The Arithmetic Hierarchy

Proofs of Claims

This document includes details of proofs that were mentioned in the lecture slides.

Arithmetic Hierarchy

Theorem 2. *Let G be a unary total function. Then the jump G' of G is G-computably enumerable but not G-computable*

Proof. Let $G : \mathbb{N} \to \mathbb{N}$ be a unary total function. Then, since

$$G' = \{n \in \mathbb{N} \mid n \in W_n^G\} = \{n \in \mathbb{N} \mid \Phi_G(n, n) \text{ is defined}\},$$

the Relativized Universality Theorem (Theorem #8 from Lecture #14) can be used to show that G' is G-computably enumerable.

Now, if $\overline{G}'$ was G-computably enumerable as well then it would follow that $\overline{G}' = W_i^G$ for some number $i \in \mathbb{N}$. However, it would then follow that

$$\begin{aligned} i \in G' &\iff i \in W_i^G && \text{(by the definition of } G'\text{)} \\ &\iff i \in \overline{G}' && \text{(by the choice of } i\text{).} \end{aligned}$$

This is certainly impossible — and this ***contradiction*** establishes that $\overline{G}'$ is not G-computably enumerable. It also follows that G' is not G-computable, as needed to complete the proof. ☐

Theorem 3. *Let $G : \mathbb{N} \to \mathbb{N}$ be a* **unary total function** *and let $A \subseteq \mathbb{N}$. Then the following are equivalent:*

(a) $A \preceq_1 G'$.

(b) $A \preceq_m G'$.

(c) A *is G-computably enumerable.*

Proof. Let $G : \mathbb{N} \to \mathbb{N}$ be a unary total function and let $A \subseteq \mathbb{N}$. Then it is immediate from the definitions of $\preceq_1$ and $\preceq_m$ that (a) $\Rightarrow$ (b).

Suppose that $A \preceq_m G'$. Then there exists a computable total function $h : \mathbb{N} \to \mathbb{N}$ such that

$$A = \{x \in \mathbb{N} \mid h(x) \in G'\}.$$

It now follows by the definition of G' that

$$A = \{x \in \mathbb{N} \mid \Phi_G(h(n), h(n)) \text{ is defined}\}.$$

The Relativized Universality Theorem can be used to show that the total function $g : \mathbb{N} \to \mathbb{N}$ such that, for all $n \in \mathbb{N}$,

$$g(n) = \Phi_G(h(n), h(n))$$

is a G-computable partial or total function. It follows by the above that A is G-computably enumerable. Thus (b) $\Rightarrow$ (c).

To see that (c) $\Rightarrow$ (a), suppose that A is G-computably enumerable. Then there exists a G-computable partial or total function $f : \mathbb{N} \to \mathbb{N}$ such that

$$A = \{n \in \mathbb{N} \mid f(n) \text{ is defined}\}.$$

Let $g : \mathbb{N}^2 \to \mathbb{N}$ such that $g(t, x) = f(x)$ for all $t, x \in \mathbb{N}$; g is certainly G-computable too. It follows by the Relativized Universality Theorem and the Relativized and Strengthened Parameter Theorem that there is a number $e \in \mathbb{N}$ such that, for all $t, x \in \mathbb{N}$,

$$g(t, x) = \Phi_G^{(2)}(t, x, e) = \Phi_G(t, S_1^1(x, e)) \tag{1}$$

where $S_1^1 : \mathbb{N}^2 \to \mathbb{N}$ is the primitive recursive function introduced in the Relativized and Strengthened Parameter Theorem. Thus

$$\begin{aligned}
A &= \{x \in \mathbb{N} \mid f(x) \text{ is defined}\} \\
&= \{x \in \mathbb{N} \mid g(S_1^1(x, e), x) \text{ is defined}\} \quad &&\text{(since g's value does not depend on its first input)} \\
&= \{x \in \mathbb{N} \mid \Phi_G(S_1^1(x, e), S_1^1(x, e)) \text{ is defined}\} &&\text{(by the equation at line (1))} \\
&= \{x \in \mathbb{N} \mid S_1^1(x, e) \in G'\}.
\end{aligned}$$

Since the function $S_1^1 : \mathbb{N}^2 \to \mathbb{N}$ is primitive recursive it is certainly a total computable function, so it follows that $A \preceq_m G'$. Furthermore, the Relativized and Strengthened Parameter Theorem implies that the function mapping x to $S_1^1(x, e)$ (for a fixed number e) is a one-one function — so it also follows from the above that $A \preceq_1 G'$. Thus $(c) \Rightarrow (a)$, as needed to complete the proof. $\square$

