Lecture #19: Creative Sets, Simple Sets, and Limits
Lecture Presentation

Proving a Result of Post

A Useful Set C

Let

$$C = \{\langle x, y\rangle \mid y \in W_x \text{ and } y > 2x\}.$$

Claim #1: The set C is computably enumerable.

Proof:

Claim #2: The set C is infinite.

Proof:

Claim #3: There is a ***one-one*** (and, hence, total) computable function $g : \mathbb{N} \to \mathbb{N}$ such that

$$C = \{g(n) \mid n \in \mathbb{N}\}.$$

Sketch of Proof:

A Related Set $\widehat{C}$

Now let $\widehat{C} \subseteq \mathbb{N}$ such that, for all $x, y \in \mathbb{N}$, $\langle x, y\rangle \in \widehat{C}$ if and only if the following conditions are satisfied:

- $\langle x, y\rangle \in C$ (so that $\widehat{C} \subseteq C$).
- For every number $z \in \mathbb{N}$ such that $z \neq y$ and $\langle x, z\rangle \in C$, $\langle x, y\rangle$ appears ***before*** $\langle x, z\rangle$ in the listing
  $$g(0), g(1), g(2), \ldots$$
  of elements of C — so that $g^{-1}(\langle x, y\rangle) < g^{-1}(\langle x, z\rangle)$.

Note that it follows from the requirements that, for every number $x \in \mathbb{N}$, there exists ***at most*** one number y such that $\langle x, y\rangle \in \widehat{C}$.

Claim #4: The set $\widehat{C}$ is also infinite.

Proof:

If you complete the ***exercises*** for this lecture then you will show that $\widehat{C}$ is also computably enumerable. Indeed, there exists a one-one computable function $h : \mathbb{N} \to \mathbb{N}$ such that $\widehat{C} = \{h(n) \mid n \in \mathbb{N}\}$.

A Simple Set S

Let
$$S = \{y \in \mathbb{N} \mid \langle x, y \rangle \in \widehat{C} \text{ for some number } x \in \mathbb{N}\}.$$

If you complete the exercises for this lecture then you also show that S ***is computably enumerable***, and that $\overline{S}$ ***is infinite***.

Claim #5: Let $B \subseteq \mathbb{N}$ such that B is computably enumerable and B is infinite. Then $B \cap S \neq \emptyset$.

Note that it follows, from the above, that S ***is a simple set***.

Proof of Claim #5:

Additional Useful Properties of S

Claim #6: The set *Cyl*(S) is computably enumerable.

Proof:

Claim #7: Neither of the sets S nor *Cyl*(S) is computable.

Proof:

Claim #8: *Cyl*$(S) \preceq_m S$, but *Cyl*$(S) \not\preceq_1 S$.

Proof:

What ***significant claim***, from the lectures, concerning the relationship between m-degrees and 1-degrees, has now been established?