Computer Science 513

Creative Sets, Simple Sets and Limits

Instructor: Wayne Eberly

Department of Computer Science
University of Calgary

Lecture #19

Where We Were

During the previous lecture ...

- ***degrees of unsolvability*** — Q-degrees for any reducibility Q — had been introduced;
- ***Post's problem*** had been raised: Does there exist a set $A \subseteq \mathbb{N}$ such that A is not computable, A is computably enumerable, and A is *not* Turing-complete for the set of computably enumerable sets? This question motivated a study of Turing degrees, but also of 1-degrees and m-degrees;
- ***Myhill's Theorem*** was stated and proved.

Where We Were

- Finally, ***cylinders*** were introduced and studied.

  It was noted that if there exists a computably enumerable set that is not computable, and not a cylinder, then there is also an *m*-degree, including computably enumerable sets that are not computable, that has more than one 1-degree as a subset.

  On the other hand, if every set that is computably enumerable but not computable is a cylinder then every 1-degree, including computably sets that are not computable, is also an *m*-degree.

Goals for Today

Goals for Today:

- Introduce ***creative sets*** and ***simple sets***.
- Prove the existence of a simple set.
- Use this to establish that there is a computably enumerable set that is not computable and not a cylinder — answering one of the questions raised during the previous lecture.

This *does not* solve Post's problem. However, it will also establish the existence of a computably enumerable set that is not computable, and also not m-complete for the set Σ_1 of computably enumerable sets.

Reference: Hartley Rogers, Jr., *Theory of Recursive Functions and Effective Computability*

Goals for Today, Continued

Goals for Today, Continued

- Introduce ***limits*** of sequences of total functions — and sets — and establish useful properties of these. These will eventually be used to prove the correctness of a solution for Post's problem.

Reference: Joseph R. Schoenfield, *Degrees of Unsolvability*

Productive Sets: Definition

Definition: A set $A \subseteq \mathbb{N}$ is ***productive*** if there exists a computable partial function $\psi_A : \mathbb{N} \to \mathbb{N}$ such that, for all $x \in \mathbb{N}$, if $W_x \subseteq A$ then $\psi_A(x)$ is defined and $\psi_A(x) \in A \setminus W_x$.

A computable partial function $\psi_A : \mathbb{N} \to \mathbb{N}$, for which this condition is satisfied, is called a ***productive partial function*** for A.

Productive Sets: Example

Recall that $K = \{n \in \mathbb{N} \mid n \in W_n\}$.

Theorem #1: There exists a productive set. In particular, the set

$$\overline{K} = \{n \in \mathbb{N} \mid n \notin W_n\}$$

is productive.

Proof: It suffices to apply the definition of a "productive function," using the identity function $I : \mathbb{N} \to \mathbb{N}$, such that $I(n) = n$ for all $n \in \mathbb{N}$, as a productive partial function for $\overline{K}$. $\square$

Note: This is an important (but potentially confusing) example of a productive function!

Productive Sets: Two Important Properties

Theorem #2: Let $A, B \subseteq \mathbb{N}$.

(a) If A is productive then A is not computably enumerable.

(b) If A is productive and $A \preceq_m B$ then B is productive.

- This is a reasonably straightforward consequence of the definitions of a "productive set" and a many-one reduction.
- Details are provided in the supplementary document for this lecture.

Productive Sets: Another Useful Property

Theorem #3: Let $A \subseteq \mathbb{N}$ be a productive set. Then there exists an infinite set $B \subseteq \mathbb{N}$ such that B is computably enumerable and $B \subseteq A$.

- A computably enumerable set B, satisfying the conditions in the claim, can be constructed from a productive partial function for A — that is, a computable partial function $\psi_A : \mathbb{N} \to \mathbb{N}$ such that for all $x \in \mathbb{N}$, if $W_x \subseteq A$ then $\psi_A(x)$ is defined and $\psi_A(x) \in A \setminus W_x$.
- The supplemental document includes a proof of this claim that is based on this.

Creative Sets: Definition and Example

Definition: A set $A \subseteq \mathbb{N}$ is ***creative*** if it satisfies the following properties:

(a) A is computably enumerable.

(b) $\overline{A} = \{x \in \mathbb{N} \mid x \notin A\}$ is productive.

