Computer Science 513

Applying the Arithmetic Hierarchy

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Lecture #17

Goals for Today

Goals for Today:

- Establish a useful alternative characterization of the sets in the Arithmetic Hierarchy.
- More accurately classify some unsolvable problems by using results from the previous lecture to locate them within the arithmetic hierarchy.

Σ_n-, Π_n-, and Δ_n-Predicates: Definition

Consider a ***total*** predicate $P : \mathbb{N}^s \to \{0, 1\}$.

- We will associate a *set*

$$S_P = \{x \in \mathbb{N} \mid P((x)_1, (x)_2, \ldots, (x)_s) = 1\} \subseteq \mathbb{N}$$

  with every such predicate.

Definition: Let P and S_P be as above and let $n \in \mathbb{N}$.

(a) P is a $\boldsymbol{\Sigma_n}$***-predicate*** if $S_P \in \Sigma_n$.

(b) P is a $\boldsymbol{\Pi_n}$***-predicate*** if $S_P \in \Pi_n$.

(c) P is a $\boldsymbol{\Delta_n}$***-predicate*** if $S_P \in \Delta_n$.

Note: All predicates considered in this lecture are total predicates.

Σ_n-, Π_n-, and Δ_n-Predicates and Sets

- Let $A \subseteq \mathbb{N}$ and consider the associated *characteristic function* $P_A : \mathbb{N} \to \{0, 1\}$ where, for all $x \in \mathbb{N}$,

$$P_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

- By the previous definition, P_A is a Σ_n-predicate (respectively, Π_n-predicate or Δ_n-predicate) if and only if $S_{P_A} \in \Sigma_n$ (respectively, Π_n, Δ_n) where

$$\begin{aligned} S_{P_A} &= \{x \in \mathbb{N} \mid P_A((x)_1) = 1\} \\ &= \{x \in \mathbb{N} \mid (x)_1 \in A\}. \end{aligned}$$

Σ_n-, Π_n-, and Δ_n-Predicates and Sets

Lemma #1: Let $A \subseteq \mathbb{N}$, let $P_A : \mathbb{N} \to \{0, 1\}$ be the characteristic function of A, and let $n \in \mathbb{N}$.

(a) $A \in \Sigma_n$ if and only if P_A is a Σ_n-predicate.

(b) $A \in \Pi_n$ if and only if P_A is a Π_n-predicate.

(c) $A \in \Delta_n$ if and only if P_A is a Δ_n-predicate.

How This is Proved: This is a straightforward consequence of Corollary #10, from Lecture #16, and the above definitions. The supplemental document includes a proof and of this and other claims included in these notes.

This, and the next sequence of lemmas, are used in the proof of ***Post's Theorem***, which is stated and proved after them.

Σ_n-, Π_n-, and Δ_n-Predicates: Closure Properties

Lemma #2: Let $s, k \geq 1$. Let $P : \mathbb{N}^s \to \{0, 1\}$ and let $Q : \mathbb{N}^k \to \{0, 1\}$ be total predicates such that there exist computable total functions $f_1, f_2, \dots, f_s : \mathbb{N}^k \to \mathbb{N}$ such that, for all $t_1, t_2, \dots, t_k \in \mathbb{N}$,

$$Q(t_1, t_2, \dots, t_k) = 1$$

if and only if

$$P(f_1(t_1, t_2, \dots, t_k), f_2(t_1, t_2, \dots, t_k), \dots, f_s(t_1, t_2, \dots, t_k)) = 1.$$

Let $n \in \mathbb{N}$.

(a) If P is a Σ_n-predicate then Q is a Σ_n-predicate as well.

(b) If P is a Π_n-predicate then Q is a Π_n-predicate as well.

(c) If P is a Δ_n-predicate then Q is a Δ_n-predicate as well.

Σ_n-, Π_n-, and Δ_n-Predicates: Closure Properties

Lemma #3: Let $s \geq 1$, $n \geq 0$, and let $P : \mathbb{N}^s \rightarrow \{0, 1\}$ be a total predicate.

(a) P is a Σ_n-predicate if and only if $\neg P$ is a Π_n-predicate.

(b) P is a Π_n-predicate if and only if $\neg P$ is a Σ_n-predicate.

Lemma #4: Let $n \in \mathbb{N}$, $s \geq 1$ and let $P, Q : \mathbb{N}^s \rightarrow \{0, 1\}$ be total predicates.

