Computer Science 513

The Arithmetic Hierarchy

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Lecture #16

Goals for Today

Goals for Today:

- Introduction to the ***Arithmetic Hierarchy*** — a significant tool for the classification of unsolvable problems. The sets included in this hierarchy will be defined and complete sets (with respect to various reducibilities) for various sets in this hierarchy will be identified.

Arithmetic Hierarchy: Definition

Definition: The sets Σ_n, Π_n and Δ_n are as follows for every number $n \geq 0$.

(a) Σ_0 is the class of all computable subsets of $\mathbb{N}$.

(b) For every number $n \geq 0$, and $A \subseteq \mathbb{N}$, $A \in \Sigma_{n+1}$ if and only if there exists a set $B \in \Sigma_n$ such that A is B-computably enumerable.

(c) For every number $n \in \mathbb{N}$, $\Pi_n = \text{co-}\Sigma_n$.

(d) For every number $n \in \mathbb{N}$, $\Delta_n = \Sigma_n \cap \Pi_n$.

These sets form the ***Arithmetic Hierarchy***.

Arithmetic Hierarchy: Initial Results

- Note that it follows from this definition that $\Sigma_0 = \Delta_0 = \Pi_0$ and these are all equal to the class of computable subsets of $\mathbb{N}$.
- It is also not hard to show that a set $A \subseteq \mathbb{N}$ is B-computably enumerable, for some computable set $B \subseteq \mathbb{N}$, if and only if A is computably enumerable — this follows from Theorem #6 from Lecture #14.

  Thus Σ_1 is the class of computably enumerable subsets of $\mathbb{N}$.

Arithmetic Hierarchy: Initial Results

Theorem #1: $\Sigma_n \subseteq \Sigma_{n+1}$ and $\Pi_n \subseteq \Pi_{n+1}$ for every number $n \in \mathbb{N}$.

Proof:

- Let $A \in \Sigma_n$. It follows by Theorem #3 from Lecture #14 that A is certainly A-computable, which implies that A is A-computably enumerable, and $A \in \Sigma_{n+1}$. Since A was arbitrarily chosen from Σ_n, $\Sigma_n \subseteq \Sigma_{n+1}$.
- Since $\Pi_n =$ co-Σ_n and $\Pi_{n+1} =$ co-Σ_{n+1} it follows that $\Pi_n \subseteq \Pi_{n+1}$ as well. $\square$

Jumps and Their Properties

Recall that if $G : \mathbb{N} \to \mathbb{N}$ is a ***unary total function*** then, for $n \in \mathbb{N}$,

$$W_n^G = \{x \in \mathbb{N} \mid \Phi_G^{(1)}(x, n) \text{ is defined}\}$$

and that, if $S \subseteq \mathbb{N}$ and $n \in \mathbb{N}$

$$W_n^S = W_n^{p_S} = \{n \in \mathbb{N} \mid \Phi_{p_S}^{(1)}(x, n) \text{ is defined}\}.$$

where $p_S : \mathbb{N} \to \mathbb{N}$ is the ***characteristic function*** of the set S.

Jumps and Their Properties

Definition: Let $G : \mathbb{N} \to \mathbb{N}$ be a unary total function. Then the ***jump*** of G is the set

$$G' = \{n \in \mathbb{N} \mid n \in W_n^G\}.$$

One can then define the ***jump*** S' of a subset S of N, as well, by replacing S with its characteristic function p_S in the above definition.

Since $K = \{n \in \mathbb{N} \mid n \in W_n = W_n^I\}$, it follows by the above definition that the jump I' of the identity function I is the set K.

Jumps and Their Properties

Theorem #2: Let G be a unary total function. Then the jump G' of G is G-computably enumerable but not G-computable.

How This is Proved: The "Relativized Universality Theorem", from Lecture #14, can be used to establish that G' is G-computably enumerable. If this set was also G-computable then it would follow that its complement is G-computably enumerable, and a contradiction is easily established.

Details of the argument are included in the supplemental document for this lecture. $\square$

Jumps and Their Properties

Theorem #3: Let $G : \mathbb{N} \to \mathbb{N}$ be a ***unary total function*** and let $A \subseteq \mathbb{N}$. Then the following are equivalent:

(a) $A \preceq_1 G'$

(b) $A \preceq_m G'$

(c) A is G-computably enumerable.

This is a reasonably straightforward consequence of results from recent lectures, including the Relativized Universality Theorem and the Relativized and Strengthened Parameter Theorem, from Lecture #14. Once again, a complete proof can be found in the supplemental document for this lecture.

Jumps and Their Properties

Theorem #4: Let $F, G : \mathbb{N} \to \mathbb{N}$ be unary total functions. If F is G-computable then $F' \preceq_1 G'$.

