Creative Sets, Simple Sets, and Limits
Proofs of Claims

This document includes details of proofs that were mentioned in the lecture slides.

Productive Sets

Theorem 2. *Let $A, B \subseteq \mathbb{N}$.*

(a) If A is productive then A is not computably enumerable.

(b) If A is productive and $A \preceq_m B$ then B is productive.

Proof. Let $A, B \subseteq \mathbb{N}$.

(a) Suppose, to the contrary, that A is productive and computably enumerable. Since A is productive, there exists a computable partial function $\psi_A : \mathbb{N} \to \mathbb{N}$ such that, for all $x \in \mathbb{N}$, if $W_x \subseteq A$ then $\psi_A(x)$ is defined and $\psi_A(x) \in A \setminus W_x$.

However, since A is computably enumerable, $A = W_x$ for some $x \in \mathbb{N}$.

Since $W_x \subseteq A$, $\psi_A(x)$ must be defined and in $A \setminus W_x$, as noted above.

However, $A = W_x$ so that $A \setminus W_x = \emptyset$, so that $\psi_A(x)$ cannot belong to this set.

This ***contradiction*** establishes part (a).

(b) Suppose that A is productive and that $A \preceq_m B$. Since A is productive there exists a computable partial function $\psi_A : \mathbb{N} \to \mathbb{N}$ such that, for all $x \in \mathbb{N}$, if $W_x \subseteq A$ then $\psi_A(x)$ is defined and $x \in A \setminus W_x$.

Since $A \preceq_m B$, there exists a computable total function $f : \mathbb{N} \to \mathbb{N}$ such that $x \in A$ if and only if $f(x) \in B$ for all $x \in \mathbb{N}$.

Let $e = \#(\mathcal{P})$, where $\mathcal{P}$ is the $\mathcal{S}$-program obtained from the following after macro expansion:

$$\begin{aligned} &X_1 \leftarrow f(X_1) \\ &Y \leftarrow \Phi^{(1)}(X_1, X_2) \end{aligned}$$

Let $g : \mathbb{N} \to \mathbb{N}$ such that $g(x) = S_1^1(e, x)$ for all $x \in \mathbb{N}$. Then g is a computable total function — indeed, g is primitive recursive.

Note that, for all $x \in \mathbb{N}$,

$$W_{g(x)} = \{y \in \mathbb{N} \mid f(y) \in W_x\}. \tag{1}$$

Now let $\psi_B : \mathbb{N} \to \mathbb{N}$ such that

$$\psi_B(x) = f \circ \psi_A \circ g(x) = f(\psi_A(g(x)))$$

for all $x \in \mathbb{N}$. Then, ψ_B is a computable partial function and, for all $x \in \mathbb{N}$,

$$\begin{aligned} W_x \subseteq B &\implies W_{g(x)} \subseteq \{y \in \mathbb{N} \mid f(y) \in B\} && \text{(by the equation at line (1))} \\ &\implies W_{g(x)} \subseteq A && \text{(by the choice of } f\text{)} \\ &\implies \psi_A(g(x)) \text{ is defined and } \psi_A(g(x)) \in A \setminus W_{g(x)} \\ &\implies \psi_B(x) = f(\psi_A(g(x))) \text{ is defined and } \psi_B(x) \in B \setminus W_x, \end{aligned}$$

by the equation at line (1). Thus B is productive, as needed to establish part (b). ☐

Theorem 3. *Let $A \subseteq \mathbb{N}$ be a productive set. Then there exists an infinite set $B \subseteq \mathbb{N}$ such that B is computably enumerable and $B \subseteq A$.*

Proof. Let $A \subseteq \mathbb{N}$ be a productive set. Let $\psi_A : \mathbb{N} \to \mathbb{N}$ be a productive partial function for A, that is, a computable partial function such that for all $x \in \mathbb{N}$, if $W_x \subseteq A$ then $\psi_A(x)$ is defined and $\psi_A(x) \in A \setminus W_x$.