Theorem 4. *Let $F, G : \mathbb{N} \to \mathbb{N}$ be unary total functions. If F is G-computable then $F' \preceq_1 G'$.*

Proof. Let $F, G : \mathbb{N} \to \mathbb{N}$ be unary total functions, and suppose that F is G-computable. By Theorem 2 F' is F-computably enumerable, so there exists an F-computable partial (or total) function $f : \mathbb{N} \to \mathbb{N}$ such that

$$F' = \{x \in \mathbb{N} \mid f(x) \text{ is defined}\}.$$

It follows by Theorem #5 in Lecture #14 that f is also a G-computable partial function.

This implies (by definition) that F' is G-computably enumerable. It now follows by Theorem 3 that $F' \preceq_1 G$, as claimed. ☐

The proofs of Theorems 5 and 6 were simple enough for their discussion to be included in the lecture slides.

Theorem 7. *For every set $A \subseteq \mathbb{N}$ and every number $n \in \mathbb{N}$, $A \in \Sigma_{n+1}$ if and only if A is $\emptyset^{(n)}$-computably enumerable.*

Proof. Let $A \subseteq \mathbb{N}$.

Suppose first that A is $\emptyset^{(n)}$-computably enumerable. Then, since $\emptyset^{(n)} \in \Sigma_n$ (by Theorem 6, it follows by the definition of Σ_{n+1} that $A \in \Sigma_{n+1}$.

In order to establish the converse, we will prove that

> For every set $A \subseteq \mathbb{N}$ and every number $n \in \mathbb{N}$, if $A \in \Sigma_{n+1}$ then A is $\emptyset^{(n)}$-computably enumerable.

by induction on n.

Basis: Suppose that $n = 0$, so that $A \in \Sigma_1$. Then A is computably enumerable, and this implies that A is $\emptyset^{(0)}$-computably enumerable, as claimed.

Inductive Step: Let n be an integer such that $n \geq 0$. It is necessary and sufficient to use the following

> Inductive Hypothesis: For every set $A \subseteq \mathbb{N}$, if $A \in \Sigma_{n+1}$ then A is $\emptyset^{(n)}$-computably enumerable.

to prove the following

> Inductive Claim: For every set $B \subseteq \mathbb{N}$, if $B \in \Sigma_{n+2}$ then B is $\emptyset^{(n+1)}$-computably enumerable.

With that noted, let B be an arbitrarily chosen set in Σ_{n+2}. Then there exists a set $C \in \Sigma_{n+1}$ such that B is C-computably enumerable. It follows by the Inductive Hypothesis that C is $\emptyset^{(n)}$-computably enumerable.

Theorem 3 now implies both that $B \preceq_1 C'$ and that $C \preceq_1 (\emptyset^{(n)})' = \emptyset^{(n+1)}$. It next follows by Theorem 4 that $C' \preceq_1 (\emptyset^{(n+1)})' = \emptyset^{(n+2)}$ as well.

Now, since $\preceq_1$ is a reducibility, $B \preceq_1 C'$ and $C' \preceq_1 \emptyset^{(n+2)}$, it follows that $B \preceq_1 \emptyset^{(n+2)}$. Finally, it follows by another application of Theorem 3 that B is $\emptyset^{(n+1)}$-computably enumerable — as needed to complete the Inductive Step, and the proof of this claim. $\square$

Corollaries 8 and 9 were stated and proved in the lecture notes.

Corollary 10. *Let $n \in \mathbb{N}$ such that $n \geq 1$.*

(a) Σ_n and Π_n are both m-closed.

(b) Σ_n and Π_n are both 1-closed.

Proof. (a) Let $A \subseteq \mathbb{N}$ and let $B \in \Sigma_n$ such that $A \preceq_m B$. It is necessary and sufficient to prove that $A \in \Sigma_n$, as well, in order to prove that Σ_n is m-closed.

Since $B \in \Sigma_n$ it follows by Corollary 8 that $B \preceq_m \emptyset^{(n)}$ and, since $A \preceq_m B$ and $\preceq_m$ is a reducibility, $A \preceq_m \emptyset^{(n)}$. It follows by another application of Corollary 8 that $A \in \Sigma_n$, as needed to prove that Σ_n is m-closed.

Since $\Pi_n =$ co-Σ_n, it follows by Corollary 8 from the previous lecture that Π_n is m-closed as well.