(a) If P and Q are both Σ_n-predicates then $P \wedge Q$ and $P \vee Q$ are both Σ_n-predicates as well.

(b) If P and Q are both Π_n-predicates then $P \wedge Q$ and $P \vee Q$ are both Π_n-predicates as well.

Σ_n-, Π_n-, and Δ_n-Predicates: Closure Properties

Lemma #5: Let $s, n \geq 1$ and let $P : \mathbb{N}^s \to \{0, 1\}$ and $Q : \mathbb{N}^{s+1} \to \{0, 1\}$ be total predicates such that, for all $x_1, x_2, \ldots, x_s \in \mathbb{N}$,

$$P(x_1, x_2, \ldots, x_s) = 1$$

if and only if

$$(\exists y)(Q(x_1, x_2, \ldots, x_s, y) = 1).$$

If Q is a Σ_n-predicate then P is a Σ_n-predicate as well.

Σ_n-, Π_n-, and Δ_n-Predicates: Closure Properties

Lemma #6: Let $s, n \geq 1$ and let $P : \mathbb{N}^s \to \{0, 1\}$ and $Q : \mathbb{N}^{s+1} \to \{0, 1\}$ be total predicates such that, for all $x_1, x_2, \ldots, x_s \in \mathbb{N}$,

$$P(x_1, x_2, \ldots, x_s) = 1$$

if and only if

$$(\forall y)(Q(x_1, x_2, \ldots, x_s, y) = 1).$$

If Q is a Π_n-predicate then P is a Π_n-predicate as well.

Post's Theorem

Theorem #7 (Post's Theorem): Let $n, s \in \mathbb{N}$, $s \geq 1$, and let $P : \mathbb{N}^s \to \{0, 1\}$ be a total predicate. Then P is a Σ_{n+1}-predicate if and only if there exists a Π_n-predicate $Q : \mathbb{N}^{s+1} \to \{0, 1\}$ such that, for all $x_1, x_2, \ldots, x_s \in \mathbb{N}$,

$$P(x_1, x_2, \ldots, x_s) = 1 \iff (\exists y)(Q(x_1, x_2, \ldots, x_s, y) = 1).$$

Once again, the supplemental document includes a proof of this result. It is follows by a straightforward (but long) sequence of applications of the minor results that have now been established — including the "Finiteness Theorem", from Lecture #14.

Post's Theorem: Useful Corollaries

Corollary #8: Let $n, s \in \mathbb{N}$, $s \geq 1$ and suppose that $P, Q : \mathbb{N}^{s+1}$ are total predicates such that, for all $x_1, x_2, \ldots, x_s, y \in \mathbb{N}$,

$$Q(x_1, x_2, \ldots, x_s, y) = 1 \iff (\forall t)_{\leq y}(P(x_1, x_2, \ldots, x_s, t) = 1).$$

If P is a Σ_n-predicate then Q is a Σ_n-predicate as well.

This is a reasonably straightforward consequence of Post's Theorem and one of the minor results used to prove it. Details are included in the supplemental document.

Post's Theorem: Useful Corollaries

Corollary #9: Let $n, s \in \mathbb{N}$, $s \geq 1$, and let $P : \mathbb{N}^s \to \{0, 1\}$ be a total predicate. Then P is a Π_{n+1}-predicate if and only if there exists a Σ_n-predicate $Q : \mathbb{N}^{s+1} \to \mathbb{N}$ such that, for all $x_1, x_2, \ldots, x_s \in \mathbb{N}$,

$$P(x_1, x_2, \ldots, x_s) = 1 \iff (\forall y)(Q(x_1, x_2, \ldots, x_s, y) = 1).$$

Proof:

- This is a straightforward consequence of Lemma #3, De Morgan's Laws, and Post's Theorem. $\square$

Post's Theorem: Useful Corollaries

Corollary #10: Let $n, s \in \mathbb{N}$, $s \geq 1$, and let $P, Q : \mathbb{N}^{s+1} \to \{0, 1\}$ be total predicates. If P is a Π_n-predicate and, for all $x_1, x_2, \ldots, x_s, y \in \mathbb{N}$,

$$Q(x_1, x_2, \ldots, x_s, y) = 1 \iff (\exists t)_{\leq y}(P(x_1, x_2, \ldots, x_s, t) = 1),$$

then Q is also a Π_n-predicate.