Example: Since $\overline{K}$ is productive, and K is computably enumerable, the set

$$K = \{n \in \mathbb{N} \mid n \in W_n\}$$

is creative.

Creative Sets: Three Important Properties

Theorem #4: Let $A, B \subseteq \mathbb{N}$.

(a) If A is creative then A is not computable.

(b) If A is creative and $A \preceq_m B$ then $\overline{B} = \{x \in \mathbb{N} \mid x \notin B\}$ is productive.

(c) If A is m-complete for the set Σ_1 of computably enumerable sets then A is creative.

Proof:

(a) If A is creative then $\overline{A}$ is productive, by definition, and it follows by part (a) of Theorem #2 that $\overline{A}$ is not computably enumerable — so that A is not computable, as needed to establish part (a).

Creative Sets: Three Important Properties

(b) Suppose that A is creative and $A \preceq_m B$. Then B is computably enumerable, since A is. $\overline{A}$ is productive and $\overline{A} \preceq_m \overline{B}$, so it follows by part (b) of Theorem #2 that $\overline{B}$ is productive, as needed to establish part (b).

(c) Suppose that A is m-complete for Σ_1. As noted above, A is computably enumerable (that is, $A \in \Sigma_1$) by the definition of "m-complete." Furthermore $K \preceq_m A$, since $K \in \Sigma_1$ and, since K is creative, it follows by part (b) that $\overline{A}$ is productive. It follows that A is creative, as needed to establish part (c). $\square$

Simple Sets: Definition

Definition: Let $A \subseteq \mathbb{N}$. Then A is ***simple*** if it satisfies the following properties:

(a) A is computably enumerable.

(b) $\overline{A} = \{x \in \mathbb{N} \mid x \notin \mathbb{N}\}$ is infinite.

(c) For every set $B \subseteq \mathbb{N}$, if B is infinite and B is computably enumerable then $B \cap A \neq \emptyset$.

- It is ***not*** obvious that "simple sets" even exist. The following property explains why they are important, if they *do* exist.

Simple Sets: An Important Property

Theorem #5: Let $A \subseteq \mathbb{N}$.

(a) If A is simple then A is not computable.

(b) If A is simple then A is not creative.

(c) If A is simple then A is not m-complete for Σ_1.

(d) If A is simple then A is not a cylinder.

Proof:

(a) Suppose that A is simple and A is computable. Then $\overline{A}$ is computable and infinite, but it would now follow by property (c) of the definition of a “simple set” that $\overline{A} \cap A = \emptyset$. This is certainly impossible, and this ***contradiction*** establishes part (a) of the claim.

Simple Sets: An Important Property

(b) Suppose next that A is both simple and creative. Then $\overline{A}$ is productive, since A is creative, and it follows by Theorem #3 that there exists a computably enumerable set $B \subseteq \mathbb{N}$ such that B is infinite and $B \subseteq \overline{A}$ — so that $B \cap A = \emptyset$.

However, this ***contradicts*** property (c) in the definition of a "simple set," as needed to establish part (b).

(c) Suppose next that A is simple and m-complete for Σ_1. Then it follows by part (c) of Theorem #4 that A is also creative, ***contradicting*** part (b) — as needed to establish part (c).

Simple Sets: An Important Property

(d) Finally, suppose that A is simple and A is a cylinder — so that

$$A \equiv Cyl(C) = \{\langle a, b\rangle \mid a \in C \text{ and } b \in \mathbb{N}\}$$

for some set $C \subseteq \mathbb{N}$. It follows by the definition of recursive invariance that there is a recursive permutation $g : \mathbb{N} \to \mathbb{N}$ such that $x \in A$ if and only if $g(x) \in Cyl(C)$ for all $x \in \mathbb{N}$.

Since $\overline{A}$ is infinite, by part (b) of the definition of a "simple set," $\overline{C} \neq \emptyset$. Let $m \in \overline{C}$.

Then, since g^{-1} is also a recursive permutation,

$$B = \{x \in \mathbb{N} \mid g(x) = \langle m, y\rangle \text{ for a number } y \in \mathbb{N}\}$$

is an infinite computably enumerable set such that $B \subseteq \overline{A}$ — so that $B \cap A = \emptyset$, ***contradicting*** part (c) of the definition of a "simple set," once again, and proving part (d). □

Simple Sets: Existence

Theorem #6 (Post): There exists a simple set.

