Lecture #12: Rice's Theorem and the Recursion Theorem
Lecture Presentation

A Generalized Fixed Point Theorem

Consider the following.

Theorem (Generalized Fixed Point Theorem)**.** *Let $f : \mathbb{N} \to \mathbb{N}$ be a Turing-computable total function and let k be a positive integer. Then there exists a number $e \in \mathbb{N}$ such that*

$$\Phi^{(k)}_{f(e)}(x_1, x_2, \ldots, x_k) = \Phi^{(k)}_{e}(x_1, x_2, \ldots, x_k)$$

for all $x_1, x_2, \ldots, x_k \in \mathbb{N}$ — that is, the $\mathcal{S}$-program with encoding $f(e)$ computes the same k-ary partial or total function as the $\mathcal{S}$-program with encoding e.

How This Generalizes the Fixed Point Theorem

Proof of This Result

Application: Proving That a Function is Computable

Consider a function $f : \mathbb{N}^2 \to \mathbb{N}$ that satisfies the following equations.

(a) $f(x, 0) = x + 2$ for all $x \in \mathbb{N}$.

(b) $f(x, 1) = 2 \times f(x, 2x)$ for all $x \in \mathbb{N}$.

(c) $f(x, 2t + 2) = 3 \times f(x, 2t)$ for all $x, t \in \mathbb{N}$.

(d) $f(x, 2t + 3) = 4 \times f(x, 2t + 1)$ for all $x, t \in \mathbb{N}$.

Proving That $f(x, y)$ is Well-Defined When y is Even:

Proving That $f(x, y)$ is Well-Defined When y is Odd:

Finding an $\mathcal{S}$-Program for f:

Consider the function $p : \mathbb{N} \to \mathbb{N}$ such that, for every non-negative integer n, $p(n)$ is the encoding of the $\mathcal{S}$-program obtained, after macro expansion, from a program that starts with

$$X_1 \leftarrow n$$

and then continues with the following one — which does not depend on n, at all:

$$
\begin{array}{ll}
 & Z_2 \leftarrow X_2 \\
 & \text{IF } Z_2 \neq 0 \text{ GOTO } A_1 \\
 & Y \leftarrow X_1 \\
 & Y \leftarrow Y + 1 \\
 & Y \leftarrow Y + 1 \\
 & \text{GOTO } B_1 \\
[A_1] & Z_2 \leftarrow X_2 - 1 \\
 & \text{IF } Z_2 \neq 0 \text{ GOTO } A_2 \\
 & Z_3 \leftarrow 2 \times X_1 \\
 & Y \leftarrow \Phi^{(2)}(X_1, Z_3, Z_1) \\
 & Y \leftarrow 2 \times Y \\
 & \text{GOTO } B_1 \\
[A_2] & Z_2 \leftarrow X_2 \bmod 2 \\
 & \text{IF } Z_2 \neq 0 \text{ GOTO } A_3 \\
 & Z_3 \leftarrow X_2 - 2 \\
 & Y \leftarrow \Phi^{(2)}(X_1, Z_3, Z_1) \\
 & Y \leftarrow 3 \times Y \\
 & \text{GOTO } B_1 \\
[A_3] & Z_3 \leftarrow X_2 \dot{-} 2 \\
 & Y \leftarrow \Phi^{(2)}(X_1, Z_3, Z_1) \\
 & Y \leftarrow 4 \times Y \\
[B_1] & Y \leftarrow Y
\end{array}
$$

What do we get, when we apply the “Generalized Fixed Point Theorem”?

Conclusions (about the “Recursion Theorem”)