Applying the The Arithmetic Hierarchy
Proofs of Claims

This document includes details of proofs that were mentioned in the lecture slides.

Alternative Characterization of the Arithmetic Hierarchy

Lemma 1. *Let $A \subseteq \mathbb{N}$, let $P_A : \mathbb{N} \to \{0, 1\}$ be the characteristic function of A, and let $n \in \mathbb{N}$.*

(a) $A \in \Sigma_n$ if and only if P_A is a Σ_n-predicate.

(b) $A \in \Pi_n$ if and only if P_A is a Π_n-predicate.

(c) $A \in \Delta_n$ if and only if P_A is a Δ_n-predicate.

Proof. Let $A \subseteq \mathbb{N}$.

(a) Suppose, first, that $A \in \Sigma_n$.

Consider the function $f : \mathbb{N} \to \mathbb{N}$ such that $f(x) = (x)_1$ for all $x \in \mathbb{N}$. Since f is a primitive recursive function, f is a computable total function. It follows by the definition of S_{P_A} that $x \in S_{P_A} \iff f(x) \in A$ for all $x \in \mathbb{N}$. Thus $S_{P_A} \preceq_m A$.

Corollary #10 of Lecture #16 implies that Σ_n is m-closed. It now follows that $S_{P_A} \in \Sigma_n$ — so that P_A is a Σ_n-predicate — since $A \in \Sigma_n$.

Suppose, instead, that P_A is a Σ_n-predicate — so that $S_{P_A} \in \Sigma_n$.

Consider the function $g : \mathbb{N} \to \mathbb{N}$ such that $g(x) = 2^x$ for all $x \in \mathbb{N}$. Since g is primitive recursive, g is a computable total function. Now, for all $x \in \mathbb{N}$,

$$\begin{aligned} g(x) \in S_{P_A} &\iff 2^x \in S_{P_A} && \text{(by the definition of } g\text{)} \\ &\iff (2^x)_1 \in A && \text{(by the definition of } S_{P_A}\text{)} \\ &\iff x \in A && \text{(since } p_1 = 2\text{).} \end{aligned}$$

Thus $A \preceq_m S_{P_A}$ and, since Σ_n is m-closed and $S_{P_A} \in \Sigma_n$, $A \in \Sigma_n$.

It now follows that $A \in \Sigma_n \iff P_A$ is a Σ_n-predicate, for all $A \subseteq \mathbb{N}$, establishing part (a).

(b) Part (b) can be proved by replacing Σ_n with Π_n in the above argument.

(c) Part (c) follows from parts (a) and (b) because $\Delta_n = \Sigma_n \cap \Pi_n$. $\square$

Lemma 2. *Let $s, k \geq 1$. Let $P : \mathbb{N}^s \to \{0, 1\}$ and let $Q : \mathbb{N}^k \to \{0, 1\}$ be total predicates such that there exist computable total functions $f_1, f_2, \dots, f_s : \mathbb{N}^k \to \mathbb{N}$ such that, for all $t_1, t_2, \dots, t_k \in \mathbb{N}$,*

$$Q(t_1, t_2, \dots, t_k) = 1$$

if and only if

$$P(f_1(t_1, t_2, \dots, t_k), f_2(t_1, t_2, \dots, t_k), \dots, f_s(t_1, t_2, \dots, t_k)) = 1.$$

Let $n \in \mathbb{N}$.

(a) If P is a Σ_n-predicate then Q is a Σ_n-predicate as well.

(b) If P is a Π_n-predicate then Q is a Π_n-predicate as well.

(c) If P is a Δ_n-predicate then Q is a Δ_n-predicate as well.

Proof. Let $s, k \geq 1$, and let $P : \mathbb{N}^k \to \{0, 1\}$, $Q : \mathbb{N}^k \to \{0, 1\}$ and $f_1, f_2, \dots, f_k : \mathbb{N}^k \to \mathbb{N}$ be as in the statement of the claim. Let $f : \mathbb{N} \to \mathbb{N}$ such that, for $t \in \mathbb{N}$,

$$f(t) = [f_1((t)_1, (t)_2, \dots (t)_k), f_2((t)_1, (t)_2, \dots, (t)_k), \dots, f_s((t)_1, (t)_2, \dots, (t)_k)].$$

Since $f_1, f_2, \dots, f_s$ are computable total functions, f is a computable total function as well.

Note that, for all $t \in \mathbb{N}$,

$$\begin{aligned} t \in S_Q &\iff Q((t)_1, (t)_2, \dots, (t)_k) = 1 && \text{(by the definition of } S_Q\text{)} \\ &\iff P(f_1((t)_1, (t)_2, \dots, (t)_k), \\ &\qquad\quad f_2((t)_1, (t)_2, \dots, (t)_k), \dots \\ &\qquad\quad f_s((t)_1, (t)_2, \dots, (t)_k)) = 1 \\ &\iff [f_1((t)_1, (t)_2, \dots, (t)_k), \\ &\qquad f_2((t)_1, (t)_2, \dots, (t)_k), \dots, \\ &\qquad f_s((t)_1, (t)_2, \dots, (t)_k)] \in S_P \\ &\iff f(t) \in S_p. \end{aligned}$$

Thus $S_Q \preceq_m S_P$.