(b) Notice that if if $A \preceq_1 B$ and $B \in \Sigma_n$ (respectively, $B \in \Pi_n$) then $A \preceq_m B$ as well. The fact that Σ_n (respectively, Π_n) is 1-closed is a straightforward consequence of the fact that Σ_n (respectively, Π_n) is m-closed. $\square$

Corollary 11. *For every set $A \subseteq \mathbb{N}$ and for all $n \in \mathbb{N}$, $A \in \Delta_{n+1}$ if and only if $A \preceq_T \emptyset^{(n)}$. Thus $\emptyset^{(n)}$ is T-complete for Δ_{n+1}.*

Proof. Let $A \subseteq \mathbb{N}$ and let $n \in \mathbb{N}$.

By definition, $A \preceq_T \emptyset^{(n)}$ if and only if there is an $\mathcal{S}$-program with an oracle that $\emptyset^{(n)}$-computes the characteristic function p_A of A — that is, if and only if A is $\emptyset^{(n)}$-computable. By Theorem 15 from the previous lecture this is true if and only if both A and $\overline{A}$ are both $\emptyset^{(n)}$-computably enumerable. By Theorem 7, this is the case if and only if $A \in \Sigma_{n+1}$ and $\overline{A} \in \Sigma_{n+1}$ — that is, if and only if

$$A \in \Sigma_{n+1} \cap \Pi_{n+1} = \Delta_{n+1},$$

as required.

It remains only to prove that $\emptyset^{(n)} \in \Delta_{n+1}$ to conclude that $\emptyset^{(n)}$ is complete for Δ_{n+1}. Now, $\emptyset^{(n)} \in \Sigma_n \subseteq \Sigma_{n+1}$ by Theorems 6 and 1. $\overline{\emptyset^{(n)}}$ is certainly $\emptyset^{(n)}$-computable, so that $\overline{\emptyset^{(n)}}$ is $\emptyset^{(n)}$-computably enumerable and $\overline{\emptyset^{(n)}} \in \Sigma_{n+1}$, by Theorem 7. Thus $\emptyset^{(n)} \in$ co-$\Sigma_{n+1} = \Pi_{n+1}$. It follows that $\emptyset^{(n)} \in \Sigma_{n+1} \cap \Pi_{n+1} = \Delta_{n+1}$, as required to complete the proof. □

Theorem 12. *For every number $n \in \mathbb{N}$, $\Sigma_n \cup \Pi_n \subseteq \Delta_{n+1}$.*

Proof. If $n = 0$ then Σ_n and Π_n are both the class of all computable sets, while Δ_{n+1} is the class of sets $A \subseteq \mathbb{N}$ such that both A and $\overline{A}$ are both computably enumerable — so that, in fact,

$$\Sigma_0 \cup \Pi_0 = \Sigma_0 = \Delta_1.$$

Suppose, instead, that n is a positive integer.

If $A \in \Sigma_n$ then, by Corollary 8, $A \preceq_1 \emptyset^{(n)}$. It follows that $A \preceq_T \emptyset^{(n)}$ so that, by Corollary 11, $A \in \Delta_{n+1}$.

On the other hand, if $A \in \Pi_n$ then $\overline{A}_n \in \Sigma_n$, so that $\overline{A} \preceq_1 \emptyset^{(n)}$ by Corollary 8, and $\overline{A} \preceq_T \emptyset^{(n)}$. It is easy to see that $A \preceq_T \overline{A}$ for any set $A \subseteq \mathbb{N}$ and, since $\preceq_T$ is a reducibility, $A \preceq_T \emptyset^{(n)}$ in this case too. Applying Corollary 11 once again we may conclude that $A \in \Delta_{n+1}$ in this case as well.

Thus $\Sigma_n \cup \Pi_n \subseteq \Delta_{n+1}$ as claimed. □

Theorem 13. *For every number $n \geq 1$, $\emptyset^{(n)} \in \Sigma_n \setminus \Delta_n$. That is, $\emptyset^{(n)} \in \Sigma_n$ and $\emptyset^{(n)} \notin \Delta_n$.*

Proof. Let n be a positive integer. It then follows by Theorem 6 that $\emptyset^{(n)} \in \Sigma_n$.

If $\emptyset^{(n)}$ also belonged to $\Delta_n = \Sigma_n \cap \Pi_n$ then it would follow that $\overline{\emptyset^{(n)}} \in \Sigma_n$ as well — so that both $\emptyset^{(n)}$ and $\overline{\emptyset^{(n)}}$ are $\emptyset^{(n-1)}$-computably enumerable (by Theorem 7). However this would imply (by Theorem 13 from the previous lecture) that $\emptyset^{(n)}$ is $\emptyset^{(n-1)}$-computable, and this would contradict Theorem 5.

