Rice's Theorem and the Recursion Theorem
Proof of the Recursion Theorem

Theorem (Recursion Theorem). *Let $g : \mathbb{N}^{m+1} \to \mathbb{N}$ be a computable partial (or total) function. Then there exists an number $e \in \mathbb{N}$ such that, for all $x_1, x_2, \ldots, x_n \in \mathbb{N}$,*

$$\Phi_e^{(m)}(x_1, x_2, \ldots, x_m) = g(e, x_1, x_2, \ldots, x_m)$$

Proof. Consider the partial (or total) function $h : \mathbb{N}^{m+1} \to \mathbb{N}$ such that, for all $v, x_1, x_2, \ldots, x_m \in \mathbb{N}$,

$$h(x_1, x_2, \ldots, x_m, v) = g(S_m^1(v, v), x_1, x_2, \ldots, x_m).$$

Note that, since the function $S_1^1 : \mathbb{N}^2 \to \mathbb{N}$ is primitive recursive (and, therefore, a Turing-computable total function), h is a Turing-computable partial function, since g is.

It follows that this is computed by some $\mathcal{S}$-program $\mathcal{P}$. Let $z_0 = \#(\mathcal{P})$. Then, for all $v \in \mathbb{N}$,

$$\begin{aligned} g(S_m^1(v, v), x_1, x_2, \ldots, x_n) &= h(x_1, x_2, \ldots, x_m, v) \\ &= \Phi^{(m+1)}(x_1, x_2, \ldots, x_m, v, z_0) \\ &= \Phi^{(m)}(x_1, x_2, \ldots, x_m, S_m^1(v, z_0)). \end{aligned}$$

Now let $v = z_0$ and let $e = S_m^1(z_0, z_0)$. Then it follows that

$$\begin{aligned} g(e, x_1, x_2, \ldots, x_n) &= g(S_m^1(z_0, z_0), x_1, x_2, \ldots, x_m) && \text{(since } e = S_m^1(z_0, z_0)\text{)} \\ &= \Phi^{(m)}(x_1, x_2, \ldots, x_m, S_m^1(v, z_0)) && \text{(as shown above)} \\ &= \Phi^{(m)}(x_1, x_2, \ldots, x_m, e) && \text{(since } S_m^1(v, z_0) = S_m^1(z_0, z_0) = e\text{)} \\ &= \Phi_e^{(m)}(x_1, x_2, \ldots, x_m) \end{aligned}$$

as claimed. $\square$