- This is result is established using a construction that does not seem easy to discover (or motivate) and seems not to be of independent interest. Once the construction is understood, the details of the proof — which is included in the supplemental document — are easily verified.

Simple Sets: Consequences

Let $S \subseteq \mathbb{N}$ be the set described in the proof of the previous result.

- It follows by the properties of ***cylinders***, established in the previous lecture, that S and $Cyl(S)$ are both computably enumerable, neither is computable, $Cyl(S) \preceq_m S$, but $Cyl(S) \not\preceq_1 S$. Thus the reducibilities "$\preceq_m$" and "$\preceq_1$" *do not* agree on the class of sets that are computably enumerable but not computable.
- It also follows that S is an example of a computably enumerable set that is not computable, and also not either m-complete or 1-complete for the class Σ_1 of computably enumerable sets.

Limits of Sequences of Total Functions

Suppose that $F_i : \mathbb{N} \to \mathbb{N}$ is a total function for all $i \in \mathbb{N}$.

Definition: The above sequence $F_0, F_1, F_2, \ldots$ ***has a limit*** if, for all $x \in \mathbb{N}$, there exists an integer $m_x > 0$ such that $F_i(x) = F_j(x)$ for all $i, j \geq m_x$.

Definition: If a sequence $F_0, F_1, F_2, \ldots$ has a limit then a ***modulus*** for this sequence is a total function $m : \mathbb{N} \to \mathbb{N}$ such, for all $x \in \mathbb{N}$, if $i, j \in \mathbb{N}$ and $i, j \geq m(x)$ then $F_i(x) = F_j(x)$.

Limits of Sequences of Total Functions

Definition: If a sequence $F_0, F_1, F_2, \ldots$ has a limit, as defined above, then the ***limit*** of this sequence is the (well-defined) total function $F : \mathbb{N} \to \mathbb{N}$ such, for all $x \in \mathbb{N}$, there exists an integer $m_x \in \mathbb{N}$ such that $F_i(x) = F(x)$ for all $i \geq m_x$.

Note: If $F_0, F_1, F_2, \ldots$ has a limit then exactly one ***limit*** $F : \mathbb{N} \to \mathbb{N}$ for this sequence exists and *infinitely many* moduli for this sequence exist — for if $m : \mathbb{N} \to \mathbb{N}$ is a modulus then so is the function $\widehat{m} : \mathbb{N} \to \mathbb{N}$ such that $\widehat{m}(x) = m(x) + 1$ for all $x \in \mathbb{N}$.

Limits of Sequences of Sets

Similarly, a sequence

$$A_0, A_1, A_2, A_3 \ldots . \tag{1}$$

of subsets of $\mathbb{N}$ ***has a limit*** if the corresponding sequence

$$p_{A_0}, p_{A_1}, p_{A_2}, p_{A_3}, \ldots \tag{2}$$

of the characteristic functions of these sets has a limit. The limit of the sequence of characteristic functions is the characteristic function of some set $A \subseteq \mathbb{N}$, and this set is called the ***limit*** of the above sequence of sets.

A ***modulus*** for the above sequence of sets (1) is the same as a modulus for the above sequence of characteristic functions (2).

G-Computable Sequences

Now let $G : \mathbb{N} \to \mathbb{N}$ be a ***total*** function.

Definition: A sequence

$$F_0, F_1, F_2, F_3, \ldots$$

of total functions $F_i : \mathbb{N} \to \mathbb{N}$, for $i \in \mathbb{N}$, is ***G-computable*** if the total function $H : \mathbb{N}^2 \to \mathbb{N}$ such that, for all $n, x \in \mathbb{N}$, $H(n, x) = F_n(x)$, is a G-computable function.