Proof:

- This is a straightforward consequence of Lemma #3, De Morgan's Laws, and Corollary #8. $\square$

Arithmetic Predicates and Alternating Quantifiers

Definition: If $s \geq 1$ then a total predicate $P : \mathbb{N}^s \to \{0, 1\}$ is an ***arithmetic predicate*** if there exists a number $n \in \mathbb{N}$ and a ***computable*** predicate $R : \mathbb{N}^{s+n} \to \{0, 1\}$ such that, for all $x_1, x_2, \ldots, x_s \in \mathbb{N}$,

$$\begin{aligned} P(x_1, x_2, \ldots, x_s) = 1 \iff \\ (Q_1 y_1)(Q_2 y_2) \ldots (Q_n y_n) \\ (R(x_1, x_2, \ldots, x_s, y_1, y_2, \ldots, y_n) = 1), \quad (1) \end{aligned}$$

where each of $Q_1, Q_2, \ldots, Q_n$ is either the symbol $\exists$ or the symbol $\forall$.

Arithmetic Predicates and Alternating Quantifiers

Furthermore, if P is an arithmetic predicate, and $Q_1, Q_2, \ldots, Q_n$ are as above, then we say that the Q_i's are ***alternating*** if, for all i such that $1 \leq i < n$,

- if Q_i is $\exists$ then Q_{i+1} is $\forall$, and
- if Q_i is $\forall$ then Q_{i+1} is $\exists$.

A proof of Theorem #11 (which follows) is included in the supplemental document. Theorem #12 is a straightforward consequence of Theorems #1 and #11.

Alternative Characterization of the Arithmetic Hierarchy

Theorem #11:

(a) Let $n \in \mathbb{N}$. Then every total predicate that is either a Σ_n- or a Π_n-predicate is an arithmetic predicate.

(b) Every arithmetic predicate is a Σ_k-predicate for some $k \in \mathbb{N}$ as well as a Π_ℓ-predicate for some $\ell \in \mathbb{N}$.

(c) Let $n, s \in \mathbb{N}$, $s \geq 1$ and let $P : \mathbb{N}^s \to \{0, 1\}$ be a total predicate. Then P is a Σ_n-predicate if and only if it can be written in the form shown on line (1) where Q_1 is $\exists$ (if $n \geq 1$) and the Q_i's are alternating.

(d) Let $n, s \in \mathbb{N}$, $s \geq 1$ and let $P : \mathbb{N}^s \to \{0, 1\}$ be a total predicate. Then P is a Π_n-predicate if and only if it can be written in the form shown on line (1) where Q_1 is $\forall$ (if $n \geq 1$) and the Q_i's are alternating.

Another Alternative Characterization of the Arithmetic Hierarchy

Theorem #12: For $n \in \mathbb{N}$, and a set $S \subseteq \mathbb{N}$,

(a) $S \in \Sigma_n$ if and only if there exists a ***computable predicate*** $P : \mathbb{N}^{n+1} \to \{0, 1\}$ such that, for all $y \in \mathbb{N}$,

$$y \in S \iff (Q_1 x_1)(Q_2 x_2) \ldots (Q_n x_n) \\ (P(x_1, x_2, \ldots, x_n, y) = 1), \quad (2)$$

where Q_i is the quantifier $\exists$ when i is odd and Q_i is $\forall$ when i is even, for $1 \leq i \leq n$.

(b) $S \in \Pi_n$ if and only if there exists a ***computable predicate*** $P : \mathbb{N}^{n+1} \to \{0, 1\}$ such that, for all $y \in \mathbb{N}$, the equation at line (2) is satisfied where Q_i is the quantifier $\forall$ when i is odd and Q_i is $\exists$ when i is even, for $1 \leq i \leq n$.

The Arithmetic Hierarchy and the Polynomial Hierarchy

Note: Students who have taken, or are currently taking CPSC 511 —*Complexity Theory* — have probably seen a definition of the ***Polynomial Hierarchy.***

Now that this alternative characterization of the Arithmetic Hierarchy is available it should not be too hard to see that the Polynomial Hierarchy really was directly inspired by the Arithmetic Hierarchy.

Furthermore, various ***conjectures*** about the Polynomial Hierarchy are inspired by the Arithmetic Hierarchy too: They are often conjectures that things about the Arithmetic Hierarchy that have been *proved* for the Arithmetic Hierarchy are true for the Polynomial Hierarchy as well.

