Lecture #19: Creative Sets, Simple Sets, and Limits

Exercises and Review

Questions for Review

Productive Sets

1. What is a ***productive set***? What is a ***productive partial function*** for a productive set?
2. Give an example of a productive set.
3. State three useful properties (included in the statements of "Theorem #2" and "Theorem #3" in the lecture notes) of productive sets.

Creative Sets

4. What is a ***creative set***?
5. Give an example of a creative set.
6. State three useful properties (included in the statement of "Theorem #4") of creative sets.

Simple Sets

7. What is a ***simple set***?
8. State four useful properties (included in the statement of "Theorem #5") of simple sets.
9. *Briefly* describe a set that has been proved to be a simple set.
10. What does the existence of a simple set (or the proof of its existence) imply about the relationship between 1-reducibility and m-reducibility?
11. What does the existence of a simple set imply about the set of m-degrees of ***computably enumerable*** sets?

Limits of Sequences of Functions and Sets

12. What does it mean for a sequence

$$F_0, F_1, F_2, F_3, \ldots$$

    to ***have a limit***? What is a ***modulus*** for a sequence, and what is the ***limit*** of such a sequence, if "has a limit?"

13. What does it mean for a sequence

$$A_0, A_1, A_2, A_3, \ldots$$

    of subsets of $\mathbb{N}$ to ***have a limit***? What is a modulus, and the limit, of *this* kind of sequence?

14. What does it mean for a sequence of functions to be ***G-computable*** when $G : \mathbb{N} \to \mathbb{N}$ is a total function? What does it mean for such a sequence to be ***A-computable*** for $A \subseteq \mathbb{N}$?
15. State the ***Relativized Modulus Lemma***.
16. State the ***Limit Lemma***.

Jumps and Leaps

17. What is the ***leap*** of a total function $G : \mathbb{N} \to \mathbb{N}$?
18. What is the leap I'' of the ***identity function*** $I : \mathbb{N} \to \mathbb{N}$? That is, which set — that has already been considered in this course, is this equal to?
19. Suppose $G : \mathbb{N} \to \mathbb{N}$ is a total function. Describe the relationship between the ***jump*** G' of G, and the ***leap*** G'' of G, as precisely as you can.

Problems To Be Solved

These problems concern properties of sets that were considered in a proof of the existence of a simple set, discussed in the lecture material. These include the set

$$C = \{\langle x, y\rangle \mid y \in W_x \text{ and } y > 2x.\}$$

As noted during the lecture presentation, this set is computably enumerable and infinite. Furthermore, there exists a ***one-one*** computable function $g : \mathbb{N} \to \mathbb{N}$ such that

$$C = \{g(n) \mid n \in \mathbb{N}\}.$$

This function is used, along with C, to define another set $\widehat{C}$ — namely, the set such that, for all $x, y \in \mathbb{N}$, $\langle x, y\rangle \in \widehat{C}$ if and only if the following conditions are satisfied:

- $\langle x, y\rangle \in C$ (so that $\widehat{C} \subseteq C$).
- For every number $z \in \mathbb{N}$ such that $z \neq y$ and $\langle x, z\rangle \in C$, $\langle x, y\rangle$ appears ***before*** $\langle x, z\rangle$ in the listing
$$g(0), g(1), g(2), \ldots$$
of elements of C — so that $g^{-1}(\langle x, y\rangle) < g^{-1}(\langle x, z\rangle)$.

As noted during the lecture presentation, it follows from the above that, for every number $x \in \mathbb{N}$, there exists ***at most*** one number $y \in \mathbb{N}$ such that $\langle x, y\rangle \in \widehat{C}$.

It was shown, in the lecture presentation, that the set $\widehat{C}$ is also infinite.

1. Using the above, prove that there is a ***one-one computable*** function $h : \mathbb{N} \to \mathbb{N}$ such that $\widehat{C} = \{h(n) \mid n \in \mathbb{N}$.

   Note that it follows, from this, that $\widehat{C}$ is also ***computably enumerable***.

Let
$$S = \{y \in \mathbb{N} \mid \langle x, y\rangle \in \widehat{C} \text{ for some number } x \in \mathbb{N}\}.$$

2. Prove that S is computably enumerable.

3. Prove that, for all $k \in \mathbb{N}$, ***at most*** k of the integers h such that $0 \leq h \leq 2k$ belong to S — so that $\overline{S}$ is infinite.