Consider a total function $g : \mathbb{N} \to \mathbb{N}$ that is recursively (or "inductively") defined as follows.

- Set z_0 to be the number of an $\mathcal{S}$-program that fails to halt on any inputs, such as the following:

  $$\begin{aligned} & X_1 \leftarrow X_1 + 1 \\ [A_1] \quad & \text{IF } X_1 \neq 0 \text{ GOTO } A_1 \end{aligned}$$

  Since $W_{z_0} = \emptyset \subseteq A$, $\psi_A(z_0)$ is defined. Set $g(0)$ to be $\psi_A(z_0)$.

- Suppose that $n \in \mathbb{N}$ and that $g(0), g(1), \dots, g(n)$ have been defined such that $g(i) \in A$ for $0 \leq i \leq n$ and $z_0, z_1, \dots, z_n$ are distinct. Set z_{n+1} to be the number of the program obtained from the following one after macro expansion:

$$
\begin{array}{ll}
 & \text{IF } X_1 = g(0) \text{ GOTO } B_1 \\
 & \text{IF } X_1 = g(1) \text{ GOTO } B_1 \\
 & \quad\quad\quad\vdots \\
 & \text{IF } X_1 = g(n) \text{ GOTO } B_1 \\
 & X_1 = X_1 + 1 \\
[A_1] & \text{IF } X_1 \neq 0 \text{ GOTO } A_1
\end{array}
$$

Since $W_{z_{n+1}} = \{g(0), g(1), \ldots, g(n)\} \subseteq A$, $\psi_A(z_{n+1})$ is defined. Furthermore, $\psi_A(z_{n+1}) \in A \setminus W_{z_{n+1}}$, so that $z_{n+1} \notin \{z_0, z_1, \ldots, z_n\}$. Set $g(n+1) = \psi_A(z_{n+1})$.

With a bit of work, it can be argued that the resulting function $g : \mathbb{N} \to \mathbb{N}$ is a computable total function and, furthermore, that g is one-one, so that

$$B = \{g(n) \mid n \in \mathbb{N}\}$$

is an infinite computably enumerable set. Furthermore $B \subseteq A$, as needed to establish the claim. $\square$

Simple Sets

Theorem 6. *There exists a simple set.*

Proof. Let

$$C = \{\langle x, y\rangle \mid y \in W_x \text{ and } y > 2x\}.$$

C is computably enumerable, since the $\mathcal{S}$-program obtained from the following, by macro expansion, halts on input $z \in \mathbb{N}$ if and only if $z \in C$:

$$
\begin{array}{ll}
 & Z_1 \leftarrow \ell(X_1) \\
 & Z_2 \leftarrow r(X_1) \\
 & \text{IF } Z_2 \leq 2 \times Z_1 \text{ GOTO } A_1 \\
 & Y \leftarrow \Phi^{(1)}(Z_2, Z_1) \\
 & \text{GOTO } A_3 \\
[A_1] & Z_3 \leftarrow Z_3 + 1 \\
[A_2] & \text{IF } Z_3 \neq 0 \text{ GOTO } A_2
\end{array}
$$

Thus there exists a computable ***total*** function $f : \mathbb{N} \to \mathbb{N}$ such that $C = \{f(n) \mid n \in \mathbb{N}\}$. Furthermore, C is infinite, and this can be used to show that there is a ***one-one*** total computable function $g : \mathbb{N} \to \mathbb{N}$ such that $C = \{g(n) \mid n \in \mathbb{N}\}$ as well – so that each element of C appears ***exactly once*** in the listing

