A Computable Total Function That is Not Primitive Recursive

Proofs of Claims

This document includes proofs of claims that are stated, but not proved, in the notes for Lecture #13.

Alternative Characterization

The beginning of the lecture notes is devoted to a proof of the following.

Theorem 1. *Let $f : \mathbb{N}^n \to \mathbb{N}$. Then f is primitive recursive if and only if it can be obtained from the (enriched) set of initial functions —*

(a) The successor function $s : \mathbb{N} \to \mathbb{N}$ *such that* $s(x) = x + 1$ *for all* $x \in \mathbb{N}$*.*

(b) The nullify function $n : \mathbb{N} \to \mathbb{N}$ *such that* $n(x) = 0$ *for all* $x \in \mathbb{N}$*.*

(c) The left and right inverses $\ell, r : \mathbb{N} \to \mathbb{N}$ *of the pairing function.*

(d) The pairing function *mapping* $x, y \in \mathbb{N}$ *to* $\langle x, y \rangle$*.*

(e) The projection functions $u_i^n : \mathbb{N}^n \to \mathbb{N}$ *such that, for* $1 \leq i \leq n$ *and all* $x_1, x_2, \ldots, x_n \in \mathbb{N}$*,*

$$u_i^n(x_1, x_2, \ldots, x_n) = x_i.$$

— using the operations **composition** *and the* **new version of primitive recursion** *that has now been defined.*

This is proved using several lemmas that are stated and proved below.

Lemma 2. *Let $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$ be primitive recursive functions, and let $h : \mathbb{N}^2 \to \mathbb{N}$ be the function obtained from f and g using the new version of primitive recursion. Then h is primitive recursive.*

Proof. Suppose, as given, that the functions $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$ are primitive recursive, and let $h : \mathbb{N} \to \mathbb{N}$ be the function obtained from f and g using the new version of primitive recursion, given in the lecture notes — so that

$$h(x, 0) = f(x)$$

and

$$h(x, t+1) = g(h(x, t))$$

for all $x, t \in \mathbb{N}$.

Let $\widehat{g} : \mathbb{N}^3 \to \mathbb{N}$ be the function obtained from the ***projection function*** $u_2^3 : \mathbb{N}^3 \to \mathbb{N}$ and the given function $g : \mathbb{N} \to \mathbb{N}$ using ***composition***: For all $t, y, x \in \mathbb{N}$,

$$\widehat{g}(t, y, x) = g(u_2^3(t, y, x)) = g(y).$$

Then $\widehat{g}$ is primitive recursive since u_2^3 and g are.

Now let $\widehat{h} : \mathbb{N}^2 \to \mathbb{N}$ be the function obtained from f and $\widehat{g}$ using the originally given (second) form of primitive recursive. Then $\widehat{h}$ is also primitive recursive, since f and $\widehat{g}$ are. Applying the definition of the second form of primitive recursion, one obtains the following.

- For all $x \in \mathbb{N}$, $\widehat{h}(x, 0) = f(x) = h(x, 0)$.
- For all $x, t \in \mathbb{N}$, $\widehat{h}(x, t+1) = \widehat{g}(t, \widehat{h}(x, t), t+1) = g(\widehat{h}(x, t)$.

On the other hand, it follows by the definition of the new version of primitive recursion that

- For all $x, t \in \mathbb{N}$, $h(x, t+1) = g(h(x, t))$ as well.

It is easily improved by induction on y that $\widehat{h}(x, y) = h(x, y)$ for all $x, y \in \mathbb{N}$ — so that h is primitive recursive, because $\widehat{h}$ is, and they are the same function. ☐

Lemma 3. *Let $f : \mathbb{N}^k \to \mathbb{N}$ for a positive integer k. If f can be constructed from the "expanded set" of initial functions, listed in parts (a)–(e) of Theorem 1, above, using a finite number of applications of composition and the new version of primitive recursion, then f is primitive recursive.*

Proof. Let $f : \mathbb{N}^k \to \mathbb{N}$ (for a positive integer k) such that f can be constructed from the "expanded set" of initial functions in parts (a)–(e) of Theorem 1, above, using a finite number of applications of composition and the new version of primitive recursion. Then there exists an integer $\ell \geq 1$ and a sequence of functions

$$f_1, f_2, \ldots, f_\ell$$

such that the following properties are satisfied.

- For $1 \leq i \leq \ell$ either f_i is one of the expanded set of initial functions in parts (a)–(e) o of Theorem 1, above, or f_i is obtained from one or more of the functions $f_1, f_2, \dots, f_{i-1}$ using either composition or the new version of primitive recursion.
- $f_\ell = f$.

Consider the following

> ***Subclaim:*** The function f_i is primitive recursive, for every integer i such that $1 \leq i \leq \ell$.

The subclaim can be proved by induction on i, using the strong form of mathematical induction, as follows.

*Basis:*If $i = 1$ then f_i must be one of the extended set of initial functions, in parts (a)–(e) of Theorem 1 — because the sequence $f_1, f_2, \dots, f_{i-1}$ is empty and there are no functions to which either composition or the new version of primitive recursion can be applied. Thus f_1 is either the successor function, the nullify function, either the left or right inverse of the pairing function, the pairing function. or a projection function — that is, either one of the initial functions included in the definition of the set of primitive recursive function, or a function proved to be primitive recursive in a previous lecture. Thus f_1 is primitive recursive, as required.

Inductive Step: Let h be an integer such that $1 \leq h \leq \ell - 1$.[1] It is necessary and sufficient to use the following

Inductive Hypothesis: f_i is primitive recursive, for $1 \leq i \leq h$.

to prove the following

Inductive Claim: f_{h+1} is primitive recursive.

