Introduction to Post's Problem
Proofs of Claims

This document includes details of proofs that were mentioned in the lecture slides.

One-One Equivalence and Recursive Permutations

Recall that if $A, B \subseteq \mathbb{N}$ then a finite sequence

$$(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n) \tag{1}$$

associates A and B if the following properties are satisfied:

(a) $a_i \neq a_j$ for $1 \leq i < j \leq n$ — so that $a_1, a_2, \ldots, a_n$ are distinct;

(b) $b_i \neq b_j$ for $1 \leq i < j \leq n$ — so that $b_1, b_2, \ldots, b_n$ are distinct;

(c) for every integer i such that $1 \leq i \leq n$, ***either*** $a_i \in A$ and $b_i \in B$ ***or*** $a_i \neq A$ and $b_i \neq B$.

Lemma 1. *Let $A, B \subseteq N$ such that $A \preceq_1 B$. Then there exists a computable total function $k : \mathbb{N}^2 \to \mathbb{N}$ such that, for all $u, a \in \mathbb{N}$, if*

- *u codes the sequence shown at line* (1)*, and this sequence associates A and B, and*
- *$a \notin \{a_1, a_2, \ldots, a_n\}$,*

then there exists a number $b \in \mathbb{N}$ such that $k(u, a)$ is the code of the finite sequence

$$(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n), (a, b) \tag{2}$$

and this sequence also associates A and B.

Proof. Suppose that $u, a \in \mathbb{N}$, and u is the code of a sequence that is as shown at line (1). Suppose, as well, that $A \preceq_1 B$ — so that there exists a one-one computable (total) function $f : \mathbb{N} \to \mathbb{N}$ such that $x \in A$ if and only if $f(x) \in B$ for all $x \in \mathbb{N}$.

Consider a function $k : \mathbb{N} \to \mathbb{N}$ such that $k(u, a)$ is as defined as follows.

If $a \in \{a_1, a_2, \ldots, a_n\}$ then let $k(u, a) = u$.

Suppose, instead, that $a \notin \{a_1, a_2, \ldots, a_n\}$. If $f(a) \notin \{b_1, b_2, \ldots, b_n\}$ then we may set b to be $f(a)$.

Suppose next that $a \notin \{a_1, a_2, \ldots, a_n\}$ but $f(a) \in \{b_1, b_2, \ldots, b_n\}$. Since the function f is one-one there is ***exactly one*** integer i_1 such that $1 \leq i_1 \leq n$ and $b_{i_1} = f(a)$. Since $a \notin \{a_1, a_2, \ldots, a_n\}$, a_1 and a are distinct.

As a generalization of this suppose that j is a positive integer and that there exist distinct integers $i_1, i_2, \ldots, i_j$ such that

(a) $1 \leq i_j \leq n$ for $1 \leq h \leq j$, and

(b) $b_{i_1} = f(a)$, and

(c) $b_{i_h} = f(a_{i_{h-1}})$ for $2 \leq h \leq j$.

The above implies that this condition is satisfied when $j = 1$.

If $f(a_{i_j}) \notin \{b_1, b_2, \ldots, b_n\}$ then it suffices to set $b = f(a_{i_j})$.

Suppose, instead, that $f(a_{i_j}) \in \{b_1, b_2, \ldots, b_n\}$ Then there exists ***exactly one*** integer i_{j+1} such that $1 \leq i_{j+1} \leq n$ and $b_{i_{h+1}} = f(a_{i_h})$. Since $a_{i_j} \notin \{a_{i_{j-1}}, a_{i_{j-2}}, \ldots, a_1, a\}$, and f is a one-one total function,

$$f(a_{i_j}) \notin \{f(a_{i_{j-1}}), f(a_{i_{j-2}}), \ldots, f(a_1), f(a)\}.$$

Since $b_{i_{h+1}} = f(a_{i_h})$ for $1 \leq h \leq j$ and $b_{i_1} = f(a)$, this implies that

$$b_{i_{j+1}} \notin \{b_{i_j}, b_{i_{j-1}}, \ldots, b_{i_2}, b_{i_1}\},$$

so that $1 \leq i_h \leq n$ for $1 \leq h \leq j+1$ and $i_1, i_2, \ldots i_{j+1}$ are distinct. It now follows that the above properties (a), (b), and (c) are satisfied when h is replaced by $h+1$.

