Lecture #20: Priority Methods and a Solution for Post's Problem

Exercises and Review

Questions for Review

Note: These notes present a single, long ***proof*** of a significant result. Students will not be tested on their understanding of this proof, in this course. Thus these "review questions" are less important than the review questions for most other lectures.

With that noted, they are intended to help you to remember the *key ideas* and *objectives* associated with this proof.

The proof is constructive — it describes how to compute a sequence

$$A_0, A_1, A_2, A_3, \ldots$$

of finite subsets of $\mathbb{N}$ that has a ***limit*** A.

Details of the Construction

1. Describe the ***objective*** I_i for each number $i \in \mathbb{N}$.
2. Describe the ***objective*** $J_{i,j}$ for each pair of numbers $i, j \in \mathbb{N}$.
3. What is the ***priority*** $\#(I_i)$ (respectively, $\#(J_{i,j})$) for each objective I_i (respectively, $J_{i,j}$)?

   Note: It should not be hard to see, from your answer, that every natural number n is the priority of ***exactly one*** of these objectives.
4. Describe a ***second*** sequence of finite subsets of $\mathbb{N}$

   $$B_0, B_1, B_2, B_3, \ldots$$

   that will also be considered as part of this constructive proof.

5. Describe a ***third*** sequence of finite subsets of $\mathbb{N}$

$$\text{Achieved}(0), \text{Achieved}(1), \text{Achieved}(2), \text{Achieved}(3), \ldots$$

that will also be considered as part of this constructive proof.

6. Describe a ***partial function***

$$\text{Neg-Required} : \mathbb{N}^2 \to \mathcal{P}(\mathbb{N}),$$

mapping ordered pairs of natural numbers to ***finite subsets*** of $\mathbb{N}$, that will also be considered during this construction.

7. Describe a ***fourth*** sequence of finite subsets of $\mathbb{N}$

$$\text{NDef}_0, \text{NDef}_1, \text{NDef}_2, \text{NDef}_3, \ldots$$

that will also be considered during this construction. How are these related to the above partial function Neg-Required?

8. These sequences of sets, and the partial function Neg-Required, will be constructed using an infinite sequence of ***steps***, including a "step s" for $s = 0, 1, 2, 3, \ldots$.

   Step s will include an attempt to ***accomplish*** exactly one objective I_i or $J_{i,j}$, for $i, j \in \mathbb{N}$. How is the number s of the set related to the objective that is considered during it?

9. *Briefly* describe how a finite set $S \subseteq \mathbb{N}$ can be "encoded" and represented using a ***variable*** whose value is a natural number.

10. *Briefly* describe how this can be used to represent *all* of the values Neg-Required(s, n) for $n \in \mathbb{N}$ such that these are defined, for any given number $s \in \mathbb{N}$.

11. Consider, now, the details of step s for $s \in \mathbb{N}$ such that $\ell(s) = n = \#(J_{i,j})$ for some numbers $n, i, j \in \mathbb{N}$. Nothing is done or changed here if $n \in$ Achieved$(s-1)$ already.

    Otherwise a reasonably straightforward ***condition*** is checked for. Again, nothing much is done or changed if the condition does not hold.

    (a) What is the ***condition*** that is checked?
    (b) How are A_{s-1} and A_s related if this condition is satisfied?
    (c) How are B_{s-1} and B_s related if this condition is satisfied?
    (d) How are Achieved$(s-1)$ and Achieved(s) related if this condition is satisfied?
    (e) How are NDef$_{s-1}$ and NDef$_s$ related if this condition satisfied?
    (f) How are Neg-Required$(s-1, m)$ and Neg-Required(s, m) related, if this condition is satisfied, for $m \in \mathbb{N}$?

12. What does it mean for the "achievement" of an objective to be ***temporary***? What does it mean for an achievement of an objective to be ***permanent***?

13. Next consider the details of step s for $s \in \mathbb{N}$ such that $\ell(s) = n = \#(I_i)$ for some numbers $n, i \in \mathbb{N}$. Once again, nothing is done or changed if $n \in$ Achieved$(s-1)$.

    Otherwise, step s proceeds with an attempt to discover the smallest number $k \in \mathbb{N}$ that satisfies a finite set of ***conditions*** — or to confirm that no number satisfying these objectives exists at all. Nothing much is changed if no number satisfying these conditions exists at all.

    (a) Describe the ***conditions*** that a number k must satisfy. One of these is the condition that k is the ***smallest*** number in $\mathbb{N}$ satisfying the other conditions that must be established.
    (b) How are A_{s-1} and A_s related if a number k, satisfying these conditions, exists?
    (c) What is the set Failed(s), if a number $k \in \mathbb{N}$ satisfying these conditions exists?
    (d) How is Achieved(s) related to Achieved$(s-1)$, $n = \ell(s)$, and Failed(s) if a number k satisfying these conditions exists?
    (e) How is NDef$_s$ related to NDef$_{s-1}$ if a number k satisfying these conditions exists?
    (f) How are Neg-Required$(s-1, m)$ and Neg-Required(s, m) related, if a number k satisfying these conditions exists, for $m \in \mathbb{N}$?
    (g) How are B_{s-1} and B_s related, if a number k satisfying these conditions exists?