Now either

(a) f_{h+1} is one of the expanded set of initial functions in parts (a)–(e) of Theorem 1,

(b) f_{h+1} has been produced using one or more of $f_1, f_2, \dots, f_h$ using composition, or

(c) f_{h+1} has been produced using one or more of $f_1, f_2, \dots, f_h$ using the new version of primitive recursion.

Each of these cases is considered separately, below.

Case: f_{h+1} is one of the expanded set of initial functions in parts (a)–(e) in Theorem 1. Then the argument used in the basis can be applied here, as well, to establish that f_{h+1} is primitive recursive.

[1] The claim is vacuous if $h \geq \ell$, since a function f_{h+1} is not defined in that case.

Case: f_{h+1} has been produced using one or more of $f_1, f_2, \dots, f_h$ using composition. In this case there exists a positive integer ℓ and functions $\varphi : \mathbb{N}^\ell \to \mathbb{N}$ and $\xi_1, \xi_2, \dots, \xi_\ell : \mathbb{N}^k \to \mathbb{N}$ such that $\varphi, \xi_1, \xi_2, \dots, \xi_\ell$ are each one of the functions $f_1, f_2, \dots, f_k$, and such that

$$f_{h+1}(x_1, x_2, \dots, x_\ell) = \varphi(\xi_1(x_1, x_2, \dots, x_k), \xi_2(x_1, x_2, \dots, x_k), \dots, \xi_\ell(x_1, x_2, \dots, x_k))$$

for all $x_1, x_2, \dots, x_k \in \mathbb{N}$.

Now, since $\varphi, \xi_1, \xi_2, \dots, \xi_\ell$ are each one of $f_1, f_2, \dots, f_k$ it follows by the Inductive Hypothesis that $\varphi, \xi_1, \xi_2, \dots, \xi_\ell$ are each primitive recursive and, since the set of primitive recursive functions is (by definition) closed under composition, it follows that f_{h+1} is primitive recursive too.

Case: f_{h+1} has been produced using one or more of $f_1, f_2, \dots, f_h$ using the new version of primitive recursion. In this case $k = 2$ and there exist functions $\varphi, \xi : \mathbb{N} \to \mathbb{N}$ such that φ and ξ are each one of $f_1, f_2, \dots, f_h$ and such that f_{h+1} is formed from φ and ξ using the new version of primitive as described (with φ replacing f and ξ replacing g) above.

Since φ and ξ are each one of $f_1, f_2, \dots, f_h$ it follows by the Inductive Hypothesis that φ and ξ are each primitive recursive functions and, since f_{h+1} is obtained from these functions using the new version of primitive recursion, it follows by Lemma 2, above, that f_{h+1} is primitive recursive in this case as well.

Since h_{h+1} is primitive recursive in every case, this establishes the Inductive Claim and completes the Inductive Step. The subclaim now follows by induction on i.

The main claim is a consequence of the subclaim, since we can set i to be ℓ, and since $f_\ell = f$. ☐

Claims similar to Lemma 2 can be established and used to prove a converse for the above result:

Lemma 4. *Let $c \in \mathbb{N}$, let $g : \mathbb{N}^2 \to \mathbb{N}$, and let $h : N \to \mathbb{N}$ such that h can be constructed from c and g using the first form of primitive recursion. Then h can be constructed from g and the extended set of initial functions in parts (a)–(e) of Theorem 1 using a finite number of applications of composition and the new version of primitive recursion.*

Proof. Let $c \in \mathbb{N}$ and let $g : \mathbb{N}^2 \to \mathbb{N}$.

- Since the nullify function $n : \mathbb{N} \to \mathbb{N}$ and the successor function $s : \mathbb{N} \to \mathbb{N}$ are in the extended set of initial functions (given in parts (b) and (a) of Theorem 1, respectively), it is easy to show by induction on ℓ that a constant function $\widehat{f_\ell} : \mathbb{N} \to \mathbb{N}$ such that $\widehat{f_\ell}(x) = \ell$, for all $x \in \mathbb{N}$, can be constructed from the extended set of initial functions using a finite number of applications of composition (and the new version of primitive recursion).

In particular, since $\langle 0, c\rangle$ is a constant if c is, this is true for the function $\widehat{f}_{\langle 0,c\rangle} : \mathbb{N} \to \mathbb{N}$ such that $\widehat{f}_{\langle 0,c\rangle}(x) = \langle 0, c\rangle$ for all $x \in \mathbb{N}$.

- Since the pairing function and the inverses of the pairing function are also in the extended set of initial functions (given in parts (d) and (c) of Theorem 1), the function $\widehat{g} : \mathbb{N} \to \mathbb{N}$ such that, for all $x \in \mathbb{N}$,
$$\widehat{g}(x) = \langle s(\ell(x)), g(\ell(x), r(x))\rangle$$
can be constructed from g and the extended set of initial functions, using a finite number of applications of composition (and the new version of primitive recursion). Note that if $t, y \in \mathbb{N}$, and $z = \langle t, y\rangle$, then
$$\begin{aligned}\widehat{g}(z) &= \langle s(\ell(z), g(\ell(z), r(z))\langle \\ &= \langle s(t), g(t, y)\rangle \\ &= \langle t+1, g(y)\rangle.\end{aligned}$$

- Consider the function $h : \mathbb{N} \to \mathbb{N}$ constructed from c and g using the (standard) first form of primitive recursion:
$$h(0) = c \tag{1}$$
and, for all $t \in$,
$$h(t+1) = g(t, h(t)). \tag{2}$$

