Lecture #5: Encodings — Sequences and Strings
Lecture Presentation

Problems To Be Solved

Encoding and Working with Integers

Encoding

Elements of the set $\mathbb{Z}$ of integers can be encoded, as natural numbers, as follows.

- The integer 0 can be encoded by the natural number 0.
- Every positive integer i can be encoded by the natural number $2i - 1$.
- For every positive integer i, the negative integer i can be encoded by the natural number $2i$.

Thus 0 is encoded by 0, 1 is encoded by 1, -1 is encoded by 2, 2 is encoded by 3, and -2 is encoded by 4.

Exercise: Show that every natural number $n \in \mathbb{N}$ is the encoding of *exactly one* integer x.

Determining the Sign of an Integer

Consider a function *sgn* : $\mathbb{N} \to \mathbb{N}$ that is defined as follows: For $n \in \mathbb{N}$, let $x \in \mathbb{Z}$ be the integer encoded by n. Then *sgn*$(n) = 1$ if $x \geq 0$, and *sgn*$(n) = 0$ if $x < 0$.

Proof that sgn is a Primitive Recursive Function:

Computing the Absolute Value of an Integer

Consider a function *abs* : $\mathbb{N} \to \mathbb{N}$ that is defined as follows: For $n \in \mathbb{N}$, let $x \in \mathbb{Z}$ be the integer encoded by n. Then abs$(n) = |x|$ — that is, the non-negative integer $|x|$, and not the (generally larger) non-negative integer that encodes it.

Proof that abs is a Primitive Recursive Function:

Mapping from a Natural Number to its Encoding

Consider a function *pEnc* : $\mathbb{N} \to \mathbb{N}$ that is defined as follows: For all $n \in \mathbb{N}$, let $m \in \mathbb{N}$ be the natural number that encodes n (when n is considered as an integer). Then *pEnc*$(n) = m$.

Proof that pEnc is a Primitive Recursive Function:

Mapping from a Natural Number to the Encoding of its Negation

Consider a function *nEnc* : $\mathbb{N} \to \mathbb{N}$ that is defined as follows: For all $n \in \mathbb{N}$, let $\widehat{m} \in \mathbb{N}$ be the natural number that encodes the negation, $-n$ of n. Then *nEnc*$(n) = \widehat{m}$.

Proof that nEnc is a Primitive Recursive Function:

Integer Addition

Consider a function *add*$_{\mathbb{Z}}$: $\mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ that is defined as follows: For $m, n \in \mathbb{N}$, if m and n encode integers x and y, respectively, then *add*$_{\mathbb{Z}}(m, n)$ is the natural number that encodes the sum, $x + y$. of x and y.

What is This When $x \geq 0$ and $y \geq 0$?

What is This When $x \geq 0$, $y < 0$, and $|x| \geq |y|$?

What is This When $x \geq 0$, $y < 0$, and $|x| < |y|$?

Other Useful Cases:

Proof That $add_{\mathbb{Z}}$ is Primitive Recursive:

Encoding and Working with Arrays of Integers

Encoding

Useful Functions — That are Primitive Recursive

Encoding and Working with Dynamic Arrays (ArrayLists) of Integers

Why Gödel Numbering isn't (Quite) Enough for an Encoding

Encoding

Useful Functions — That are Primitive Recursive