Lecture #11: Reducibilities — and Many-One Reductions
Exercises and Review

Questions for Review

Reducibilities

1. What is a ***reducibility***?
2. Suppose that "$\preceq_Q$" is a reducibility and that $\mathcal{W}$ is a collection of subsets of $\mathbb{N}$. Say what it means for $\mathcal{W}$ to be ***Q-closed***.

Many-One Reductions

3. Define a ***many-one reduction*** from a set $A \subseteq \mathbb{N}$ to another set $B \subseteq \mathbb{N}$.
4. Sketch a proof that the set of many-one reductions, between subsets of $\mathbb{N}$, forms a ***reducibility***.
5. Sketch a proof that if $A, B \subseteq \mathbb{N}$, $A \preceq_m B$, and B is computable, then A is computable too — so that the collection of computable subsets of $\mathbb{N}$ is "m-closed".
6. Sketch a proof that if $A, B \subseteq \mathbb{N}$, $A \preceq_m B$, and B is computably enumerable, then A is computably enumerable too — so that the collection of computably enumerable subsets of $\mathbb{N}$ is also "m-closed".
7. Describe processes, involving many-one reductions, that one can use
   (a) to prove that a given set $S \subseteq \mathbb{N}$ is ***not*** computable, and
   (b) to prove that a given set $S \subseteq \mathbb{N}$ is ***not*** computably enumerable.
8. Describe at least one common mistake that students make when using reductions to prove that sets are not computable, or not computably enumerable — that you should watch for and avoid.

Completeness and Equivalence

9. Let "$\preceq_Q$" be a reducibility and let $\mathcal{W}$ be a collection of subsets of $\mathbb{N}$ that is Q-closed.

   Say — as precisely as you can — what it means for a set $A \subseteq \mathbb{N}$ to be ***Q-complete*** for $\mathcal{W}$.

10. Describe two subsets of $\mathbb{N}$ that are m-complete for the collection of computably enumerable subsets of $\mathbb{N}$.

11. Let "$\preceq_Q$" be a reducibility, and let $\mathcal{W}_1$ and $\mathcal{W}_2$ be collections of subsets of $\mathbb{N}$ such that $\mathcal{W}_1$ is Q-closed, and such that $\mathcal{W}_1 \subsetneq \mathcal{W}_2$.

    Let $A \subseteq \mathbb{N}$ such that A is Q-complete for $\mathcal{W}_2$. Can A belong to $\mathcal{W}_1$?

12. Let "$\preceq_Q$" be a reducibility, and let $\mathcal{W}$ be a collection of subsets of $\mathbb{N}$ that is Q-closed. Describe a way to use one set $A \subseteq \mathbb{N}$, that is Q-complete for $\mathcal{W}$, to prove that another set, $B \subseteq \mathbb{N}$, is Q-complete for $\mathcal{W}$, as well.

13. Let "$\preceq_Q$" be a reducibility. Say — as precisely as you can — what it means for a pair of sets $A, B \subseteq \mathbb{N}$ to be ***Q-equivalent***.

Practice Problems

These problems — which continue activities in the lecture presentation — will not be discussed during the lecture, and solutions for these problems will not generally be made available. They can be used as "practice" problems that can help you to practice skill considered in the lecture presentation for Lecture #11.

1. Consider the functions

   $$\text{Total} = \{n \in \mathbb{N} \mid \Phi_n \text{ is total}\}$$

   and

   $$\text{Monotone} = \{n \in \mathbb{N} \mid \Phi_n \text{ is total and } \Phi_n(x) \leq \Phi_n(x+1) \text{ for all } x \in \mathbb{N}\}.$$

   (a) For each natural number $n \in \mathbb{N}$, let $p(n)$ be the index of the $\mathcal{S}$-program obtained from the following after macro expansion:

   $$\begin{array}{ll} & Z_1 \leftarrow \Phi^{(1)}(X_1+1, n) \\ & Z_2 \leftarrow \Phi^{(1)}(X_1, n) \\ & Z_3 \leftarrow Z_2 - Z_1 \\ [\mathrm{A}_1] & \text{IF } Z_3 \neq 0 \text{ GOTO } \mathrm{A}_1 \end{array}$$

   Use the Parameter Theorem to give a reasonably ***short*** proof that the function $p : \mathbb{N} \to \mathbb{N}$, defined above, is a computable total function.

   (b) Use the above to prove that Monotone $\preceq_m$ Total.