It follows by Corollary #10 from Lecture #16 that Σ_n is m-closed for $m \geq 1$, and this establishes part (a) of the claim in this case.

Part (b) of the claim also follows by Corollary #10 from Lecture #16, because this established that Π_n is m-closed as well.

Part (c) follows from parts (a) and (b), along with the fact that

$$\Delta_n = \Sigma_n \cap \Pi_n$$

for every number $n \in \mathbb{N}$. $\square$

Lemma 3. *Let $s \geq 1$, $n \geq 0$, and let $P : \mathbb{N}^s \to \{0, 1\}$ be a total predicate.*

(a) P is a Σ_n-predicate if and only if $\neg P$ is a Π_n-predicate.

(b) P is a Π_n-predicate if and only if $\neg P$ is a Σ_n-predicate.

Proof. It suffices to note that

$$S_{\neg P} = \overline{S_P} = \{x \in \mathbb{N} \mid P((x)_1, (x)_2, \dots, (x)_s) = 0\}$$

and to recall that $\Pi_n =$ co-Σ_n. $\square$

Lemma 4. *Let $n \in \mathbb{N}$, $s \geq 1$ and let $P, Q : \mathbb{N}^s \to \{0, 1\}$ be total predicates.*

(a) If P and Q are both Σ_n-predicates then $P \wedge Q$ and $P \vee Q$ are both Σ_n-predicates as well.

(b) If P and Q are both Π_n-predicates then $P \wedge Q$ and $P \vee Q$ are both Π_n-predicates as well.

Proof. This is straightforward if $n = 0$ because Σ_0 and Π_0 are both equal to the class of computable sets. Suppose, therefore, that $n \geq 1$.

Note that

$$S_P = \{x \in \mathbb{N} \mid P((x)_1, (x)_2, \dots, (x)_s) = 1\},$$
$$S_Q = \{x \in \mathbb{N} \mid Q((x)_1, (x)_2, \dots, (x)_s) = 1\},$$

and — by a consideration of the definitions of these sets, as well as the definition of $P \wedge Q$ and $P \vee Q$ —

$$S_{P \wedge Q} = S_P \cap S_Q \quad \text{and} \quad S_{P \vee Q} = S_P \cup S_Q.$$

Suppose (in order to prove part (a)) that P and Q are both Σ_n-predicates — so that $S_P \in \Sigma_n$ and $S_Q \in \Sigma_n$.

- It follows by Theorem #7 from Lecture #10 that S_P and S_Q are both $\emptyset^{(n-1)}$-computably enumerable.
- It follows by Theorem #14 from Lecture #15 that $S_{P \wedge Q} = S_P \cap S_Q$ and $S_{P \vee Q} = S_P \cup S_Q$ are both $\emptyset^{(n-1)}$-computably enumerable too.
- It follows by another application of Theorem #7 from Lecture #10 that $S_{P \wedge Q} \in \Sigma_n$ and $S_{P \vee Q} \in \Sigma_n$ — so that $P \wedge Q$ and $P \vee Q$ are both Σ_n-predicates, establishing part (a).

Suppose (in order to prove part(b)) that P and Q are both Π_n-predicates — so that $S_P \in \Pi_n$, $\overline{S_P} \in \Sigma_n$, $S_Q \in \Pi_n$, and $\overline{S_Q} \in \Sigma_n$.

- It follows by Theorem #7 from Lecture #10 that $\overline{S_P}$ and $\overline{S_Q}$ are both $\emptyset^{(n-1)}$-computably enumerable.
- It follows by Theorem #14 from Lecture #15, and De Morgan's laws, that the sets

  $$\overline{S_{P\wedge Q}} = \overline{(S_P \cap S_Q)} = \overline{S_P} \cup \overline{S_Q}$$

  and

  $$\overline{S_{P\vee Q}} = \overline{(S_P \cup S_Q)} = \overline{S_P} \cap \overline{S_Q}$$

  are both $\emptyset^{(n-1)}$-computably enumerable too.
- It now follows by another application of Theorem #7 from Lecture #10 that $\overline{S_{P\wedge Q}} \in \Sigma_n$ and $\overline{S_{P\vee Q}} \in \Sigma_n$ as well.
- Thus $S_{P\wedge Q} \in \Pi_n$ and $S_{P\vee Q} \in \Pi_n$ — establishing that both $P \wedge Q$ and $P \vee Q$ are both Π_n-predicates, as needed to complete the proof of part (b). $\square$

Lemma 5. *Let $s, n \geq 1$ and let $P : \mathbb{N}^s \to \{0,1\}$ and $Q : \mathbb{N}^{s+1} \to \{0,1\}$ be total predicates such that, for all $x_1, x_2, \dots, x_s \in \mathbb{N}$,*

$$P(x_1, x_2, \dots, x_s) = 1$$

if and only if

$$(\exists y)(Q(x_1, x_2, \dots, x_s, y) = 1).$$

If Q is a Σ_n-predicate then P is a Σ_n-predicate as well.