- Consider, as well, the function $\widehat{h} : \mathbb{N}^2 \to \mathbb{N}$ constructed from $\widehat{f}_{\langle 0,c\rangle}$ and $\widehat{g}$ using the new version of primitive recursion. Since $\widehat{f}_{\langle 0,c\rangle}$ and $\widehat{g}$ can each be constructed from g and the extended set of initial functions using a finite number of applications of composition and the new version of primitive recursion, this is certainly true for $\widehat{h}$ as well. Now, it follows by the definition of the new version of primitive recursion that, for all $x \in \mathbb{N}$,
$$\widehat{h}(x, 0) = \widehat{f}_{\langle 0,c\rangle}(x) = \langle 0, c\rangle = \langle 0, h(0)\rangle \tag{3}$$
and, for all $x, t \in \mathbb{N}$,
$$\widehat{h}(x, t+1) = \widehat{g}(\widehat{h}(x, t)).$$
Note that it follows from the above that if $\widehat{h}(x, t) = \langle t, h(t)\rangle$ then
$$\begin{aligned}\widehat{h}(x, t+1) &= \widehat{g}(\widehat{h}(x, t)) \\ &= \widehat{g}(\langle t, h(t)\rangle) \\ &= \langle t+1, g(t, h(t))\rangle \\ &= \langle t+1, h(t+1)\rangle\end{aligned}$$
by equation (2), above.

- This observation, and the equation at line (3), above, can be used to show by induction on y that
$$\widehat{h}(x, y) = \langle y, h(y) \rangle$$
for all $x, y \in \mathbb{N}$.
- Since the projection functions are in the extended set of initial functions (given in part (e) of Theorem 1), it now suffices to note that if $\widetilde{h} : \mathbb{N} \to \mathbb{N}$ such that
$$\widetilde{h}(x) = r(\,\widehat{h}(u_1^1(x), u_1^1(x)))$$
then $\widetilde{h}$ can also be constructed from g and the extended set of initial functions using a finite number of applications of composition and the new version of primitive recursion, since $\widehat{h}$ can be constructed in this way. Now, for all $x \in \mathbb{N}$,
$$\begin{aligned}\widetilde{h}(x) &= r(\,\widetilde{h}(u_1^1(x), u_1^1(x)))\\ &= r(\,\widetilde{h}(x, x))\\ &= r(\langle x, h(x) \rangle)\\ &= h(x),\end{aligned}$$
so that $\widetilde{h}$ and h are the same function, establishing the claim. $\square$

Lemma 5. *Let k be a positive integer, let $f : \mathbb{N}^k \to N$, let $g : \mathbb{N}^{k+2} \to \mathbb{N}$, and let $h : \mathbb{N}^{k+1} \to \mathbb{N}$ be the function obtained from f and g using the second (standard) form of primitive recursion. Then h can be obtained from f, g, and the extended set of initial functions (given in parts (a)–(e) of Theorem 1) using a finite number of applications of composition and the new version of primitive recursion.*

Proof. By induction on k. The standard form of mathematical induction will be used.

Basis: Suppose that $k = 1$, so that $f : \mathbb{N} \to \mathbb{N}$, $g : \mathbb{N}^3 \to \mathbb{N}$, and $h : \mathbb{N}^2 \to \mathbb{N}$ such that the following properties are satisfied.

- $h(x, 0) = f(x)$ for all $x \in \mathbb{N}$.
- $h(x, t + 1) = g(t, h(x, t), x)$ for all $x, t \in \mathbb{N}$.

Consider the following sequence of functions, as well. An examination of the definition of each confirms that each can be constructed from f, g, and the extended set of initial functions using a finite number of applications of composition and the new version of primitive recursion.

- $\widehat{f} : \mathbb{N} \to \mathbb{N}$ such that, for all $x \in \mathbb{N}$,
$$\widehat{f}(x) = \langle n(x), \langle f(x), u_1^1(x) \rangle\rangle = \langle 0, \langle f(x), x \rangle\rangle.$$

Note the that nullify function, the pairing function and the projection function are in the extended set of initial functions, since they are introduced in parts (b), (d) and (e) of Theorem 1.

- $\widehat{g} : \mathbb{N} \to \mathbb{N}$ such that, for all $z \in \mathbb{N}$,

$$\widehat{g}(z) = \langle s(\ell(z)), \langle g(\ell(z), \ell(r(z)), r(r(z))), r(r(z))\rangle\rangle.$$

  Note that the successor function and the left and right inverses of the pairing function are in the extended set of initial functions, since they are introduced in parts (a), Note, as well, that if $z = \langle t, \langle y, x\rangle\rangle$ for $t, y, x \in \mathbb{N}$ then

$$\begin{aligned}
\widehat{g}(z) &= \langle s(\ell(z)), \langle g(\ell(z), \ell(r(z)), r(r(z))), r(r(z))\rangle\rangle \\
&= \langle s(\ell(\langle t, \langle y, x\rangle\rangle)), \langle g(\ell(\langle t, \langle y, x\rangle\rangle), \ell(r(\langle t, \langle y, x\rangle\rangle)), \\
&\qquad\qquad r(r(\langle t, \langle y, x\rangle\rangle))), r(r(\langle t, \langle y, x\rangle\rangle))\rangle\rangle \\
&= \langle s(t), \langle g(t, \ell(\langle y, x\rangle), r(\langle y, x\rangle)), r(\langle y, x\rangle)\rangle\rangle \\
&= \langle t+1, \langle g(t, y, x), x\rangle\rangle.
\end{aligned}$$