Classifying Unsolvable Problems

The previous results have been used to "classify" various unsolvable problems by placing corresponding sets at various levels in the arithmetic hierarchy. The rest of this set of notes include several such classifications.

Classifying Total

Recall the set

$$\text{Total} = \{n \in \mathbb{N} \mid \text{the program encoded by } n \text{ computes a total function } f : \mathbb{N} \to \mathbb{N}\}.$$

- It was shown in Lecture #11 that $K \preceq_m \text{Total}$, so that the above set is not computable.

Classifying Total

Recall the ***total predicate*** $\mathrm{Halt}_1 : \mathbb{N}^3 \to \{0, 1\}$ such that, for all $x, y, t \in \mathbb{N}$, if $\mathcal{P}$ is an $\mathcal{S}$-program such that $\#(\mathcal{P}) = x$, then

$$\mathrm{Halt}_1(x, y, t) = 1$$

if and only if $\mathcal{P}$ halts, when executed on input y, after taking at most t steps.

- It was shown in Lecture #9 that "Halt_1" is a primitive recursive predicate.
- Notice that

$$\mathrm{Total} = \{x \in \mathbb{N} \mid (\forall y)(\exists t)(\mathrm{Halt}(x, y, t))\}$$

so it follows by Theorem #12, above, that $\mathrm{Total} \in \Pi_2$.

Classifying Total

Theorem #13: Total is 1- complete for Π_2. Therefore Total $\notin \Sigma_2$.

Proof: Since no 1-complete function for Π_2 can belong to Σ_2, and since Total $\in \Pi_2$, it is necessary and sufficient to prove that $A \preceq_1$ Total for every set $A \subseteq \mathbb{N}$ such that $A \in \Pi_2$.

With that noted, let $A \subseteq \mathbb{N}$ such that $A \in \Pi_2$.

- Since $A \in \Pi_2$, there exists a computable total predicate $R : \mathbb{N}^3 \to \mathbb{N}$ such that, for all $x \in \mathbb{N}$, $x \in A$ if and only if

$$(\forall y)(\exists z)(R(x, y, z) = 1).$$

Classifying Total

- Let $h : \mathbb{N}^2 \to \mathbb{N}$ such that, for all $x, y \in \mathbb{N}$,

$$h(y, x) = \mu z R(x, y, z)$$

  so that h is obtained from R using ***unbounded minimization*** and is, therefore, a computable partial or total function (and $h(y, x)$ is undefined if and only if $R(x, y, z) = 0$ for all $z \in \mathbb{N}$).
- Then h is computed by some $\mathcal{S}$ program, $\mathcal{P}$, with a number e for some $e \in \mathbb{N}$.
- Thus, for all $x, y \in \mathbb{N}$,

$$\begin{aligned} &(\exists z)(R(x, y, z) = 1) \\ &\quad\Longleftrightarrow\; h(y, x) \text{ is defined} \\ &\quad\Longleftrightarrow\; \Phi^{(2)}(y, x, e) \text{ is defined} \\ &\quad\Longleftrightarrow\; \Phi^{(1)}(y, S_1^1(x, e)) \text{ is defined} \\ &\qquad\qquad\qquad \text{(by the Parameter Theorem).} \end{aligned}$$

Classifying Total

- Let $g : \mathbb{N} \to \mathbb{N}$ such that, for all $x \in \mathbb{N}$, $g(x) = S_1^1(x, e)$. Then it follows by the *Strengthened Parameter Theorem* that g is a one-one computable function.
- It follows that, for all $x \in \mathbb{N}$,

$$\begin{aligned} x \in A &\iff (\forall y)(\exists z)(R(x, y, z) = 1) \\ &\iff (\forall y)(h(y, x) \text{ is defined}) \\ &\iff (\forall y)(\Phi^{(1)}(y, g(x)) \text{ is defined}) \\ &\iff g(x) \in \text{Total} \end{aligned}$$

so that $A \preceq_1$ Total, as required to establish the claim. □

Classifying Infinite

Recall that, for $n \in \mathbb{N}$, W_n is the the set

$$W_n = \{x \in \mathbb{N} \mid \text{the } \mathcal{S} \text{ program } \mathcal{P} \text{ encoded by } n \text{ halts on input } x\}.$$

