Lecture #13: A Computable Total Function That is Not Primitive Recursive

Questions for Review

1. Describe an ***alternative definition*** (or "characterization") of the set of ***primitive recursive functions*** that is provably equivalent to the definition given at the beginning of this course.

2. Describe an ***alternative definition*** (or "characterization") of the set of ***unary primitive recursive functions***.

3. Describe an ***enumeration*** of the set of all unary primitive recursive functions that can be obtained, using the above alternative definition of them.

4. This was used to define a pair of functions $\xi : \mathbb{N}^2 \to \mathbb{N}$ and $\rho : \mathbb{N}^2 \to \mathbb{N}$. Describe these functions as precisely as you can.

5. Describe, as precisely as you can, how both the ***Universality Theorem*** and the ***Recursion Theorem*** were used to prove that the above functions are both computable.

6. Describe how the ***Diagonalization method*** was then used to establish that ρ is a computable total function is not primitive recursive.

7. Describe the ***Ackerman function***. Explain — informally — why *it* is not primitive recursive, either.

Problems To Be Solved

1. Consider the function $f : \mathbb{N} \to \mathbb{N}$ such that, for all $x \in \mathbb{N}$,

$$f(x) = \ell(x) + r(x).$$

   This function can be shown to be ***primitive recursive*** as follows.

(a) The function $f_1 : \mathbb{N} \to \mathbb{N}$ such that, for all $x \in \mathbb{N}$,

$$f_1(x) = u_1^1(x) = x$$

is primitive recursive because it is one of the initial functions (in particular, one of the projection functions).

(b) The function $f_2 : \mathbb{N}^3 \to \mathbb{N}$ such that, for all $t, z, x \in \mathbb{N}$,

$$f_2(t, z, x) = z + 1 = s(u_2^3(t, z, x))$$

is also primitive recursive, because it is obtained from initial functions (namely, the successor function and a projection function) using *composition*.

(c) Consider the function $f_3 : \mathbb{N}^2 \to \mathbb{N}$ obtained from f_1 and f_2 using the second form of *primitive recursion*: This function is primitive recursive as well. Now, for $x \in \mathbb{N}$,

$$f_3(x, 0) = f_1(x) = x$$

and, for every integer $t \in \mathbb{N}$,

$$f_3(x, t + 1) = g(t, f(x, t), x) = f(x, t) + 1.$$

It is easily shown, by induction on y, that, for all $x, y \in \mathbb{N}$,

$$f_3(x, y) = x + y.$$

(d) Finally, it suffices to note that if $f : \mathbb{N} \to \mathbb{N}$ is as given above then

$$f(x) = f_3(\ell(x), r(x))$$

so that f is primitive recursive functions because it is obtained from the primitive functions f_3, ℓ and r by primitive recursion.

Note: This construction is not, quite, complete, because it does not include the constructions to show that the functions $\ell, r : \mathbb{N} \to \mathbb{N}$ are primitive recursive. See Lecture #5 for the constructions needed to establish this.

During Lecture #12 an *alternative* definition of the primitive recursive functions (which included more “initial” functions and composition, but only allowed a special form of primitive recursion) was given. Show that the above function f could also be constructed using the expanded set of initial functions, included in Theorem #1 from the lecture notes, and using a finite number of applications of composition and this new form of primitive recursion.