It also follows that there must exist an integer j such that $1 \leq j \leq n$, $f(a_{i_\ell}) \in \{b_1, b_2, \ldots, b_n\}$ for $1 \leq \ell < j$, and $f(a_{i_j}) \notin \{b_1, b_2, \ldots, b_n\}$ — for, otherwise, it would follow from the above that $i_1, i_2, \ldots, i_n, i_{n+1}$ are distinct integers such that $1 \leq i \leq n$, which is certainly impossible. We can now set $b = f(a_{i_j})$ for this integer j.

Next suppose that the sequence shown at line (1) associates A and B. Then

$$a \in A \iff f(a) \in B \iff a_{i_1} \in A$$

since $b_{i_1} = f(a)$. Similarly, for $1 \leq h \leq j-1$,

$$a_{i_h} \in A \iff f(a_{i_h}) \in B \iff b_{i_{h+1}} \in B \iff a_{i_{h+1}} \in A.$$

Finally,

$$a_{i_j} \in A \iff f(a_{i_j}) \in B \iff b \in B,$$

so that $a \in A \iff b \in B$, and the sequence at line (2) associates A and B if the sequence at line (1) does.

It remains only to notice that the predicate "$a \in \{a_1, a_2, \ldots, a_n\}$" is easily computable from the code, u, of the sequence at line (1), and a. $k(u, a) = u$ if $a \in \{a_1, a_2, \ldots, a_n\}$, so that $k(u, a)$ is certainly computable from u and a in this case. If $a \notin \{a_1, a_2, \ldots, a_n\}$ then $k(u, a)$ can be produced using the value b in the sequence at line (2) — which is computable from u and a, using the process described above. In particular, $k(u, a)$ is the code of this longer sequence obtained by adding the pair a, b, so that

$$k(u, a) = \langle \ell(u) + 1, r(u) \times p_{\ell(u)+1}^{\langle a,b \rangle} \rangle$$

and $k(u, a)$ is computable from u and a in this case well. Thus $k : \mathbb{N}^2 \to \mathbb{N}$ is a computable total function, as needed to complete the proof of the lemma. $\square$

Theorem 2 (Myhill's Theorem)**.** *Let $A, B \subseteq \mathbb{N}$. Then $A \equiv_1 B$ if and only if $A \equiv B$.*

Proof. Let $A, B \subseteq \mathbb{N}$.

As previously noted, if $A, B \subseteq \mathbb{N}$ and $A \equiv B$ then $A \equiv_1 B$, so it remains only to establish the converse. With that noted, let $A, B \subseteq \mathbb{N}$ such that $A \equiv_1 B$.

Then $A \preceq_1 B$, so that there exists a one-one (and, therefore, total) computable function $f : \mathbb{N} \to \mathbb{N}$ such that $x \in A$ if and only if $f(x) \in B$ for all $x \in \mathbb{N}$ — and a corresponding function $k : \mathbb{N}^2 \to \mathbb{N}$ that can be used to extend finite sequences that associate A and B, as described in Lemma 1.

$B \preceq_1 A$ as well, so that there also exists a one-one (thus, total) computable function $\widehat{f} : \mathbb{N} \to \mathbb{N}$ such that $x \in B$ if and only $\widehat{f}(x) \in A$ for all $x \in \mathbb{N}$ — and a corresponding function $\widehat{k} : \mathbb{N}^2 \to \mathbb{N}$ that can be used to extend finite sequences that associate B and A, as described in Lemma 1.

As in the proof of Lemma 1, we may assume that $k(u, a) = \widehat{k}(u, a) = u$ if u encodes a sequence, as shown at line (1), such that $a \in \{a_1, a_2, \ldots, a_n\}$.