- The function $\widehat{h} : \mathbb{N} \to \mathbb{N}$ obtained from $\widehat{f}$ and $\widehat{g}$ using the new version of primitive recursion — so that

$$\widehat{h}(x, 0) = \widehat{f}(x) = \langle 0, \langle f(x), x\rangle\rangle = \langle 0, \langle h(x, 0), x\rangle\rangle$$

  for all $x \in \mathbb{N}$ and, for all $x, t \in \mathbb{N}$,

$$\widehat{h}(x, t+1) = \widehat{g}(\widehat{h}(x, t))$$

  so that if $\widehat{h}(x, t) = \langle t, \langle h(x, t), x\rangle\rangle$ then

$$\begin{aligned}
\widehat{h}(x, t+1) &= \langle t+1, \langle g(t, h(x, t), x), x\rangle\rangle \\
&= \langle t+1, \langle h(x, t+1), x\rangle\rangle.
\end{aligned}$$

  It is easily proved from the above, by induction on t, that

$$\widehat{h}(x, y) = \langle y, \langle h(x, y), x\rangle\rangle$$

  for all $x, y \in \mathbb{N}$.

- The function $\widetilde{h} : \mathbb{N}^2 \to \mathbb{N}$ such that, for all $x, y \in \mathbb{N}$,

$$\widetilde{h}(x, y) = \ell(r(\widehat{h}(x, y)))$$

  …since this as been obtained from the above function $\widehat{h} : \mathbb{N}^2 \to \mathbb{N}$ and the inverses of the pairing function using ***composition***.

Since

$$\begin{aligned}\widetilde{h}(x,y) &= \ell(r(\widehat{h}(x,y))) \\ &= \ell(r(\langle y, \langle h(x,y), x\rangle\rangle) \\ &= \ell(\langle h(x,y), x\rangle) \\ &= h(x,y)\end{aligned}$$

for all $x, y \in \mathbb{N}$, $\widetilde{h}$ and h are the same function, so that h can be constructed from f, g, and the extended set of initial functions using a finite number of applications of composition and the new version of primitive recursion — as needed to establish the result to be proved in the basis.

Inductive Step: Let m be an integer such that $m \geq 1$. It is necessary and sufficient to use the following

> Inductive Hypothesis: Let $f : \mathbb{N}^m \to \mathbb{N}$, let $g : \mathbb{N}^{m+2} \to \mathbb{N}$, and let $h : \mathbb{N}^{m+1} \to \mathbb{N}$ be the function obtained from f and g using the second (standard) form of primitive recursion. Then h can be obtained from f, g, and the extended set of initial functions (given in parts (a)–(e) of Theorem 1) using a finite number of applications of composition and the new version of primitive recursion.

to prove the following

> Inductive Claim: Let $f : \mathbb{N}^{m+1} \to \mathbb{N}$, let $g : \mathbb{N}^{m+3} \to \mathbb{N}$, and let $h : \mathbb{N}^{m+2} \to \mathbb{N}$ be the function obtained from f and g using the second (standard) form of primitive recursion. Then h can be obtained from f, g, and the extended set of initial functions (given in parts (a)–(e) of Theorem 1) using a finite number of applications of composition and the new version of primitive recursion.

With that noted, let $f : \mathbb{N}^{m+1} \to \mathbb{N}$, let $g : \mathbb{N}^{m+3} \to \mathbb{N}$, and let $h : \mathbb{N}^{m+2} \to \mathbb{N}$ be the function obtained from f and g using the second (standard) form of primitive recursion — so that the following properties are satisfied.

- For all $x_1, x_2, \ldots, x_m, x_{m+1} \in \mathbb{N}$,

$$h(x_1, x_2, \ldots, x_m, x_{m+1}, 0) = f(x_1, x_2, \ldots, x_m, x_{m+1}).$$

- For all $x_1, x_2, \ldots, x_m, x_{m+1}, t \in \mathbb{N}$,

$$\begin{aligned}&h(x_1, x_2, \ldots, x_m, x_{m+1}, t+1) \\ &\qquad = g(t, h(x_1, x_2, \ldots, x_m, x_{m+1}, t), x_1, x_2, \ldots, x_m, x_{m+1}).\end{aligned}$$

Consider the following sequence of functions. It is not hard to check that each can be constructed from the above functions f and g, and the extended set of initial functions, using a finite number of applications of composition and the new version of primitive recursion.

- Since the pairing functions.its inverses, and the projection functions are included in the extended set of initial functions, listed in Theorem 1, the function $\widehat{f} : \mathbb{N}^m \to \mathbb{N}$ such that, for all $x_1, x_2, \ldots, x_m \in \mathbb{N}$,

  $$\begin{aligned}\widehat{f}(x_1, x_2, \ldots, x_m) = f(\ell(u_1^m(x_1, x_2, \ldots, x_m)), r(u_1^m(x_1, x_2, \ldots, x_m)),\\ u_2^m(x_1, x_2, \ldots, x_m), u_3^m(x_1, x_2, \ldots, x_m), \ldots, u_m^m(x_1, x_2, \ldots, x_m))\end{aligned}$$

  can be created from f, g, and the extended set of initial functions using a finite number of applications of composition (and the new version of primitive recursive). Note that

  $$\widehat{f}(x_1, x_2, \ldots, x_m) = f(\ell(x_1), r(x_1), x_2, x_3, \ldots, x_m)$$

  for all $x_1, x_2, \ldots, x_m \in \mathbb{N}$.