Proof. Let P and Q be as above, and suppose that Q is a Σ_n predicate — so that the set $S_Q \in \Sigma_n$. It follows by another application of Theorem #7 from Lecture #16 that S_Q is $\emptyset^{(n-1)}$-computably enumerable. It follows by Theorem #15 from Lecture #15 that there exists an $\emptyset^{(n-1)}$-computable total predicate $R : \mathbb{N}^2 \to \{0,1\}$ such that

$$S_Q = \{x \in \mathbb{N} \mid (\exists t)(R(x,t) = 1)\}.$$

Now, for all $x_1, x_2, \dots, x_s, y \in \mathbb{N}$,

$$\begin{aligned} &Q(x_1, x_2, \dots, x_s, y) = 1 \\ &\qquad \iff [x_1, x_2, \dots, x_s, y] \in S_Q && \text{(by the definition of } S_Q) \\ &\qquad \iff (\exists t)(R([x_1, x_2, \dots, x_s, y], t) = 1). \end{aligned}$$

It follows that, for $x \in \mathbb{N}$,

$$\begin{aligned} x \in S_P &\iff P((x)_1, (x)_2, \ldots, (x)_s) = 1 && \text{(by the definition of } S_P\text{)} \\ &\iff (\exists y)(Q((x)_1, (x)_2, \ldots, (x)_s, y) = 1) && \text{(as given in the claim)} \\ &\iff (\exists y)(\exists t)(R([(x)_1, (x)_2, \ldots, (x)_s, y], t) = 1) && \text{(by the above)} \\ &\iff (\exists z)(R[((x)_1, (x)_2, \ldots, (x)_s, \ell(z)], r(z)) = 1) \end{aligned}$$

Theorem #4 from Lecture #14 and Theorem #15 from Lecture #15 establish that S_P is $\emptyset^{(n-1)}$-computably enumerable. Theorem #7 from Lecture #16 now implies that $S_P \in \Sigma_n$, so that P is a Σ_n-predicate, as claimed. □

Lemma 6. *Let $s, n \geq 1$ and let $P : \mathbb{N}^s \to \{0, 1\}$ and $Q : \mathbb{N}^{s+1} \to \{0, 1\}$ be total predicates such that, for all $x_1, x_2, \ldots, x_s \in \mathbb{N}$,*

$$P(x_1, x_2, \ldots, x_s) = 1$$

if and only if

$$(\forall y)(Q(x_1, x_2, \ldots, x_s, y) = 1).$$

If Q is a Π_n-predicate then P is a Π_n-predicate as well.

Proof. Suppose that Q is a Π_n-predicate, as in the statement of the claim. Then S_Q is in Π_n, so that $S_{\neg Q} = \overline{S_Q}$ is in Σ_n, and $\neg Q$ is a Σ_n-predicate.

It follows that, for all $x_1, x_2, \ldots, x_s \in \mathbb{N}$,

$$\begin{aligned} &\neg P(x_1, x_2, \ldots, x_s) = 1 \\ &\qquad \iff \neg(\forall y)(Q(x_1, x_2, \ldots, x_s, y) = 1) \\ &\qquad \iff (\exists y)(\neg Q(x_1, x_2, \ldots, x_s, y) = 1). \end{aligned}$$

Since $\neg Q$ is a Σ_n-predicate, it follows by Lemma 5 that $\neg P$ is also a Σ_n-predicate. Since $S_{\neg P} = \overline{S_P}$ it follows that $\overline{S_P} \in \Sigma_n$, so that $S_P \in \Pi_n$ and P is a Π_n-predicate, as claimed. □

$\exists^{n+1}$-Predicates

Theorem 7 (Post's Theorem)**.** *Let $n, s \in \mathbb{N}$, $s \geq 1$, and let $P : \mathbb{N}^s \to \{0,1\}$ be a total predicate. Then P is a Σ_{n+1}-predicate if and only if there exists a Π_n-predicate $Q : \mathbb{N}^{s+1} \to \{0,1\}$ such that, for all $x_1, x_2, \dots, x_n \in \mathbb{N}$,*

$$P(x_1, x_2, \dots, x_s) = 1 \iff (\exists y)(Q(x_1, x_2, \dots, x_s, y) = 1).$$

Definition: A total predicate $P : \mathbb{N}^s \to \{0,1\}$ is an $\boldsymbol{\exists^{n+1}}$***-predicate*** if there is a Π_n-predicate $Q : \mathbb{N}^{s+1} \to \{0,1\}$ such that, for all $x_1, x_2, \dots, x_s \in \mathbb{N}$,

$$P(x_1, x_2, \dots, x_s) = 1 \iff (\exists y)(Q(x_1, x_2, \dots, x_s, y) = 1).$$

Post's Theorem is, therefore, the result that a total predicate $P : \mathbb{N}^s \to \{0,1\}$ is a Σ_{n+1}-predicate if and only if it is an $\exists^{n+1}$-predicate, for all $n \in \mathbb{N}$. The following minor results, concerning $\exists^{n+1}$-predicates, are used to prove Post's theorem but will not be used after that.