If $A \subseteq \mathbb{N}$ then the above sequence is ***A-computable*** if the above function $H : \mathbb{N}^2 \to \mathbb{N}$ is p_A-computable, where $p_A : \mathbb{N} \to \{0, 1\}$ is the characteristic function of A so that, for all $x \in \mathbb{N}$,

$$p_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

Relativized Modulus Lemma

Theorem #7 (Relativized Modulus Lemma): Let $G : \mathbb{N} \to \mathbb{N}$ and let $A \subseteq \mathbb{N}$. If A is G-computably enumerable and a total function $F : \mathbb{N} \to \mathbb{N}$ is A-computable, then there exists a G-computable sequence $F_0, F_1, F_2, F_3, \ldots$ of total functions $F_i : \mathbb{N} \to \mathbb{N}$, for $i \in \mathbb{N}$, such that F is the limit of this sequence.

This sequence of functions also has an A_G-computable modulus function $m : \mathbb{N} \to \mathbb{N}$, where $A_G : \mathbb{N} \to \mathbb{N}$ such that, for all $x \in \mathbb{N}$,

$$A_G(2x) = G(x),$$

and

$$A_G(2x+1) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

Limit Lemma

Theorem #8 (Limit Lemma): Let $F, G : \mathbb{N} \to \mathbb{N}$ be total functions. Then F is G'-computable if and only if there exists a sequence

$$F_0, F_1, F_2, F_3, \ldots$$

of total functions, such that $F_i : \mathbb{N} \to \mathbb{N}$ for $i \in \mathbb{N}$, satisfying the following properties:

(a) The sequence $F_0, F_1, F_2, F_3, \ldots$ is G-computable.

(b) The sequence $F_0, F_1, F_2, F_3, \ldots$ has a limit. In particular, F is the limit of this sequence.

- The (not very interesting) proofs of Theorems #7 and #8 are given in the supplemental document for this lecture.

Jumps and Leaps

While the reference *Degrees of Unsolvability* defines the “jump” of a total function G, this is ***not*** the same as the jump G' of G that has already been defined in *Computability, Complexity and Languages* and in these notes. The version of “jump” used in *Degrees of Unsolvability* will, therefore, be given a different name here.

Definition: Let $G : \mathbb{N} \to \mathbb{N}$ be a total function. Then the ***leap*** G'' of G is the set

$$G'' = \{x \in \mathbb{N} \mid \ell(x) \in W^G_{r(x)}\}.$$

Jumps and Leaps

Recall that if $I : \mathbb{N} \to \mathbb{N}$ is the identity function (so that $I(n) = n$ for all $n \in \mathbb{N}$) then the ***jump*** I' of I is the set

$$K = \{n \in \mathbb{N} \mid n \in W_n\}.$$

Similarly, it is easily verified that the ***leap*** I'' of I is the set

$$K_0 = \{x \in \mathbb{N} \mid \ell(x) \in W_{r(x)}\}$$

considered in Lecture #10 (on "Diagonalization and Reducibility").

It was shown, in the above lecture, that $K_0 \equiv_m K$ and, indeed, that K_0 and K are both m-complete for the collection Σ_1 of computably enumerable sets.

Jumps and Leaps

Theorem #9: Let $G : \mathbb{N} \to \mathbb{N}$.

(a) $G' \equiv_1 G''$.

(b) G' and G'' are both 1-complete for the set of G-computably enumerable sets.

Proof:

- The proof of Theorem #11 in Lecture #14 (on "Reductions and Reducibilities") relativizes, and this establishes that the jump G' of G is 1-complete for the set of G-computably enumerable sets.
- It can be shown, using the Relativized Universality Theorem, that G'' is G-computably enumerable, so that $G'' \preceq_1 G'$.

Jumps and Leaps

- Let $f : \mathbb{N} \to \mathbb{N}$ such that, for all $x \in \mathbb{N}$,

$$f(x) = \langle x, x \rangle.$$

  Then f is primitive recursive, so that f is certainly a computable total function. It is also easily checked that f is a ***one-one*** computable total function.
- Furthermore $x \in G'$ if and only if $f(x) \in G''$ for all $x \in \mathbb{N}$, so that $G' \preceq_1 G''$ as well.
- Thus $G' \equiv_1 G''$, as needed to establish part (a). Since G' is 1-complete for the set of G-computably enumerable sets, and G'' is G-computably enumerable, this also establishes that G'' is also 1 complete for the set of G-computably enumerable sets, establishing part (b). $\square$