- Since the pairing functions.its inverses, and the projection functions are included in the extended set of initial functions, listed in Theorem 1, the function $\widehat{g} : \mathbb{N}^{m+2} \to \mathbb{N}$ such that, for all $t, y, x_1, x_2, \ldots, x_m \in \mathbb{N}$,

  $$\begin{aligned}\widehat{g}(t, y, x_1, x_2, \ldots, x_m) = g(u_1^{m+2}(t, y, x_1, x_2, \ldots, x_m), u_2^{m+2}(t, y, x_1, x_2, \ldots, x_m),\\ \ell(u_3^{m+2}(t, y, x_1, x_2, \ldots, x_m), r(u_3^{m+2}(x_1, x_2, \ldots, x_m), u_4^{m+2}(t, y, x_1, x_2, \ldots, x_m),\\ u_5^{m+2}(x_1, x_2, \ldots, x_{m+2}), u_6^{m+2}(x_1, x_2, \ldots, x_{m+2}), \ldots, u_{m+2}^{m+2}(x_1, x_2, \ldots, x_{m+2}))\end{aligned}$$

  can be created from f, g, and the extended set of initial functions using a finite number of applications of composition (and the new version of primitive recursive). Note that

  $$\widehat{g}(t.y, x_1, x_2, \ldots, x_m) = g(t, y, \ell(x_1), r(x_1), x_2, x_3, \ldots, x_m)$$

  for all $t, y, x_1, x_2, \ldots, x_m \in \mathbb{N}$.

- It follows by the Inductive Hypothesis that the function $\widehat{h} : \mathbb{N}^{m+1} \to \mathbb{N}$, obtained from $\widehat{f}$ and $\widehat{g}$ using the (standard) second form of primitive recursion, can be formed from $\widehat{f}$, $\widehat{g}$, and the extended set of initial functions using a finite number of applications of composition and the new version of primitive recursion.

  Since $\widehat{f}$ and $\widehat{g}$ can each be obtained from f, g, and the extended set of initial functions using a finite number of applications of composition and the new version of primitive recursion, this implies that $\widehat{h}$ can be obtained from f, g, and the extended set of initial functions, using a finite number of applications of composition and the new version of primitive recursion, as well.

The definition of the second form of primitive recursion implies the following.

- For all $x_1, x_2, \ldots, x_m \in \mathbb{N}$,

$$\begin{aligned}\widehat{h}(x_1, x_2, \ldots, x_m, 0) &= \widehat{f}(x_1, x_2, \ldots, x_n) \\ &= f(\ell(x_1), r(x_1), x_2, x_3, \ldots, x_m) \\ &= h(\ell(x_1), r(x_1), x_2, x_3, \ldots, x_m, 0).\end{aligned}$$

- For all $t, y, x_1, x_2, \ldots, x_m \in \mathbb{N}$,

$$\begin{aligned}\widehat{h}(x_1, x_2, \ldots, x_m, t+1) &= \widehat{g}(t, \widehat{h}(x_1, x_2, \ldots, x_m, t), x_1, x_2, \ldots, x_m) \\ &= g(t, \widehat{h}(x_1, x_2, \ldots, x_m, t), \ell(x_1), r(x_1), x_2, x_3, \ldots, x_m).\end{aligned}$$

Note that if

$$\widehat{h}(x_1, x_2, \ldots, x_m, t) = h(\ell(x_1), r(x_1), x_2, x_3, \ldots, x_m, t)$$

then it follows by the above that

$$\begin{aligned}\widehat{h}(x_1, x_2, \ldots, x_m, t+1) &= g(t, h(\ell(x_1), r(x_1), x_2, x_3, \ldots, x_m, t), \\ &\qquad\qquad \ell(x_1), r(x_1), x_2, x_3, \ldots, x_m) \\ &= h(\ell(x_1), \ell(x_1), r(x_1), x_2, x_3, , \ldots, x_m, t+1)\end{aligned}$$

as well. It is, therefore, easily established by induction on y that

$$\widehat{h}(x_1, x_2, \ldots, x_m, y) = h(\ell(x_1), r(x_1), x_2, x_3, \ldots, x_m, y)$$

for all $x_1, x_2, \ldots, x_m, y \in \mathbb{N}$.

- It follows that the function $\widetilde{f} : \mathbb{N}^{m+2} : \mathbb{N}$ such that, for all $x_1, x_2, \ldots, x_{m+2} \in \mathbb{N}$,

$$\begin{aligned}\widetilde{h}(x_1, x_2, \ldots, x_{m+2}) &= \widehat{h}(\langle u_1^{m+2}(x_1, x_2, \ldots, x_{m+2}), u_2^{m+2}(x_1, x_2, \ldots, x_{m+2})\rangle, \\ &\quad u_3^{m+2}(x_1, x_2, \ldots, x_{m+2}), u_4^{m+2}(x_1, x_2, \ldots, x_{m+2}), \ldots, u_{m+2}^{m+2}(x_1, x_2, \ldots, x_{m+2})\end{aligned}$$

can be composed from f, g, and the extended set of transition functions using a finite number of applications of composition and the new version of primitive recursion, as well.

It now suffices to note that, for all $x_1, x_2, \ldots, x_{m+2} \in \mathbb{N}$,

$$\widetilde{h}(x_1, x_2, \ldots, x_{m+2}) = \widehat{h}(\langle x_1, x_2\rangle, x_3, x_4, \ldots, x_{m+2}) = h(x_1, x_2, \ldots, x_{m+2}),$$

so that $\widetilde{h} = h$, in order to establish the Inductive Claim — completing the Inductive Step, and the proof. $\square$

Lemma 6. *Let $f : \mathbb{N}^k \to \mathbb{N}$. If f is primitive recursive then f can be constructed from the "expanded set" of initial functions, listed in parts (a)–(e), of Theorem 1, above, using a finite number of applications of composition and the new version of primitive recursion.*

This can be proved by induction on the length of the derivation of f from the initial functions using composition and both (standard) forms of primitive recursion. The proof is left as a (straightforward) exercise, using the lemmas that have been established before this.