$$g(0), g(1), g(2), g(3), \ldots$$

Consider the set $\widehat{C} \subseteq C$ such that, for $x, y \in \mathbb{N}$, $\langle x, y\rangle \in \widehat{C}$ if and only if $\langle x, y\rangle \in C$ and, furthermore, if $z \in \mathbb{N}$, $z \neq y$, and $\langle x, z\rangle \in C$, then $\langle x, y\rangle$ appears ***before*** $\langle x, z\rangle$ in the above listing — that is, $g^{-1}(\langle x, y\rangle) < g^{-1}(\langle x, z\rangle)$. With a bit of work, the above function g can be used to describe a one-one computable total function $h : \mathbb{N} \to \mathbb{N}$ such that $\widehat{C} = \{h(n) \mid n \in \mathbb{N}\}$ — so that $\widehat{C}$ is also computably enumerable.

Let

$$S = \{y \in \mathbb{N} \mid \langle x, y\rangle \in \widehat{C} \text{ for some } x \in \mathbb{N}\} \subseteq \mathbb{N}.$$

The above function h (and an $\mathcal{S}$-program that computes it) can be used to produce an $\mathcal{S}$-program $\mathcal{P}$ such that, for all $y \in \mathbb{N}$, $\mathcal{P}$ halts, when executed on the input y, if and only if $y \in S$. Thus S is also computably enumerable.

Note that if $\langle x, y\rangle \in \widehat{C}$ then $y > 2x$ and there is ***at most one*** $y \in \mathbb{N}$ such that $\langle x, y\rangle \in \widehat{C}$ for any $x \in \mathbb{N}$. This can be used to argue that, for all $k \in \mathbb{N}$, at most k of the integers in the set

$$\{0, 1, 2, \ldots, 2k\}$$

can be included in S. Thus $\overline{S}$ is infinite.

Finally, let $B \subseteq \mathbb{N}$ such that B is infinite and computably enumerable. Then $B = W_x$ for some $x \in \mathbb{N}$ and, since B is infinite, there are infinitely many integers $y \in \mathbb{N}$ such that $y > 2x$ and $y \in W_x = B$. It follows, by the definition of $\widehat{C}$, that there exists an integer $y \in \mathbb{N}$ such that $\langle x, y\rangle \in \widehat{C}$. It now follows by the definition of S that $y \in S$. Furthermore $\langle x, y\rangle \in \widehat{C} \subseteq C$ and it follows by the definition of C that $y \in W_x = B$. Thus $y \in S \cap B$.

Since B was an arbitrarily chosen infinite computably enumerable set, it now follows by the definition of a "simple set" that S is simple, as needed to establish the claim. $\square$

Limits of Sequences of Functions and Sets

Theorem 7 (Relativized Modulus Lemma)**.** *Let $G : \mathbb{N} \to \mathbb{N}$ and let $A \subseteq \mathbb{N}$. If A is G-computably enumerable and a total function $F : \mathbb{N} \to \mathbb{N}$ is A-computable, then there exists a G-computable sequence $F_0, F_1, F_2, F_3, \ldots$ of total functions $F_i : \mathbb{N} \to \mathbb{N}$, for $i \in \mathbb{N}$, such that F is the limit of this sequence.*

This sequence of functions also has an A_G-computable modulus function $m : \mathbb{N} \to \mathbb{N}$, where $A_G : \mathbb{N} \to \mathbb{N}$ such that, for all $x \in \mathbb{N}$,

$$A_G(2x) = G(x),$$

and

$$A_G(2x+1) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

Proof. Let $G : \mathbb{N} \to \mathbb{N}$ and $A \subseteq \mathbb{N}$ such that A is G-computably enumerable. Suppose, as well, that $F : \mathbb{N} \to \mathbb{N}$ is a total function that is A-computable.

Then, since A is G-computably enumerable, there exists a number $e \in \mathbb{N}$ such that $A = W_e^G$, the set of numbers $x \in \mathbb{N}$ such that $\mathcal{P}$ halts when executed on input x — where $\mathcal{P}$ is an $\mathcal{S}$-program, with oracle for G, such that $\#(\mathcal{P}) = e$.