Lemma 7(a). *Let $n \geq 0$ and $s \geq 1$. If a total predicate $P : \mathbb{N}^s \to \{0,1\}$ is a Σ_n-predicate then P is an $\exists^{n+1}$-predicate.*

Proof. Suppose, first, that $n = 0$. If $P : \mathbb{N}^s \to \{0,1\}$ is a Σ_0-predicate then $S_P \in \Sigma_0$, that is, S_P is a computable set. In this case it suffices to set $Q : \mathbb{N}^{s+1} \to \{0,1\}$ to be the total predicate such that, for all $x_1, x_2, \dots, x_s, y \in \mathbb{N}$,

$$Q(x_1, x_2, \dots, x_s, y) = 1 \iff P(x_1, x_2, \dots, x_s) = 1.$$

It should not be hard to see that S_Q is a computable set because S_P is, so that $S_Q \in \Pi_0$. Thus Q is a Π_0-predicate and P is an $\exists^1$-predicate, as claimed.

Suppose, next, that $n \geq 1$. If $P : \mathbb{N}^s \to \{0,1\}$ is a Σ_n-predicate then $S_P \in \Sigma_n$ so that, by Theorem #7 from Lecture #16, S_P is $\emptyset^{(n-1)}$-computably enumerable. It follows by Theorem #15 from Lecture #15 that there exists a $\emptyset^{(n-1)}$-computable total predicate $R : \mathbb{N}^2 \to \mathbb{N}$ such that

$$S_P = \{x \in \mathbb{N} \mid (\exists t)(R(x,t) = 1)\}.$$

It now follows by the definition of S_P that, for all $x_1, x_2, \dots, x_s \in \mathbb{N}$,

$$P(x_1, x_2, \dots, x_s) = 1 \iff (\exists t)(R([x_1, x_2, \dots, x_s], t) = 1)$$

and the result follows if it can be shown that the predicate

$$R([x_1, x_2, \dots, x_s], t)$$

(when viewed as a function of $x_1, x_2, \dots, x_s, t$) is a Π_n-predicate.

Theorem #4 from Lecture #14 implies that, since the predicate $R : \mathbb{N}^2 \to \{0,1\}$ is a $\emptyset^{(n-1)}$-computable predicate, the above predicate is $\emptyset^{(n-1)}$-computable too. It follows, by Corollary #11 from Lecture #16, that this is a Δ_n-predicate and, since $\Delta_n = \Sigma_n \cap \Pi_n \subseteq \Pi_n$, this implies that this is a Π_n-predicate, as needed to complete the proof. $\square$

Lemma 7(b). *Let $n \geq 0$ and $s \geq 1$. If a total predicate $P : \mathbb{N}^s \to \{0,1\}$ is a Π_n-predicate then P is an $\exists^{n+1}$-predicate.*

Proof. Suppose that P is a Π_n-predicate, as in the claim. Set $Q : \mathbb{N}^{s+1} \to \{0,1\}$ to be the total predicate such that, for all $x_1, x_2, \dots, x_s, y \in \mathbb{N}$,

$$Q(x_1, x_2, \dots, x_s, y) = 1 \iff P(x_1, x_2, \dots, x_s) = 1.$$

It follows that, for all $x_1, x_2, \dots, x_s \in \mathbb{N}$,

$$P(x_1, x_2, \dots, x_s) = 1 \iff (\exists y)(Q(x_1, x_2, \dots, x_s, y) = 1).$$

Since Q is as above,

$$\begin{aligned} S_P &= \{x \in \mathbb{N} \mid P((x)_1, (x)_2, \dots, (x)_s) = 1\} \\ &= \{x \in \mathbb{N} \mid Q((x)_1, (x)_2, \dots, (x)_s, (x)_{s+1})) = 1\} \\ &= S_Q. \end{aligned}$$

Since P is a Π_n-predicate, $S_Q = S_P \in \Pi_n$, so that Q is a Π_n-predicate and P is an $\exists^{n+1}$-predicate, as claimed. $\square$

Lemma 7(c). *Let $n \geq 0$ and $s \geq 1$. If $P : \mathbb{N}^{s+1} \to \{0,1\}$ and $Q : \mathbb{N}^s \to \{0,1\}$ are total predicates such that, for all $x_1, x_2, \dots, x_s \in \mathbb{N}$,*

$$Q(x_1, x_2, \dots, x_s) = 1 \iff (\exists z)(P(x_1, x_2, \dots, x_s, z) = 1)$$

and P is an $\exists^{n+1}$-predicate then Q is an $\exists^{n+1}$-predicate as well.