Proof of Theorem 1. The result now follows from Lemmas 3 and 6. ☐

Unary Primitive Recursive Functions

A set PR of unary functions is defined in the lecture notes.

Theorem 7. PR *is the set of unary primitive recursive functions.*

The lemmas that follow will be used to prove this.

Lemma 8. *Suppose that $f, g : \mathbb{N} \to \mathbb{N}$ are primitive recursive. If $h : \mathbb{N} \to \mathbb{N}$ is the function such that*

$$h(0) = 0$$

and, for all $t \in \mathbb{N}$,

$$h(t+1) = \begin{cases} f(t/2) & \textit{if } t \textit{ is even,} \\ g(h((t+1)/2)) & \textit{if } t \textit{ is odd,} \end{cases}$$

then h is primitive recursive as well.

Proof. Consider the function $\widehat{g} : \mathbb{N}^2 \to \mathbb{N}$ such that, for all $t, y \in \mathbb{N}$,

$$\widehat{g}(t, y) = \begin{cases} y \times p_{t+2}^{f(t \text{ quo } 2)} & \text{if } t \text{ is even,} \\ y \times p_{t+2}^{g((y)_{((t+3) \text{ quo } 2)})} & \text{if } t \text{ is odd} \end{cases}$$

— where p_n is the n^{th} prime, for $n \geq 1$, and $(y)_n$ is one of the inverses of the Gödel function, namely, the largest integer m such that p_n^m divides y.

After reviewing the lecture notes about ***encodings*** as needed, it should not be hard to see that $\widehat{g}$ is primitive recursive because f and g are — and because the predicate "t is even" (that is, $f \pmod 2 = 0$) is primitive recursive as well.

Let $\widehat{h} : \mathbb{N} \to \mathbb{N}$ be obtained from $\widehat{g}$ using primitive recursion. Specifically, suppose that

$$\widehat{h}(0) = 1$$

and, for all $t \in \mathbb{N}$,

$$\begin{aligned}
\widehat{h}(t+1) &= \widehat{g}(t, \widehat{h}(t)) \\
&= \begin{cases} \widehat{h}(t) \times p_{t+2}^{f(t \text{ quo } 2)} & \text{if } t \text{ is even,} \\ \widehat{h}(t) \times p_{t+2}^{g((\widehat{h}(t))_{((t+3) \text{ quo } 2)})} & \text{if } t \text{ is odd,} \end{cases} \\
&= \begin{cases} \widehat{h}(t) \times p_{t+2}^{f(t/2)} & \text{if } t \text{ is even,} \\ \widehat{h}(t) \times p_{t+2}^{g((\widehat{h}(t))_{((t+3)/2)})} & \text{if } t \text{ is odd.} \end{cases}
\end{aligned}$$

Notice that it follows from the above that

- $\widehat{h}(0) = 1 = 2^0 = [h(0)]$, since $h(0) = 0$.
- $\widehat{h}(1) = \widehat{g}(0, \widehat{h}(0)) = \widehat{h}(0) \times p_2^{f(0)} = \widehat{h}(0) \times p_2^{h(1)} = [h(0), h(1)]$.
- $\widehat{h}(2) = \widehat{g}(1, \widehat{h}(1))$
  $= \widehat{h}(1) \times p_3^{g((\widehat{h}(1))_{(2)})}$
  $= \widehat{h}(1) \times p_3^{g(h(1))}$
  $= \widehat{h}(1) \times p_3^{h(2)}$
  $= [h(0), h(1), h(2)]$.

Indeed, it can be proved by induction on t (using the strong form of mathematical induction) that

$$\widehat{h}(t) = [h(0), h(1), h(2), \ldots, h(t)]$$

for every integer $t \geq 0$.

Since $\widehat{h}$ was obtained from $\widehat{g}$ and the constant function 0 using the first form of primitive recursion, $\widehat{h}$ is primitive recursive as well.

Since

$$h(t) = (\widehat{h}(t))_{(t+1)}$$

for every integer $t \geq 0$, the function h is primitive recursive as well, as claimed. $\square$

Lemma 9. *Every function in* PR *is a unary primitive recursive function.*

Proof. Notice that all the functions mentioned in parts (a)–(c) of the definition of PR, included in the lecture notes, are primitive recursive.

If $f, g : \mathbb{N} \to \mathbb{N}$ are primitive recursive then the functions $f \circ g$ and h described in parts (d) and (e) of the definition are certainly primitive recursive as well. It follows by Lemma 8 that the function described in part (f) is also primitive recursive.

The claim now follows by induction on the number of applications of parts (d), (e) and (f) to show that a given function $f \in$ PR. $\square$

Lemma 10. *The function* $u_1^1 : \mathbb{N} \to \mathbb{N}$ *is in* PR*.*

Proof. It follows by part (c) of the definition of PR that the inverses $\ell, r : \mathbb{N} \to \mathbb{N}$ of the pairing function are in PR. However, since

$$u_1^1(x) = \langle \ell(x), r(x) \rangle$$

for all $x \in \mathbb{N}$, it follows by part (e) of the definition of PR that $u_1^1 \in$ PR too. $\square$

See the lecture notes for the definition of a ***satisfactory*** function.

Lemma 11. *Let* $f : \mathbb{N} \to \mathbb{N}$ *be a* unary *primitive recursive function. Then* f *is satisfactory* if and only if $f \in$ PR

Proof. Suppose, first, that f is satisfactory. Then, since $u_1^1 \in$ PR, by the previous lemma, it follows that the function $\widehat{f} : \mathbb{N} \to \mathbb{N}$ such that, for all $x \in \mathbb{N}$,

$$\widehat{f}(x) = f(u_1^1(x)) = f(x)$$

belongs to PR. Since $\widehat{f} = f$, $f \in$ PR.