How This is Proved: This is a reasonably straightforward consequence of Theorem #5 from Lecture #14, and Theorems #2 and #3, above. $\square$

Iterated Jumps

Let $G \subseteq \mathbb{N}$ and let $n \geq 0$. Then $G^{(n)}$ is "the jump, iterated n times." That is,

$$G^{(0)} = G$$

and, for every integer $n \geq 0$,

$$G^{(n+1)} = (G^{(n)})'.$$

Consider the empty set $\emptyset$ (whose characteristic function is the constant function 0). The following is a now a direct consequence of Theorem #2:

Theorem #5: For every integer $n \geq 0$ the set $\emptyset^{(n+1)}$ is $\emptyset^{(n)}$-computably enumerable but not $\emptyset^{(n)}$-computable. $\square$

Iterated Jumps and the Arithmetic Hierarchy

Theorem #6: $\emptyset^{(n)} \in \Sigma_n$ for every number $n \in \mathbb{N}$.

Proof:

- $\emptyset^{(0)}$ is, by definition, the characteristic function of $\emptyset$ — that is, the function $f : \mathbb{N} \to \mathbb{N}$ such that $f(n) = 0$ for all $n \in \mathbb{N}$. This function is certainly computable, so $\emptyset^{(0)} \in \Sigma_0$.
- It follows by Theorem #2 that $\emptyset^{(n+1)}$ is $\emptyset^{(n)}$-computably enumerable for every number $n \in \mathbb{N}$ — so that $\emptyset^{(n+1)} \in \Sigma_{n+1}$ if $\emptyset^{(n)} \in \Sigma_n$. This can be used to prove (by induction on n) that

  $$\emptyset^{(n)} \in \Sigma_n$$

  for every number $n \in \mathbb{N}$, as claimed. $\square$

Iterated Jumps and the Arithmetic Hierarchy

Theorem #7: For every set $A \subseteq \mathbb{N}$ and every number $n \in \mathbb{N}$, $A \in \Sigma_{n+1}$ if and only if A is $\emptyset^{(n)}$-computably enumerable.

How This is Proved: One of the containments between sets, asserted here, is a straightforward consequence of previous results. The other containment can be established by induction on n, using recently established results as well.

Iterated Jumps and the Arithmetic Hierarchy

Corollary #8: For every number $n \in \mathbb{N}$ such that $n \geq 1$ the following are equivalent:

(a) $A \preceq_1 \emptyset^{(n)}$.

(b) $A \preceq_m \emptyset^{(n)}$.

(c) $A \in \Sigma_n$.

Proof:

- This follows by the above result and Theorem #3. $\square$

Iterated Jumps and the Arithmetic Hierarchy

Corollary #9: For every number $n \geq 1$, $\emptyset^{(n)}$ is 1-complete for Σ_n.

Proof:

- It follows by Theorem #6 that $\emptyset^{(n)} \in \Sigma_n$, and it follows by the above corollary that $A \preceq_1 \emptyset^{(n)}$ for every set $A \in \Sigma_n$. The claim now follows by the definition of "1-complete." $\square$

Arithmetic Hierarchy: Closure Properties

Corollary #10: Let $n \in \mathbb{N}$ such that $n \geq 1$.

(a) Σ_n and Π_n are both *m*-closed.

(b) Σ_n and Π_n are both 1-closed.

This is another reasonably straightforward consequence of results that have now been established. Details are in the supplemental document for this lecture.

Iterated Jumps and the Arithmetic Hierarchy

Each of the following results can now be established by a sequence of results that have been proved in this and the previous lecture. Details of their proofs are included in the supplemental document.

Corollary #11: For every set $A \subseteq \mathbb{N}$ and for all $n \in \mathbb{N}$, $A \in \Delta_{n+1}$ if and only if $A \preceq_T \emptyset^{(n)}$. Thus $\emptyset^{(n)}$ is T-complete for Δ_{n+1}.

Theorem #12: For every number $n \in \mathbb{N}$, $\Sigma_n \cup \Pi_n \subseteq \Delta_{n+1}$.

Theorem #13 For every number $n \geq 1$, $\emptyset^{(n)} \in \Sigma_n \setminus \Delta_n$. That is, $\emptyset^{(n)} \in \Sigma_n$ and $\emptyset^{(n)} \notin \Delta_n$.

Kleene's Hierarchy Theorem

The next result — which provides a considerable amount of information about the structure of the hierarchy being defined — is also a reasonably straightforward of results that have now been obtained. The supplemental document includes a complete proof.

Theorem #14 (Kleene's Hierarchy Theorem): Let $n \in \mathbb{N}$ such that $n \geq 1$.

(a) $\Delta_n \subset \Sigma_n$ and $\Delta_n \subset \Pi_n$.

(b) $\Sigma_n \subset \Sigma_{n+1}$ and $\Pi_n \subset \Pi_{n+1}$.

(c) $\Sigma_n \cup \Pi_n \subset \Delta_{n+1}$.

Kleene's Hierarchy Theorem

Corollary #15: $\Sigma_n \neq \Pi_n$ for every number $n \in \mathbb{N}$ such that $n \geq 1$.