Proof. Suppose that P is an $\exists^{n+1}$-predicate; then (by definition) there exists a Π_n-predicate $R : \mathbb{N}^{s+2} \to \{0,1\}$ such that, for all $x_1, x_2, \dots, x_s, z \in \mathbb{N}$,

$$P(x_1, x_2, \dots, x_s, z) = 1 \iff (\exists y)(R(x_1, x_2, \dots, x_s, z, y) = 1).$$

It follows that,, for all $x_1, x_2, \dots, x_s \in \mathbb{N}$,

$$\begin{aligned} &Q(x_1, x_2, \dots, x_s) = 1 \\ &\qquad \iff (\exists z)(P(x_1, x_2, \dots, x_s, z) = 1) \\ &\qquad \iff (\exists z)(\exists y)(R(x_1, x_2, \dots, x_s, z, y) = 1) \\ &\qquad \iff (\exists t)(R(x_1, x_2, \dots, x_s, \ell(t), r(t)) = 1). \end{aligned}$$

Since the identity function and the left and right inverses of the pairing function are computable total functions, it follows by Lemma 2 that the predicate

$$R(x_1, x_2, \ldots, x_s, \ell(t), r(t))$$

is a Π_n-predicate, when considered as a function of $x_1, x_2, \ldots, x_s, t$. It follows (by definition) that Q is an $\exists^{n+1}$-predicate, as claimed. $\square$

Lemma 7(d). *Let $n, s \in \mathbb{N}$ such that $s \geq 1$. If $P, Q : \mathbb{N}^s \to \{0, 1\}$ are total predicates that are both $\exists^{n+1}$-predicates, then $P \wedge Q$ and $P \vee Q$ are $\exists^{n+1}$ predicates as well.*

Proof. Let $P, Q : \mathbb{N}^s \to \{0, 1\}$ be $\exists^{n+1}$-predicates. Then there exist Π_n-predicates $R, S : \mathbb{N}^{s+1} \to \{0, 1\}$ such that, for all $x_1, x_2, \ldots, x_s \in \mathbb{N}$,

$$P(x_1, x_2, \ldots, x_s) = 1 \iff (\exists y)(R(x_1, x_2, \ldots, x_s, y) = 1)$$

and

$$Q(x_1, x_2, \ldots, x_s) = 1 \iff (\exists z)(S(x_1, x_2, \ldots, x_s, z) = 1).$$

In this case

$$\begin{aligned}
&(P \wedge Q)(x_1, x_2, \ldots, x_s) = 1 \\
&\qquad \iff (\exists y)(\exists z)(R(x_1, x_2, \ldots, x_s, y) = 1 \\
&\qquad\qquad\qquad\qquad \wedge S(x_1, x_2, \ldots, x_s, z) = 1) \\
&\qquad \iff (\exists w)(T(x_1, x_2, \ldots, x_s, w) = 1)
\end{aligned}$$

where $T : \mathbb{N}^{s+1} \to \{0, 1\}$ is the predicate such that, for all $x_1, x_2, \ldots, x_s, w \in \mathbb{N}$,

$$T(x_1, x_2, \ldots, x_s, w) = R(x_1, x_2, \ldots, x_s, \ell(w)) \wedge S(x_1, x_2, \ldots, x_s, r(w)).$$

R and S are both Π_n-predicates, when viewed as functions of $x_1, x_2, \ldots, x_s, y, z$, and it follows by Lemma 4 that $R \wedge S$ is a Π_n-predicate, when viewed as a function of these variables as well Since

$$T(x_1, x_2, \ldots, x_s, w) = (R \wedge S)(x_1, x_2, \ldots, x_s, \ell(w), r(w))$$

it follows by Lemma 2 that T is a Π_n-predicate (viewed as a function of $x_1, x_2, \ldots, x_s, w$) too.

Since

$$(P \wedge Q)(x_1, x_2, \ldots, x_s) = 1 \iff (\exists w)(T(x_1, x_2, \ldots, x_s, w) = 1)$$

it follows that $P \wedge Q$ is a $\exists^{n+1}$-predicate (when viewed as a function of $x_1, x_2, \ldots, x_s$), as claimed.