Suppose, on the other hand, that $f \in$ PR. Let $h_1 : \mathbb{N} \to \mathbb{N}$ be an arbitrarily chosen function in PR. It is necessary and sufficient to prove that the function $\widehat{f} : \mathbb{N} \to \mathbb{N}$ such that, for all $n \in \mathbb{N}$,

$$\widehat{f}(x) = f(h_1(x))$$

is in PR. However, this is just the function $f \circ h_1$ mentioned in part (d) of the definition PR, so this follows immediately from this definition. $\square$

Lemma 12. *Let* $h : \mathbb{N} \to \mathbb{N}$ *be primitive recursive. Then* h *is satisfactory.*

Proof. Let $h : \mathbb{N} \to \mathbb{N}$ be primitive recursive. Then it follows by Theorem 1 that h can be constructed from the "enriched" set of initial functions, given in the statement of that theorem, using a finite number of applications of composition and the new version of "primitive recursion" (introduced shortly before the statement of that result). The claim can be established by induction on the number of applications of composition and the new form of primitive recursion are needed to obtain h; the strong form of mathematical induction will be used.

Basis: Suppose that composition and the special form of primitive recursion are not used to obtain h at all. Then h is one of the functions described in parts (a)–(e) of Theorem 1:

(a) If h is the *successor function* $s : \mathbb{N} \to \mathbb{N}$ then $h \in$ PR, by part (a) of the definition of PR and, since h is unary, it follows by Lemma 11 that h is satisfactory.

(b) If h is the *nullify function* $n : \mathbb{N} \to \mathbb{N}$ then $h \in$ PR, by part (b) of the definition of PR and, since h is unary, it follows by Lemma 11 that h is satisfactory once again.

(c) Similarly, if h is one of the inverses $\ell, r : \mathbb{N} \to \mathbb{N}$ of the pairing function then it follows by part (c) of the definition of PR and Lemma 11 that h is satisfactory in this case as well.

(d) Suppose next that $h : \mathbb{N}^2 \to \mathbb{N}$ is the *pairing function* — that is, $h(x, y) = \langle x, y \rangle$ for all $x, y \in \mathbb{N}$. Let $h_1, h_2 : \mathbb{N} \to \mathbb{N}$ be arbitrary elements of PR. Then it follows by part (e) of the definition of PR that the function $\widehat{h} : \mathbb{N} \to \mathbb{N}$ such that, for all $t \in \mathbb{N}$,

$$\widehat{h}(t) = h(h_1(t), h_2(t)) = \langle h_1(t), h_2(t) \rangle$$

is in PR as well — establishing that the pairing function, h, is also satisfactory.

(e) Let $1 \leq i \leq n$ and suppose that $h : \mathbb{N}^n \to \mathbb{N}$ is the *projection function* u_i^n. Let $h_1, h_2, \dots, h_n$ be arbitrarily chosen functions in PR. It is necessary and sufficient to show that the function $\widehat{h} : \mathbb{N} \to \mathbb{N}$ such that, for all $t \in \mathbb{N}$,

$$\widehat{h}(t) = u_i^n(h_1(t), h_2(t), \dots, h_n(t))$$

is in PR. However, this is just the function h_i so that is certainly the case — establishing that each one of the *projection functions* is satisfactory as well.

Inductive Step: Let k be an integer such that $k \geq 0$. It is necessary and sufficient to use the following

> Inductive Hypothesis: If ℓ is an integer such that $0 \leq \ell \leq k$, and $h : \mathbb{N} \to \mathbb{N}$ is a primitive recursive function that has been obtained using ℓ applications of composition and the special form of primitive recursion, then h is satisfactory.

to prove the following

> Inductive Claim: If $h : \mathbb{N} \to \mathbb{N}$ is a primitive recursive function that has been obtained using $k + 1$ applications of composition and the special form of primitive recursion, then h is satisfactory.

With that noted, let $h : \mathbb{N} \to \mathbb{N}$ be a primitive recursive function that has been obtained using $k + 1$ applications of composition and the special form of primitive recursion.

Either h was produced by a (final) application of composition, or by an application of the special form of primitive recursion. These cases are considered separately.

Case: h was produced using a (final) application of the composition. Then there exists a positive integer m and primitive recursive functions $f : \mathbb{N}^m \to \mathbb{N}$ and $g_1, g_2, \ldots, g_m : \mathbb{N}^n \to \mathbb{N}$ such that, for all $x_1, x_2, \ldots, x_n \in \mathbb{N}$,

$$h(x_1, x_2, \ldots, x_n) = f(g_1(x_1, x_2, \ldots, x_n), g_2(x_1, x_2, \ldots, x_n), \ldots, g_m(x_1, x_2, \ldots, x_n)).$$

Furthermore, each of f and $g_1, g_2, \ldots, g_m$ were obtained using at most k applications of composition and the special form of primitive recursion — so it follows by the inductive hypothesis that f and $g_1, g_2, \ldots, g_m$ are all satisfactory.

Now let $h_1, h_2, \ldots, h_n : \mathbb{N} \to \mathbb{N}$ be arbitrarily chosen elements of PR.

Since $g_1, g_2, \ldots, g_m$ are all satisfactory, the functions $\widehat{g}_1, \widehat{g}_2, \ldots, \widehat{g}_m : \mathbb{N} \to \mathbb{N}$ such that, for all $t \in \mathbb{N}$ (and $1 \leq i \leq m$)

$$\widehat{g}_i(t) = g_i(h_1(t), h_2(t), \ldots, h_n(t))$$

are all in PR.