14. Why is this construction called a ***priority method***?

Note: If your answers for the above questions are correct, and sufficiently detailed, then it should not be hard to see that each of $\#(A_s)$, $\#(B_s)$, $\#($Achieved(s), $\#($NDef$_s)$, and a number encoding

$$\{\text{Neg-Required}(s, m) \mid m \in \mathbb{N} \text{ and } m \in \text{NDef}_s\}$$

are all computable from s.

Useful Properties

15. What is the limit, A, of the sequence $A_0, A_1, A_2, A_3, \dots$? What property of these sets (or the relationship between A_i and A_{i+1}, for $i \in \mathbb{N}$) makes it reasonably easy to show that this sequence *has* a limit?

16. A consideration of the process that is being used confirms that if $i \in \mathbb{N}$ then any achievement of the objective I_i is ***permanent,*** that is, if the priority, n of the objective I_i is included in a set Achieved(s), then $n \in$ Achieved(t) as well for all $t \in \mathbb{N}$ such that $t \geq s$.

On the other hand, an achievement of an objective $J_{i,j}$ (for $i, j \in \mathbb{N}$) can sometimes be ***temporary*** instead: The priority n of the objective $J_{i,j}$ can be added to a set Achieved(s) but then removed from Achieved(t), where $s, t \in \mathbb{N}$ and $s < t$.

Explain *briefly* why there can be at most $\lfloor (n-1)/2 \rfloor + 1$ temporary achievements of an objective $J_{i,j}$ such that $\#(J_{i,j}) = n$.

17. Explain briefly why the above information (discussed in Question #16) can be used to argue that the sequence

$$\text{Achieved}(0), \text{Achieved}(1), \text{Achieved}(2), \text{Achieved}(3), \ldots$$

also has a limit Achieved $\subseteq \mathbb{N}$.

18. After considering the process that has been described, explain briefly how the result, discussed in Question #17, can be used to establish each of the following.

(a) The sequence $B_0, B_1, B_2, B_3, \ldots$ has a limit, $B \subseteq \mathbb{N}$.

(b) The sequence $\text{NDef}_0, \text{NDef}_1, \text{NDef}_2, \text{NDef}_3, \ldots$ has a limit, NDef $\subseteq \mathbb{N}$.

(c) For every number $n \in$ NDef there exists a number $m \in \mathbb{N}$ such that $n \in \text{NDef}_s$ for all $s \in \mathbb{N}$ such that $s \geq m$ and, furthermore, if $s, t \in \mathbb{N}$ and $s, t \geq m$, then Neg-Required(s, n) = Neg-Required(t, n).

Proving That A is not Computable

19. What has to be proved about the $\mathcal{S}$-program $\mathcal{P}$ such that $\#(\mathcal{P}) = i$, for all $i \in \mathbb{N}$, in order to establish that A is ***not computable***?

20. How is this property proved about $\mathcal{P}$ when $i = \#(\mathcal{P})$ and $i \in$ Achieved, where Achieved is the set described in Question #17?

21. How is this property proved about $\mathcal{P}$ when $i = \#(\mathcal{P})$ and $i \notin$ Achieved?

Proving That A is Computably Enumerable

22. What property of the sets $A_0, A_1, A_2, A_3, \ldots$ — specifically, the relationship between A_i and A_{i+1} for $i \in \mathbb{N}$ — makes it reasonably easy to prove that the limit $A \subseteq \mathbb{N}$ of this sequence is ***computably enumerable***?

Proving That A is not Turing-Complete for Σ_1

23. Suppose that $S, T \subseteq \mathbb{N}$ such that $S \preceq_T T$. How is the ***leap*** S'' of S related to the ***leap*** T'' of T?

24. It can be shown that if $S \subseteq \mathbb{N}$ and S'' is $\emptyset^{(1)}$-computable, then S is not Turing-complete for Σ_1, by assuming otherwise and obtaining a ***contradiction*** of something that has already been proved about the Arithmetic Hierarchy. What, specifically, is the fact about the Arithmetic Hierarchy that is contradicted?

    Note that the proof of this depends on the result discussed in Question #23, along with the fact that $S' \equiv_1 S''$ for any set $S \subseteq \mathbb{N}$.

25. Explain, briefly, how it is shown that A'' is $\emptyset^{(1)}$-computable, in order to complete a proof that A is not Turing-complete for Σ_1.

Exercises

There will be no "problems to be solved", this time. Making your way through the above review questions should be more than enough!