Virtually the same argument (with "$\wedge$" replaced by "$\vee$," throughout) establishes that $P \vee Q$ is an $\exists^{n+1}$-predicate of $x_1, x_2, \ldots, x_s$ as well. $\square$

Lemma 7(e). *If $s \geq 1$, $P : \mathbb{N}^{s+1} \to \{0,1\}$ is a $\exists^{n+1}$-predicate, and $Q : \mathbb{N}^{s+1} :\to \{0,1\}$ is a total predicate such that, for all $x_1, x_2, \dots, x_s, y \in \mathbb{N}$,*

$$Q(x_1, x_2, \dots, x_s, y) \iff (\forall t)_{\leq y} P(x_1, x_2, \dots, x_s, t),$$

then Q is also an $\exists^{n+1}$-predicate.

Proof. Since P is an $\exists^{n+1}$ predicate, there is a Π_n-predicate $R : \mathbb{N}^{s+2} \to \{0,1\}$ such that, for all $x_1, x_2, \dots, x_s, t, z \in \mathbb{N}$,

$$P(x_1, x_2, \dots, x_s, t) \iff (\exists z) R(x_1, x_2, \dots, x_s, t, z).$$

Thus

$$\begin{aligned} &Q(x_1,x_2,\dots,x_s,y) \\ &\qquad \iff (\forall t)_{\leq y}(\exists z) R(x_1, x_2, \dots, x_s, t, z) \\ &\qquad \iff (\exists u)(\forall t)_{\leq y} R(x_1, x_2, \dots, x_s, t, (u)_{t+1}), \end{aligned}$$

where u is the Gödel number encoding the sequence of values of z corresponding to $t = 0, 1, \dots, y$. It follows that

$$\begin{aligned} &Q(x_1,x_2,\dots,x_s,y) \\ &\qquad \iff (\exists u)(\forall t)((t > y) \vee R(x_1, x_2, \dots, x_s, t, (u)_{t+1})) \\ &\qquad \iff (\exists u) S(x_1, x_2, \dots, x_s, y, u) \end{aligned}$$

where

$$S(x_1, x_2, \dots, x_s, y, u) \iff (\forall t)((t > y) \vee R(x_1, x_2, \dots, x_s, t, (u)_{t+1}).$$

Now, if $n = 0$ then R is a computable total predicate, since it is a Π_n-predicate, and it is easily checked that S is a computable predicate, as well — as needed to establish that Q is an $\exists^{n+1}$-predicate in this case.

If $n \geq 1$ then it follows by Lemma 2 that the predicate $R(x_1, x_2, \dots, x_s, t, (u)_{t+1})$ is also a Π_n-predicate, when viewed as a function of $x_1, x_2, \dots, x_s$, t, and u. The predicate "$t > y$" is a computable predicate, so it is certainly a Π_n-predicate as well. Lemmas 4 and 6 can now be used to show that the above predicate S is also a Π_n-predicate. Thus Q is an $\exists^{n+1}$-predicate in this case too, as needed to establish the claim. $\square$

Recall, from Lecture #14, that if $u \in \mathbb{N}$ then $\{u\} : \mathbb{N} \to \mathbb{N}$ is a partial function such that, for all $i \in \mathbb{N}$,

- if $i < \ell(u)$ then $\{u\}(i) = (r(u))_{i+1}$, and
- if $i \geq \ell(u)$ then $\{u\}(i)$ is undefined.

Furthermore, if $G : \mathbb{N} \to \mathbb{N}$ is a unary total function then

$$u \prec G$$

if $\{u\}(i) = G(i)$ for every integer i such that $0 \leq i < \ell(u)$. Since any unary total predicate R is also a unary total function, this provides a definition of a total predicate

$$u \prec R.$$

Lemma 7(f). *Let $n \geq 0$ and let $R : \mathbb{N} \to \{0, 1\}$ be a Σ_n-predicate. Then the predicate*

$$u \prec R$$

is an $\exists^{n+1}$-predicate (when considered as a function of u).

Proof. For all $u \in \mathbb{N}$,

$$\begin{aligned} u \prec R \iff & (\forall i)_{<\ell(u)}(((r(u))_{i+1} = 1 \wedge R(i)) \\ & \qquad \vee ((r(u))_{i+1} = 0 \wedge \neg R(i))) \\ \iff & ((\ell(u) = 0) \vee (\exists z)((z + 1 = \ell(u)) \wedge \\ & \qquad (\forall i)_{\leq z}((((r(u))_{i+1} = 1) \wedge R(i)) \\ & \qquad\qquad \vee ((r(u))_{i+1} = 0) \wedge \neg R(i)))). \end{aligned}$$

Since R is a Σ_n-predicate it follows by Lemma 7(a) that R is also an $\exists^{n+1}$ predicate. On the other hand, since $S_{\neg R} = \overline{S_R}$, it is not hard to see that $\neg R$ is a Π_n-predicate — and it follows by Lemma 7((b) that $\neg R$ is an $\exists^{n+1}$-predicate as well.

The predicates "$((r(u))_{i+1} = 1)$" and "$((r(u))_{i+1} = 0)$" are both computable predicates, so they are certainly Σ_n predicates, and it follows by Lemma 7(a) that these are $\exists^{n+1}$-predicates as well.