Thus $\emptyset^{(n)} \notin \Delta_n$, as claimed. □

Hierarchy Theorem

Theorem 14 (Kleene's Hierarchy Theorem)**.** *Let $n \in \mathbb{N}$ such that $n \geq 1$.*

(a) $\Delta_n \subset \Sigma_n$ *and* $\Delta_n \subset \Pi_n$.

(b) $\Sigma_n \subset \Sigma_{n+1}$ *and* $\Pi_n \subset \Pi_{n+1}$.

(c) $\Sigma_n \cup \Pi_n \subset \Delta_{n+1}$.

Proof. Let n be a positive integer.

(a) It follows by the definition of Δ_n that $\Delta_n \subseteq \Sigma_n$ and $\Delta_n \subseteq \Pi_n$, so it suffices to show that $\Delta_n \neq \Sigma_n$ and $\Delta_n \neq \Pi_n$.

By Theorem 13, $\emptyset^{(n)} \in \Sigma_n \setminus \Delta_n$, and this implies that $\Delta_n \neq \Sigma_n$. This implies that $\overline{\emptyset^{(n)}} \in \Pi_n \setminus \Delta_n$, so that $\Delta_n \neq \Pi_n$ as well.

(b) It follows by Theorem 1 that $\Sigma_n \subseteq \Sigma_{n+1}$ and $\Pi_n \subseteq \Pi_{n+1}$, so it suffices to show that $\Sigma_n \neq \Sigma_{n+1}$ and $\Pi_n \neq \Pi_{n+1}$.

By Theorem 6, $\emptyset^{(n+1)} \in \Sigma_{n+1}$. If $\emptyset^{(n+1)}$ also belonged to Σ_n then it would follow by Theorem 12 that $\emptyset^{(n+1)} \in \Delta_{n+1}$, contradicting Theorem 13. Thus $\emptyset^{(n+1)} \in \Sigma_{n+1} \setminus \Sigma_n$ and $\Sigma_n \neq \Sigma_{n-1}$.

Since $\Pi_n =$ co-Σ_n and $\Pi_{n+1} =$ co-Σ_{n+1} this implies that $\Pi_n \neq \Pi_{n+1}$ as well.

(c) It follows by Theorem 12 that $\Sigma_n \cup \Pi_n \subseteq \Delta_{n+1}$, so it suffices to show that $\Sigma_n \cup \Pi_n \neq \Delta_{n+1}$.

Let

$$A_n = \emptyset^{(n)} \oplus \overline{\emptyset^{(n)}} = \{2m \mid m \in \emptyset^{(n)}\} \cup \{2m+1 \mid m \in \overline{\emptyset^{(n)}}\},$$

as defined in the previous lecture.

It should not be hard to see that $A_n \preceq_T \emptyset^{(n)}$, and it follows by Corollary 11 that $A_n \in \Delta_{n+1}$.

Notice that

$$\emptyset^{(n)} = \{x \in \mathbb{N} \mid 2x \in A_n\}$$

and

$$\overline{\emptyset(n)} = \{x \in \mathbb{N} \mid 2x+1 \in A_n\}$$

so that (considering the one-one computable functions $f(x) = 2x$ and $g(x) = 2x+1$, respectively) $\emptyset^{(n)} \preceq_1 A_n$ and $\overline{\emptyset^{(n)}} \preceq_1 A_n$.

Suppose that $A_n \in \Sigma_n$. Then, since Σ_n is 1-closed (by Corollary 10), it would follow that $\overline{\emptyset^{(n)}} \in \Sigma_n$, and that $\emptyset^{(n)} \in \Delta_n$. This contradicts Theorem 13 — so $A_n \notin \Sigma_n$.

Suppose that $A_n \in \Pi_n$. Then, since Π_n is also 1-closed (by Corollary 10), it would follow that $\emptyset^{(n)} \in \Pi_n$. Once again, it would follow that $\emptyset^{(n)} \in \Delta_n$, contradicting Theorem 13. Thus $A_n \notin \Pi_n$.

It follows that $A_n \in \Delta_{n+1} \setminus (\Sigma_n \cup \Pi_n)$ as needed to show that $\Sigma_n \cup \Pi_n \neq \Delta_{n+1}$ and to complete the proof. $\square$