Let $A_n = W_{e,n}^G$ — the set of numbers $x \in \mathbb{N}$ such that $\mathcal{P}$ halts, when executed on input x, after at most n steps, for the above program $\mathcal{P}$. Since A is the set W_e^G it is not hard to show that A is the limit of the sequence $A_0, A_1, A_2, A_3, \ldots$ of sets.

The Relativized Step Counter Theorem can be used to argue that the corresponding sequence

$$p_{A_0}, p_{A_1}, p_{A_2}, p_{A_3}, \ldots$$

of characteristic functions of these sets if G-computable.

Consider the total function $\widehat{m} : \mathbb{N} \to \mathbb{N}$ such that, for all $x \in \mathbb{N}$,

- If $x \in A$ — which is true if and only if $A_G(2x+1) = 1$ — then $\widehat{m}(x)$ is the smallest integer $n \in \mathbb{N}$ such that $x \in A_n$.
- If $x \notin A$ — which is true if and only if $A_G(2x+1) = 0$ — then $\widehat{m}(x) = 0$.

It is not hard to show that (by definition) $\widehat{m}$ is a modulus for the sequence of sets $A_0, A_1, A_2, \ldots$. Furthermore, since $G(x) = A_G(2x)$ for all $x \in \mathbb{N}$, it is reasonably easy to show that $\widehat{m}$ is A_G-computable.

Now, since F is A-computable, there exists a number $d \in \mathbb{N}$ such that $\mathcal{Q}$ A-computes F, where $\mathcal{Q}$ is the $\mathcal{S}$ program, with an oracle for A, such that $\#(\mathcal{Q}) = d$. Since $\mathcal{Q}$ A-computes a total function, $y \in W_d^A$ for all $y \in \mathbb{N}$.

It follows that $y \in W_{d,n}^{A_n}$ for all sufficiently large n as well. With that noted, for $n \in \mathbb{N}$, let $F_n : \mathbb{N} \to \mathbb{N}$ be the total function such that, for all $y \in \mathbb{N}$, if $y \in W_{d,n}^{A_n}$ then $F_n(y)$ is the value that is A_n-computed by $\mathcal{Q}$. On the other hand, if $y \notin W_{d,n}^{A_n}$ then $F_n(y) = 0$. Since the sequence $p_{A_0}, p_{A_1}, p_{A_2}, p_{A_3} \ldots$ is G-computable it can be shown, with a bit of work, that the sequence $F_0, F_1, F_2, F_3, \ldots$ is G-computable as well.

Let $m : \mathbb{N} \to \mathbb{N}$ such that, for all $y \in \mathbb{N}$, $m(y)$ is the smallest number $z \in \mathbb{N}$ such that $y \in W_{d,z}^A$ and, furthermore, such that $z \geq \widehat{m}(x)$ for every A-query x (that is, number x such that it is asked whether $x \in A$) during the A-computation of $F(y)$ using program $\mathcal{Q}$. Since $\widehat{m}$ is a modulus for the sequence $A_0, A_1, A_2, A_3, \ldots$ it is not hard to used this definition to show that the sequence $F_0, F_1, F_2, F_3, \ldots$ has a limit and that m is a modulus for this sequence.

Furthermore, since $\widehat{m}$ is A_G-computable, and since of evaluation of A_G can be used both to decide membership in A and evaluate G, it can be shown that m is A_G-computable.

Finally, let $y \in \mathbb{N}$ and suppose that $z \geq m(y)$. Then it can be argued, using the definition of m, that $F(y) = F_z(y)$. Thus F is the limit of the sequence $F_0, F_1, F_2, F_3, \dots$, as needed to establish the claim. $\square$

Theorem 8 (Limit Lemma)**.** *Let $F, G : \mathbb{N} \to \mathbb{N}$ be total functions. Then F is G'-computable if and only if there exists a sequence*

$$F_0, F_1, F_2, F_3, \dots$$

of total functions, such that $F_i : \mathbb{N} \to \mathbb{N}$ for $i \in \mathbb{N}$, satisfying the following properties:

(a) The sequence $F_0, F_1, F_2, F_3, \dots$ is G-computable.