There is also a well-defined total computable function rev : $\mathbb{N} \to \mathbb{N}$ such that, for all $x \in \mathbb{N}$, if x is the code for a sequence

$$(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)$$

then rev(x) is the code for the corresponding sequence

$$(b_1, a_1), (b_2, a_2), \ldots, (b_n, a_n).$$

Indeed,

$$\text{rev}(u) = \langle \ell(u), \prod_{i=1}^{\ell(u)} p_i^{\langle r((r(u))_i), \ell((r(u))_i) \rangle} \rangle.$$

Consider a function $\nu : \mathbb{N} \to \mathbb{N}$ that is defined as follows.

- $\nu(0) = 0$, so that $\nu(0)$ is the code for the empty sequence.
- If $x \geq 0$ then

  $$\nu(2x+1) = k(\nu(2x), x)$$

  and

  $$\nu(2x+2) = \text{rev}(\widehat{k}(\text{rev}(\nu(2x+1)), x)).$$

It is not hard to show that ν is a total computable function. Furthermore, ν satisfies the following properties:

- For each $x \in \mathbb{N}$, $\nu(x)$ is the code of a sequence that associates A and B.
- For each $x \in \mathbb{N}$ the sequence with code $\nu(x+1)$ is either equal to the sequence encoded by $\nu(x)$ or is an "extension" of this sequence, produced by including one more ordered pair.
- For all $a \in \mathbb{N}$ there is a number $x \in \mathbb{N}$ encoding a sequence as at line (1) such that $a \in \{a_1, a_2, \ldots, a_n\}$. Indeed, this is the case when $x = 2a+1$ (and for all $x \geq 2a+1$).
- For all $b \in \mathbb{N}$ there is a number $x \in \mathbb{N}$ encoding a sequence as at line (1) such that $b \in \{b_1, b_2, \ldots, b_n\}$. Indeed, this is the case when $x = 2a+2$ (and for all $x \geq 2a+2$).

Now let $g : \mathbb{N} \to \mathbb{N}$ such that for all $a \in \mathbb{N}$, $g(a)$ is the number b such that (a, b) appears in the sequence encoded by $\nu(2a+1)$. Then

$$g(a) = \min_b (\exists i)_{i \leq \ell(\nu(2a+1))} [(r(\nu(2a+1)))_i = \langle a, b \rangle]$$

so that g is a computable total function. Indeed, it follows by the above properties that g is a recursive permutation such that $a \in A$ if and only if $g(a) \in B$ for all $a \in \mathbb{N}$.

It follows that $A \equiv B$. Since A and B were arbitrarily chosen such that $A \equiv_1 B$, this completes the proof of the theorem. ☐

m-Degrees versus 1-Degrees

Recall that if $S \subseteq \mathbb{N}$ then the ***cylindrication*** of S is the set

$$\textit{Cyl}(S) = \{\langle a, b \rangle \mid a \in S \text{ and } b \in \mathbb{N}\}.$$

A set $A \subseteq \mathbb{N}$ is a ***cylinder*** if $A \equiv \textit{Cyl}(S)$ for some set $S \subseteq \mathbb{N}$.

Theorem 3. *Let $A \subseteq \mathbb{N}$.*

(a) $A \preceq_1$ *Cyl*(A)*.*

(b) *Cyl*$(A) \preceq_m A$ *(and, therefore,* $A \equiv_m$ *Cyl*(A)*).*

(c) A *is a cylinder if and only if and only if the following is true for every set* $B \subseteq \mathbb{N}$*: If* $B \preceq_m A$ *then* $B \preceq_1 A$*.*

(d) $A \preceq_m B$ *if and only if Cyl*$(A) \preceq_1$ *Cyl*(B)*, for all* $B \subseteq \mathbb{N}$*.*

Proof. Let $A \subseteq \mathbb{N}$.

(a) The function $f : \mathbb{N} \to \mathbb{N}$ such that $f(n) = \langle n, 0 \rangle$ for all $n \in \mathbb{N}$ is a one-one total computable function such that $x \in A$ if and only if $f(x) \in$ *Cyl*(A). It follows that $A \preceq_1$ *Cyl*(A), as needed to establish part (a).

(b) The function $g : \mathbb{N} \to \mathbb{N}$ such that $f(n) = \ell(n)$ for all $n \in \mathbb{N}$ is a computable total function such that $x \in$ *Cyl*(A) if and only if $g(x) \in A$. It follows that *Cyl*$(A) \preceq_m A$, as needed to establish part (b).