Lemma 7(d) can now be applied, several times, to establish that the predicate

$$((((r(u))_{i+1} = 1) \wedge R(i)) \vee (((r(u))_{i+1} = 0) \wedge \neg R(i)))$$

is an $\exists^{n+1}$-predicate as well. Lemma 7(e) can be applied, next, to establish that

$$(\forall i)_{\leq z}((((r(u))_{i+1} = 1) \wedge R(i)) \vee (((r(u))_{i+1} = 0) \wedge \neg R(i)))$$

is an $\exists^{n+1}$ predicate of u and z.

The predicates "$(\ell(u) = 0)$" and "$(z + 1 = \ell(u))$" are both computable predicates, so they are certainly Σ_n-predicates, and Lemma 7(a) can be applied to establish that these are both $\exists^{n+1}$ predicates too.

Lemmas 7(c) and 7(d) can now be applied, several more times, to establish that the expression shown above, as equivalent to "$u \prec R$" is an $\exists^{n+1}$-predicate. (This is left as a straightforward ***exercise***.)

It follows that "$u \prec R$" is an $\exists^{n+1}$-predicate, as claimed. $\square$

Post's Theorem and Corollaries

Proof of Post's Theorem. Let $n, s \in \mathbb{N}$ such that $s \geq 1$ and let $P : \mathbb{N}^s \to \{0, 1\}$ be a total predicate, as in the statement of the claim.

Suppose first that $P : \mathbb{N}^s \to \{0, 1\}$ is an $\exists^{n+1}$-predicate and, in particular, that $Q : \mathbb{N}^{s+1} \to \{0, 1\}$ is a Π_n-predicate such that the relationship between P and Q, stated in the theorem, holds. Then, since $\Pi_n \subseteq \Delta_{n+1} \subseteq \Sigma_{n+1}$, Q is also a Σ_{n+1}-predicate, and it follows by Lemma 5 that P is a Σ_{n+1} predicate too. Thus every $\exists^{n+1}$-predicate is a Σ_{n+1}-predicate.

Now suppose that P is a Σ_{n+1}-predicate. Then

$$S_P = \{x \in \mathbb{N} \mid P((x)_1, (x)_2, \dots, (x)_s) = 1\}$$

is in Σ_{n+1}, so that (by definition) it is B-computably enumerable for some set $B \subseteq \mathbb{N}$ such that $B \in \Sigma_n$ — that is, the characteristic function P_B of B is in Σ_n — and, by Lemma 1 can also be considered to be a Σ_n-predicate. It follows that there is a P_B-computable (partial or total) function $f : \mathbb{N} \to \mathbb{N}$ such that

$$S_P = \{x \in \mathbb{N} \mid f(x) \text{ is defined}\}.$$

Let $y_0 \in \mathbb{N}$ such that y_0 is the encoding of an $\mathcal{S}$-program $\mathcal{P}$, with an oracle for P_B[1], that P_B-computes f. It now follows by Theorem #11 from Lecture #14 — the "Finiteness Theorem" — that

$$\begin{aligned} x \in S_P \iff & (\exists t)(\text{halted}^{(1)}_{P_B}(x, y_0, t) = 1) \\ \iff & (\exists t)(\exists u)((u \prec P_B) \wedge (\text{halted}^{(1)}_{\{u\}}(x, y_0, t) = 1)). \end{aligned}$$

Thus, for all $x_1, x_2, \dots, x_s \in \mathbb{N}$,

$$P(x_1, x_2, \dots, x_s) = 1 \iff (\exists t)(\exists u)((u \prec P_B) \wedge (\text{halted}^{(1)}_{\{u\}}([x_1, x_2, \dots, x_s], y_0, t) = 1)).$$

It follows by Theorem #10 from Lecture #14 that the predicate

$$\text{halted}^{(1)}_{\{u\}}(x, y_0, t)$$

is a computable predicate (when considered as a function of u, x, y_0 and t) so it is certainly an $\exists^{n+1}$-predicate. Since P_B is a Σ_n-predicate it follows by Lemma 7(f) that the predicate

$$u \prec p_B$$

is an $\exists^{n+1}$-predicate (considered as a function of u). Lemmas 7(d) and 7(c) can now be applied, using the above, to establish that P is an $\exists^{n+1}$-predicate. Thus every Σ_{n+1}-predicate is an $\exists^{n+1}$-predicate, as needed to complete a proof of Post's Theorem. $\square$

[1] expressed as a characteristic function for S_B instead of B

Corollary 8. *Let $n, s \in \mathbb{N}$, $s \geq 1$ and suppose that $P, Q : \mathbb{N}^{s+1}$ are total predicates such that, for all $x_1, x_2, \ldots, x_s, y \in \mathbb{N}$,*

$$Q(x_1, x_2, \ldots, x_s, y) = 1 \iff (\forall t)_{\leq y}(P(x_1, x_2, \ldots, x_s, t) = 1).$$

If P is a Σ_n-predicate then Q is a Σ_n-predicate as well.