Proof:

- Suppose that $n \geq 1$ and $\Sigma_n = \Pi_n$.
- Then $\Delta_n = \Sigma_n \cap \Pi_n = \Sigma_n$, contradicting part (a) of the previous result. $\square$

Kleene's Hierarchy Theorem

Theorem #16: Let $n \in \mathbb{N}$ such that $n \geq 1$.

(a) If A is m-complete for Σ_n then $A \notin \Pi_n$.

(b) If A is m-complete for Π_n then $A \notin \Sigma_n$.

Proof:

- Suppose, in order to obtain a contradiction, that A is m-complete for Σ_n and that $A \in \Pi_n = \text{co-}\Sigma_n$.
- Then, since Π_n is m-closed (by Corollary #10) it would follow by Theorem #4 from the previous lecture that $\Sigma_n = \Pi_n$ — contradicting Corollary #15. Thus $A \notin \Pi_n$, as needed to prove part (a).
- The proof of part (b) is virtually identical — the rules of Σ_n and Π_n in the argument just need to be switched. $\square$

$\mathcal{AH}$: Definition

Definition: Let

$$\mathcal{AH} = \bigcup_{n \geq 1} \Sigma_n.$$

Since $\Pi_n \subseteq \Delta_{n+1} \subseteq \Sigma_{n+1}$, Σ_n, Π_n and Δ_n are all subsets of $\mathcal{AH}$ for all $n \in \mathbb{N}$.

$\mathcal{AH}$: Lack of Complete Sets

Theorem #17: There is no set $S \subseteq \mathbb{N}$ that is T-complete for $\mathcal{AH}$.

Proof: By contradiction: Suppose $S \subseteq \mathbb{N}$ is a set that is T-complete for $\mathcal{AH}$.

- Then $S \in \mathcal{AH}$, so that $S \in \Sigma_n \subseteq \Delta_{n+1}$ for some $n \in \mathbb{N}$.
- However, it now follows by Corollary #11 that $S \preceq_T \emptyset^{(n)}$. Furthermore, since S is T-complete for $\mathcal{AH}$, $A \preceq_T S$ for all $A \in \mathcal{AH}$, and it now follows that $A \preceq_T \emptyset^{(n)}$ for all $A \in \mathcal{AH}$ as well.

$\mathcal{AH}$: Lack of Complete Sets

- It follows by Corollary #11 that $A \in \Delta^{n+1}$ for all $A \in \mathcal{AH}$ — but it now follows by Kleene's Hierarchy Theorem that

$$\mathcal{AH} \subseteq \Delta_{n+1} \subsetneq \Sigma_{n+1} \subseteq \mathcal{AH}.$$

A ***contradiction*** has now been obtained, as needed to establish the claim. $\square$

A Set Outside of $\mathcal{AH}$

Let

$$K_1 = \{\langle n, x\rangle \mid n \in \mathbb{N} \text{ and } x \in \emptyset^{(n)}\} \subseteq \mathbb{N}.$$

Theorem #18: $K_1 \notin \mathcal{AH}$.

Proof: By contradiction. Suppose, instead, that $K_1 \in \mathcal{AH}$.

- Let $A \in \mathcal{AH}$. Then $A \in \Sigma_m \subseteq \Delta_{m+1}$ for some $m \in \mathbb{N}$ and it follows, again, by Corollary #11, that $A \preceq_T \emptyset^{(m)}$.
- Consider the function $f : \mathbb{N} \to \mathbb{N}$ such that, for all $x \in \mathbb{N}$,

  $$f(x) = \langle m, x\rangle.$$

  This is a primitive recursive function (since m is a constant) so it is certainly a computable total function. Furthermore, $x \in \emptyset^{(m)}$ if and only if $f(x) \in K_1$ for all $x \in \mathbb{N}$. Thus $\emptyset^{(m)} \preceq_T K_1$, and it follows that $A \preceq_T K_1$ as well.

A Set Outside of $\mathcal{AH}$

- Since A was arbitrarily chosen from $\mathcal{AH}$ it would now follow that K_1 is T-complete for $\mathcal{AH}$ — ***contradicting*** the previous result. Thus $K_1 \notin \mathcal{AH}$. $\square$

What We Know So Far

- We have now proved the existence of an infinite hierarchy of classes of subsets of $\mathbb{N}$. With the exception of the set at the very bottom, each of these classes includes unsolvable problems.
- The classes at the bottom levels are shown on the following slide. All classes that are shown as separate are provably different — with classes that are at a higher level containing all the classes that are at a lower level.

What We Know So Far

$$
\begin{tikzcd}[arrows=-]
 & \Delta_3 & \\
\Sigma_2 \arrow[ur] & & \Pi_2 \arrow[ul] \\
 & \Delta_2 \arrow[ul] \arrow[ur] & \\
\Sigma_1 \arrow[ur] & & \Pi_1 \arrow[ul] \\
 & \Sigma_0 = \Pi_0 = \Delta_0 = \Delta_1 \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$