(c) Suppose that A is a cylinder, so that $A \equiv$ *Cyl*(S) for some set $S \subseteq \mathbb{N}$. Suppose, as well, that $B \subseteq \mathbb{N}$ such that $B \preceq_m A$.

- Since $A \equiv$ *Cyl*(S), $A \preceq_m$ *Cyl*(S), and it follows that $B \preceq_m$ *Cyl*(S). Since *Cyl*$(S) \preceq_m S$, by part (b), it follows that $B \preceq_m S$. Thus there exists a computable total function $f : \mathbb{N} \to \mathbb{N}$ such that $x \in B$ if and only if $f(x) \in S$ for all $x \in \mathbb{N}$.
- Let $g : \mathbb{N} \to \mathbb{N}$ such that, for all $n \in \mathbb{N}$, $g(n) = \langle f(n), n \rangle$. Then g is a one-one computable function, and $x \in B$ if and only if $g(x) \in$ *Cyl*(S) for all $x \in \mathbb{N}$. Thus $B \preceq_1$ *Cyl*(S).
- Since $A \equiv$ *Cyl*(S), $A \equiv_1$ *Cyl*(S) and *Cyl*$(S) \preceq_1 A$. It now follows that that $B \preceq_1 A$.
- Since B was arbitrarily chosen such that $B \preceq_m A$, it follows (when A is a cylinder) that if $B \preceq_m A$ then $B \preceq_1 A$ for all $B \subseteq \mathbb{N}$.

Suppose, instead, that $A \subseteq \mathbb{N}$ such that, for all $B \subseteq \mathbb{N}$, if $B \preceq_m A$ then $B \preceq_1 A$.

- Then, since *Cyl*$(A) \preceq_m A$, by part (b), *Cyl*$(A) \preceq_1 A$, so that $A \equiv_1$ *Cyl*(A), by part (a), and $A \equiv$ *Cyl*(A) by Myhill's Theorem. Thus A is a cylinder.

Part (c) now follows.

(d) Suppose next that $A, B \subseteq \mathbb{N}$ and $A \preceq_m B$.

- Then there exists a computable total function $f : \mathbb{N} \to \mathbb{N}$ such that $x \in A$ if and only if $f(x) \in B$ for all $x \in \mathbb{N}$.
- Consider the function $g : \mathbb{N} \to \mathbb{N}$ such that, for all $x \in \mathbb{N}$,

$$g(x) = \langle f(\ell(x)), x \rangle.$$

- Then g is also a computable total function and, furthermore, g is one-one, since $\langle a, b\rangle \neq \langle c, d\rangle$ for all $a, b, c, d \in \mathbb{N}$ such that $b \neq d$. Furthermore

$$\begin{aligned} x \in \textit{Cyl}(A) &\iff \ell(x) \in A \\ &\iff f(\ell(x)) \in B \\ &\iff g(x) \in \textit{Cyl}(B). \end{aligned}$$

  Thus $\textit{Cyl}(A) \preceq_1 \textit{Cyl}(B)$.

Suppose, on the other hand, that $A, B \subseteq \mathbb{N}$ and $\textit{Cyl}(A) \preceq_1 \textit{Cyl}(B)$.

- Then there exists a one-one computable function $f : \mathbb{N} \to \mathbb{N}$ such that $x \in \textit{Cyl}(A)$ if and only if $f(x) \in \textit{Cyl}(B)$ for all $x \in \mathbb{N}$.
- Consider the function $g : \mathbb{N} \to \mathbb{N}$ such that, for all $x \in \mathbb{N}$,

$$g(x) = \ell(f(\langle x, 0\rangle)).$$

- This is certainly a total computable function and, for all $x \in \mathbb{N}$,

$$\begin{aligned} x \in A &\iff \langle x, 0\rangle \in \textit{Cyl}(A) \\ &\iff f(\langle x, 0\rangle) \in \textit{Cyl}(B) \\ &\iff g(x) = \ell(f(\langle x, 0\rangle)) \in B. \end{aligned}$$

- Thus $A \preceq_m B$, as needed to complete the proof of part (d). $\square$