) $\langle 2, 0\rangle$
   (e) $\langle 1, 1\rangle$
   (f) $\langle 0, 2\rangle$

2. Compute each of the following values.
   (a) $\ell(0)$
   (b) $\ell(1)$
   (c) $\ell(2)$
   (d) $\ell(3)$
   (e) $\ell(4)$
   (f) $r(0)$
   (g) $r(1)$
   (h) $r(2)$
   (i) $r(3)$
   (j) $r(4)$

3. Compute each of the following values.

   (a) $\langle 1, 2, 3 \rangle_3$
   (b) $\rho_1^3(3583)$
   (c) $\rho_2^3(3583)$
   (d) $\rho_3^3(3583)$

4. Compute each of the following values.

   (a) $[1, 2, 3, 4]$
   (b) Lt$(5, 402, 250)$
   (c) $(5, 402, 250)_1$
   (d) $(5, 402, 250)_2$
   (e) $(5, 402, 250)_3$
   (f) $(5, 402, 250)_4$
   (g) $(5, 402, 250)_5$

5. Consider the alphabet $\Sigma = \{\sigma_1, \sigma_2, \sigma_3\}$ where $\sigma_1 =$ "a", $\sigma_2 =$ "b", and σ_3 = "c".

   Compute #(`abba`).

6. Compute each of the following values.

   (a) upchange$_{3,5}(55)$
   (b) downchange$_{5,3}(196)$.

A More Challenging Problem — Addressing a Problem with Gödel Numbers

As the lecture notes note, there is a problem that arises when Gödel numbers are used to encode finite sequences — multiple sequences have the same Gödel number. This problem introduces a related encoding scheme that eliminates this.

7. Recall that the ***Gödel number*** of a finite sequence

$$\alpha_1, \alpha_2, \dots, \alpha_k \tag{1}$$

   of integers (for a non-negative integer k) is the product

$$\prod_{i=1}^{k} p_i^{\alpha_i}$$

   where p_i is the i^{th} prime for every positive integer i (so that $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, and so on). Thus the Gödel number of the empty sequence (with length 0 is 1.

Unfortunately, multiple finite sequences have the same Gödel number. In particular, the sequence

$$\alpha_1, \alpha_2, \ldots, \alpha_k, 0 \tag{2}$$

has the same Gödel number as the sequence shown at line (1).

In order to deal with this problem, suppose that we "encode" finite sequences a little bit differently:

- Let us encode the empty sequence by the natural number 1.
- For a positive integer ℓ, let us encode a finite sequence of natural numbers, with length ℓ, by the product of p_ℓ and the Gödel number for that sequence. Thus the sequence at line (1) would be encoded by the natural number

$$\left(\prod_{i=1}^{k-1} p_i^{\alpha_i}\right) \cdot p_k^{\alpha_k+1}$$

while the sequence at line (2) would be encoded by the natural number

$$\left(\prod_{i=1}^{k-1} p_i^{\alpha_i}\right) \cdot p_k^{\alpha_k} \cdot p_{k+1}$$

instead.

(a) Show that, if this new encoding scheme is used, then 0 does not encode any finite sequence, but every positive natural number encodes *exactly one* finite sequence.

(b) Confirm that, for every positive integer k, the function mapping a finite sequence with with length k, to the natural number that encodes it, is a primitive recursive function.

(c) Consider a function $\widehat{\mathsf{Lt}} : \mathbb{N} \to \mathbb{N}$ such that $\widehat{\mathsf{Lt}}(0) = \widehat{\mathsf{Lt}}(1) = 0$ and such that if $x \in \mathbb{N}$ and $x \geq 2$, then $\widehat{\mathsf{Lt}}(x)$ is the largest natural number k such that x is divisible by p_k. Show that the function $\widehat{\mathsf{Lt}}$ is a primitive recursive function such that, for every positive integer x, $\widehat{\mathsf{Lt}}(x)$ is the length of the finite sequence that is encoded by x.

(d) Consider functions $\widehat{(\cdot)}_i : \mathbb{N} \to \mathbb{N}$ that are defined as follows.

- $\widehat{(x)}_0 = 1$ for all $x \in \mathbb{N}$.
- $\widehat{(0)}_i = \widehat{(1)}_i = 0$ for every positive integer i.
- For every integer x such that $x \geq 2$, and for every integer i such that $i > \widehat{\mathsf{Lt}}(x)$, $\widehat{(x)}_i = 0$.

- For every integer x such that $x \geq 2$, the natural numbers $\widehat{(x)}_1, \widehat{(x)}_2, \ldots, \widehat{(x)}_{\widehat{\mathsf{Lt}}(x)}$ are the unique natural numbers such that the finite sequence

  $$\widehat{(x)}_1, \widehat{(x)}_2, \ldots, \widehat{(x)}_{\widehat{\mathsf{Lt}}(x)}$$

  is encoded by x.

Describe — as precisely as you can — how the functions $(\cdot)_i$ and $\widehat{(\cdot)}_i$ are related (they are *not*, quite, identical). Then sketch a proof that the function $\widehat{(\cdot)}_i : \mathbb{N} \to \mathbb{N}$ is primitive recursive, for every natural number i.