Since f is also satisfactory, the function $\widehat{f} : \mathbb{N} \to \mathbb{N}$ such that, for all $t \in \mathbb{N}$,

$$\widehat{f}(t) = f(\widehat{g}_1(t), \widehat{g}_2(t), \ldots, \widehat{g}_m(t))$$

is in PR as well.

However, $\widehat{f}(t) = h(h_1(t), h_2(t), \ldots, h_n(t))$ for all $t \in \mathbb{N}$ — so this implies that h is satisfactory, as claimed.

Case: h was produced using a (final) application of the special form of primitive recursion.

Then there exist primitive recursive functions $f, g : \mathbb{N} \to \mathbb{N}$ such that, for all $x \in \mathbb{N}$,

$$h(x, 0) = f(x)$$

and, for all $x, t \in \mathbb{N}$,

$$h(x, t+1) = g(h(x, t)).$$

Furthermore, each of f and g were obtained using at most k applications of composition and the special form of primitive recursion — so it follows by the inductive hypothesis that f and g are both satisfactory. Since f and g are both *unary* it follows by Lemma 11 that $f, g \in$ PR.

Now consider the function $\psi : \mathbb{N} \to \mathbb{N}$ such that

$$\psi(0) = 0$$

and, for all $t \in \mathbb{N}$,

$$\psi(t+1) = h(r(t), \ell(t)).$$

We will show that $\psi \in$ PR, so that ψ is also satisfactory, by Lemma 11. This will be used to show that if $h_1, h_2 \in$ PR then the function $\widehat{h} : \mathbb{N} \to \mathbb{N}$ such that, for all $t \in \mathbb{N}$,

$$\widehat{h}(t) = h(h_1(t), h_2(t))$$

is in PR — as needed to establish that h is satisfactory, and complete the proof.

Recall that, for $a, b \in \mathbb{N}$,

$$\langle a, b \rangle = (2^a \cdot (2b + 1)) - 1$$

and notice that $\langle a, b \rangle$ is even *if and only if* $a = 0$.

Let $t = \langle a, b \rangle$.

Suppose that t is odd. Then, as noted above, $t = \langle a, b \rangle$ where $a \geq 1$, and

$$\begin{aligned}
\psi(t+1) &= h(r(t), \ell(t)) && \text{(by the definition of } \psi\text{)} \\
&= h(b, a) && \text{(by the definition of } t\text{)} \\
&= g(h(b, a-1)) && \text{(by the definition of } h\text{, since } a \geq 1\text{).}
\end{aligned}$$

Let $\widehat{t} = \langle a - 1, b \rangle = 2^{a-1}(2b+1) - 1$. Then it is easily checked that $\widehat{t} = (t-1)/2 \geq 0$, and it now follows that

$$\begin{aligned}
\psi((t+1)/2) &= \psi((t-1)/2 + 1) \\
&= h(r((t-1)/2), \ell((t-1)/2)) && \text{(by the definition of } \psi\text{)} \\
&= h(r(\widehat{t}\,), \ell(\widehat{t}\,)) && \text{(since } \widehat{t} = (t-1)/2\text{)} \\
&= h(b, a-1) && \text{(since } \widehat{t} = \langle a-1, b \rangle\text{).}
\end{aligned}$$

It now follows by the above that if t is odd then

$$\psi(t+1) = g(\psi((t+1)/2)).$$

Suppose, instead, that t is even. Then, as noted above, $t = \langle a, b \rangle$ where $a = 0$, so that

$$t = 2^0 \cdot (2b+1) - 1 = 2b,$$

and

$$\begin{aligned}
\psi(t+1) &= h(r(t), \ell(t)) && \text{(by the definition of } \psi\text{)} \\
&= h(b, a) \\
&= h(t/2, 0) \\
&= f(t/2) && \text{(by the definition of } h\text{).}
\end{aligned}$$

In summary, $\psi : \mathbb{N} \to \mathbb{N}$ such that $\psi(0) = 0$ and, for all $t \in \mathbb{N}$,

$$\psi(t+1) = \begin{cases} f(t/2) & \text{if } t \text{ is even,} \\ g(\psi((t+1)/2)) & \text{if } t \text{ is odd.} \end{cases}$$

Since $f, g \in$ PR it follows by part (f) of the definition of PR that $\psi \in$ PR as well. Consequently, by Lemma 11, ψ is satisfactory.

Now let $h_1, h_2 : \mathbb{N} \to \mathbb{N}$ be arbitrarily chosen functions in PR. In order to complete this proof if is sufficient to show that the function $\widehat{h} : \mathbb{N} \to \mathbb{N}$ such that, for all $t \in \mathbb{N}$,

$$\widehat{h}(t) = h(h_1(t), h_2(t))$$

is in PR.

It now suffices to note that

$$\begin{aligned} \widehat{h}(t) &= h(h_1(t), h_2(t)) \\ &= \psi(\langle h_2(t), h_1(t) \rangle + 1) && \text{(by the definition of } \psi\text{)} \\ &= \psi(s(\langle h_2(t), h_1(t) \rangle)). \end{aligned}$$

Since the pairing function, successor function, and ψ are all satisfactory, it follows that $\widehat{h} \in$ PR — as needed to complete the inductive step, and the proof of the claim. ☐

Lemma 13. *If f is a unary primitive recursive function then $f \in$ PR.*

Proof. This now follows immediately from Lemmas 11 and 12. ☐

Proof of Theorem 7. This follows immediately from Lemmas 9 and 13. ☐