Let

$$\text{Infinite} = \{n \in \mathbb{N} \mid W_n \text{ is an infinite set}\}.$$

Notice that

$$\begin{aligned} \text{Infinite} &= \{n \in \mathbb{N} \mid (\forall x)(\exists y)((y > x) \wedge (y \in W_n))\} \\ &= \{n \in \mathbb{N} \mid (\forall x)(\exists y)(\exists t)((y > x) \wedge (\text{Halt}_1(n, y, t)))\} \\ &= \{n \in \mathbb{N} \mid (\forall x)(\exists z)((\ell(z) > x) \wedge (\text{Halt}_1(n, \ell(z), r(z))))\} \end{aligned}$$

Since the predicate "$((\ell(z) > x) \wedge (\text{Halt}_1(n, \ell(z), r(z))))$" is a primitive recursive predicate of n, x and z, it follows that Infinite $\in \Pi_2$.

Classifying Infinite

Theorem #14: Infinite is 1-complete for Π_2. Therefore Infinite $\notin \Sigma_2$.

Proof:

- Since Total is 1-complete for Π_2, by Theorem #13, it suffices to prove that Total $\preceq_1$ Infinite.
- Consider the set

$$S = \{x \in \mathbb{N} \mid (x)_2 \in W_{(x)_1}\}.$$

The Universality Theorem can be used to establish that S is computably enumerable, since it can be used to establish that there is a computable partial function whose domain is S. Thus $S \in \Sigma_1$.

Classifying Infinite

- Consider the predicate $Q : \mathbb{N}^2 \to \{0, 1\}$ such that, for all $x, k \in \mathbb{N}$, $Q(x, k) = 1$ if and only if $k \in W_x$.
- As defined at the beginning of this set of notes, S_Q is the above set S.
- Since $S \in \Sigma_1$, Q is (by definition) a Σ_1-predicate.

Classifying Infinite

- Corollary #8 now implies that the total predicate $P : \mathbb{N}^2 \to \{0, 1\}$ such that, for all $x, z \in \mathbb{N}$, $P(x, z) = 1$ if and only if
$$(\forall k)_{\leq z}(k \in W_x)$$
is also a Σ_1-predicate. Note that if $P(x, z) = 0$ then $P(x, z + 1) = 0$ and, indeed, $P(x, \widehat{z}) = 0$ for all $\widehat{z} \geq z$, for all $x, z \in \mathbb{N}$.

- It follows that, for some number $e \in \mathbb{N}$, and for all $x, z \in \mathbb{N}$,
$$\begin{aligned} P(x, z) = 1 &\iff \Phi^{(2)}(z, x, e) \text{ is defined} \\ &\iff \Phi^{(1)}(z, S_1^1(x, e)) \text{ is defined} \\ &\iff z \in W_{S_1^1(x,e)}. \end{aligned}$$

Classifying Infinite

- Let $g : \mathbb{N} \to \mathbb{N}$ such that, for all $x \in \mathbb{N}$, $g(x) = S^1_1(x, e)$. Once again, the Strengthened Parameter Theorem implies that g is a one-one computable function.
- Note that, for all $x \in \mathbb{N}$,

$$\begin{aligned} x \in \text{Total} &\Rightarrow P(x, z) = 1 \text{ for all } z \in \mathbb{N} \\ &\Rightarrow z \in W_{g(x)} \text{ for all } z \in \mathbb{N} \\ &\Rightarrow g(x) \in \text{Total} \\ &\Rightarrow g(x) \in \text{Infinite}. \end{aligned}$$

Classifying Infinite

- As noted above, if $P(x, z) = 0$ for any $z \in \mathbb{N}$ then $P(x, z) = 0$ for all but finitely $z \in \mathbb{N}$. Thus

$$\begin{aligned} x \notin \text{Total} &\Rightarrow P(x, z) = 0 \text{ for some } z \in \mathbb{N} \\ &\Rightarrow P(x, z) = 0 \text{ for all but finitely many } z \\ &\Rightarrow W_{g(x)} \text{ is finite} \\ &\Rightarrow g(x) \notin \text{Infinite.} \end{aligned}$$

- Thus $x \in$ Total if and only if $g(x) \in$ Infinite for all $x \in \mathbb{N}$, as needed to prove that Total $\preceq_1$ Infinite and establish the claim. $\square$