Lecture #16: The Arithmetic Hierarchy
Exercises and Review

Questions for Review

1. What is the set Σ_0?
2. How are the sets Σ_{n+1} and Σ_n related for every number $n \in \mathbb{N}$?
3. What is the set Π_n for $n \in \mathbb{N}$?
4. What is the set Δ_n for $n \in \mathbb{N}$?
5. What is the ***Arithmetic Hierarchy***?
6. Suppose that $G : \mathbb{N} \to \mathbb{N}$ is a unary total function. What is the ***jump*** of G?
7. If G is as above, how are G and G' related?
8. Suppose that G is as above and $A \subseteq \mathbb{N}$. Give two other conditions that are provably equivalent to the condition that $A \preceq_1 G'$.
9. Suppose $F, G : \mathbb{N} \to \mathbb{N}$ are unary total functions such that F is G-computable. How are F' and G' related?
10. Now let $G \subseteq \mathbb{N}$ and let $n \in \mathbb{N}$. What is the "jump, iterated n times" of G denoted $G^{(n)}$)?
11. For $A \subseteq \mathbb{N}$ and $n \in \mathbb{N}$, state another condition that is provably equivalent to the condition that A is $\emptyset^{(n)}$-computably enumerable.
12. How are the set $\emptyset^{(n)}$ and the collection of sets Σ_n related?
13. Are Σ_n and Π_n m-closed? (Note that this would imply that they are 1-closed as well.)
14. How are the set $\emptyset^{(n)}$ and the collection of sets Δ_{n_1} related?
15. State ***Kleene's Hierarchy Theorem***.
16. What is $\mathcal{AH}$?
17. Are there T-complete sets for $\mathcal{AH}$? Why (or why not)?
18. Name and describe a set that is *not* in $\mathcal{AH}$.

Problems To Be Solved

A subset of $\mathbb{N}$ that is not in the Arithmetic Hierarchy was introduced in Lecture #15. The goal of this set of problems is to produce another such set $D \subseteq \mathbb{N}$.

Consider a number $x \in \mathbb{N}$. Note that this can be used to encode a set A_x in the Arithmetic Hierarchy as follows:

- If Lt$(x) = 0$ (so that $x = 0$ or $x = 1$) then $A_x = \emptyset$.
- It Lt$(x) = 1$ then A_x is the domain of the (partial or total) function computed by the $\mathcal{S}$-program $\mathcal{P}$ such that $\#(\mathcal{P}) = (x)_1$ — so that A_x (a computably enumerable set) is in Σ_1.
- If Lt$(x) = k \geq 2$ then a sequence of sets

  $$A^{(1)}, A^{(2)}, \ldots, A^{(k)}$$

  can be defined as follows: $A^{(1)}$ is the domain of the (partial or total) function computed by the $\mathcal{S}$-program $\mathcal{P}_1$ such that $\#(\mathcal{P}_1) = (x)_1$. For $1 \leq \ell \leq k-1$, $A^{(\ell+1)}$ is the domain of the (partial or total) function that is $A^{(\ell)}$-computed by the $\mathcal{S}$-program with a macro, $\mathcal{P}_{\ell+1}$, such that $\#(\mathcal{P}_{\ell+1}) = (x)_{\ell+1}$. Now $A_x = A^{(k)}$.

1. Sketch a proof that every set $A \in \mathcal{AH}$ is equal to A_ℓ, for some number $\ell \in \mathbb{N}$.
2. Use this describe a set $D \subseteq \mathbb{N}$ that can be proved — using a ***diagonalization argument*** — not to be in $\mathcal{AH}$.

Note: In fact, a diagonalization argument can be used to prove that the set of all subsets of $\mathbb{N}$ is ***uncountable***. The number of sets in the Arithmetic Hierarchy can be shown by the above to be *countable*, and it follows that the number of subsets of $\mathbb{N}$ that are not in the Arithmetic Hierarchy is uncountable as well.