Proof. If $n = 0$ then it follows that P is a computable predicate and the result follows by the fact that the class of computable functions is closed under bounded quantification. If $n \geq 1$ then this follows directly from Post's Theorem and Lemma 7(e). $\square$

The proofs of Corollaries 9 and 10 are short and simple enough to be included in the lecture notes.

Recall that If $s \geq 1$ then a total predicate $P : \mathbb{N}^s \to \{0, 1\}$ is an ***arithmetic predicate*** if there exists a number $n \in \mathbb{N}$ and a ***computable*** predicate $R : \mathbb{N}^{s+n} \to \{0, 1\}$ such that, for all $x_1, x_2, \ldots, x_s \in \mathbb{N}$,

$$P(x_1, x_2, \ldots, x_s) = 1 \iff (Q_1 y_1)(Q_2 y_2) \ldots (Q_n y_n)(R(x_1, x_2, \ldots, x_s, y_1, y_2, \ldots, y_n) = 1), \quad (1)$$

where each of $Q_1, Q_2, \ldots, Q_n$ is either the symbol $\exists$ or the symbol $\forall$. We say that the Q_i's are ***alternating*** if, for all i such that $1 \leq i < n$, if Q_i is $\exists$ then Q_{i+1} is $\forall$, and if Q_i is $\forall$ then Q_{i+1} is $\exists$.

Theorem 11.

(a) *Let $n \in \mathbb{N}$. Then every total predicate that is either a Σ_n- or a Π_n-predicate is an arithmetic predicate.*

(b) *Every arithmetic predicate is a Σ_k-predicate for some $k \in \mathbb{N}$ as well as a Π_ℓ-predicate for some $\ell \in \mathbb{N}$.*

(c) *Let $n, s \in \mathbb{N}$, $s \geq 1$ and let $P : \mathbb{N}^s \to \{0, 1\}$ be a total predicate. Then P is a Σ_n-predicate if and only if it can be written in the form shown on line* (1) *where Q_1 is $\exists$ (if $n \geq 1$) and the Q_i's are alternating.*

(d) *Let $n, s \in \mathbb{N}$, $s \geq 1$ and let $P : \mathbb{N}^s \to \{0, 1\}$ be a total predicate. Then P is a Π_n-predicate if and only if it can be written in the form shown on line* (1) *where Q_1 is $\forall$ (if $n \geq 1$) and the Q_i's are alternating.*

Proof. Let $n \in \mathbb{N}$.

(a) This first part of the claim is easily proved by induction on n.

For the *basis*, note that, since Σ_0-predicates and Π_0-predicates are computable predicates they are (trivially) arithmetic predicates.

Let $n \geq 0$ and suppose that all Σ_n-predicates and all Π_n-predicates are arithmetic predicates. Post's Theorem and Corollary 9 can now be applied to prove that all Σ_{n+1}-predicates and all Π_{n+1}-predicates are arithmetic predicates as well — as needed to complete the *inductive step* and prove part (a) of the claim.

(b) Suppose that $P : \mathbb{N}^s \to \{0, 1\}$ is an arithmetic predicate, so that P has the form shown at line (1). Part (b) of the claim can be established by induction on the number n of quantifiers shown in the expression at line (1).

For the *basis*, note that if $n = 0$ then P is a computaable predicate, so that P is both a Σ_0-predicate and a Π_0-predicate.

For the *inductive step*, suppose that $n \geq 0$ and that that every predicate P having the form as shown at line (1) is a Σ_k-predicate as well as a Π_ℓ-predicate for some $k, \ell \in \mathbb{N}$, when the expression on the right hand side at line (1) includes n quantifiers.

Let P be an arithmetic predicate, as shown at line (1), where the expression on the right hand side of this line includes $n + 1$ quantifiers instead. Then (by inspection of line (1)) there exists an arithmetic predicate $\widehat{P} : \mathbb{N}^s \to \{0, 1\}$ such that, for all $x_1, x_2, \ldots, x_s \in \mathbb{N}$,

$$P(x_1, x_2, \ldots, x_s) = 1 \iff (Q_1 y_1)(\widehat{P}(x_1, x_2, \ldots, x_s, y_1) = 1)$$

such that $\widehat{P}$ has the form shown at line (1), with n quantifiers on the right hand side of the expression. It follows by the inductive hypothesis that $\widehat{P}$ is both a Σ_k-predicate and a Π_ℓ-predicate for some $k, \ell \in \mathbb{N}$. Lemmas 5 and 6, Theorem 7 and Corollary 9 can now be used to complete the inductive step and prove part (b) of the claim.

(c) Finally, parts (c) and (d) are easy to prove using mathematical induction, applying Theorem 7 and Corollary 9 as needed. $\square$