(b) The sequence $F_0, F_1, F_2, F_3, \dots$ has a limit. In particular, F is the limit of this sequence.

Proof. Let $F : G \to \mathbb{N} \to \mathbb{N}$ be total functions.

(a) Suppose, first, that a sequence $F_0, F_1, F_2, F_3, \dots$ of total functions $F_i : \mathbb{N} \to \mathbb{N}$ (for $i \in \mathbb{N}$) that satisfy properties (a) and (b) exists.

Define a set $A \subseteq \mathbb{N}$ such that, for all $x, n \in \mathbb{N}$, $\langle n, x \rangle \in A$ if and only if $F_m(x) = F_n(x)$ for every integer m such that $m \geq n$. Then, for all $m, n \in \mathbb{N}$, $\langle n, x \rangle \in \overline{A} = \{z \in \mathbb{N} \mid z \notin A\}$ if and only if there exists an integer m such that $m \geq n$ and $F_m(x) \neq F_n(x)$.

Since the sequence $F_0, F_1, F_2, F_3, \dots$ is G-computable, the function $H : \mathbb{N} \to \mathbb{N}$ such that, for all $n, x \in \mathbb{N}$ $H(n, x) = F_n(x)$ is (by definition) G-computable.

It is not hard to use the above properties to argue that $\overline{A}$ is G-computably enumerable, and it follows by Theorem #3 from Lecture #16 (on the "Arithmetic Hierarchy") that $\overline{A} \preceq_m G'$. Thus $\overline{A} \preceq_T G'$ as well and, since $A \preceq_T \overline{A}$ as well, $A \preceq_T G'$.

Since the sequence $F_0, F_1, F_2, F_3, \dots$ has a limit it follows by the definition of A that, for all $x \in \mathbb{N}$, there exists a number $n \in \mathbb{N}$ such that $\langle n, x \rangle \in A$. The function $J : \mathbb{N} \to \mathbb{N}$ such that, for all $x \in \mathbb{N}$, $J(n)$ is the *smallest* natural number such that $\langle n, x \rangle \in A$ is, therefore, a well-defined ***total*** function, and it is not hard to argue that J is G'-computable, since (the characteristic function of) A is G'-computable.

Now recall that the function $H : \mathbb{N}^2 \to \mathbb{N}$ such that, for all $n, x \in \mathbb{N}$, $H(n, x) = F_n(x)$, is G-computable. Since G is G-computable, so that G is also G-computably enumerable, $G \preceq_T G'$ as well — see, again, Theorem #3 in Lecture #16. It follows that H is also G'-computable.

Since F is the limit of the sequence $F_0, F_1, F_2, F_3, \dots$ it is easily checked that, for all $x \in \mathbb{N}$,

$$F(x) = H(J(x), x).$$

Since H and J are both G'-computable, F is certainly G'-computable as well, as required.

(b) Suppose, conversely, that $F : \mathbb{N} \to \mathbb{N}$ is a total function such that F is G'-computable.

Set $A = G' \subseteq \mathbb{N}$. Then A is G-computably enumerable and F is A-computable, so it follows by the Relativized Modulus Lemma that there exists a sequence

$$F_0, F_1, F_2, F_3, \ldots$$

of total functions $F_i : \mathbb{N} \to \mathbb{N}$ for $i \in \mathbb{N}$ such that

(a) the sequence $F_0, F_1, F_2, F_3, \ldots$ is G-computable, and

(b) the sequence $F_0, F_1, F_2, F_3, \ldots$ has a limit. Indeed, F is the limit of this sequence,

as needed to establish the claim. $\square$