Lecture #21: Chomsky Hierarchy II — Context-Free Languages

Exercises and Review

Questions for Review

1. What is a ***context-free grammar***? What is a ***context-free language***?
2. Describe how ***regular languages*** and ***context-free languages*** are related.
3. What does it mean for a context-free grammar to be in ***Chomsky normal form***? How is the set of languages of context-free grammars, in Chomsky normal form, related to the set of context-free languages?
4. ***Pushdown automata*** were also introduced in the notes for this lecture.
   (a) Give the definition of a ***pushdown automaton***. How is this similar to the kinds of abstract machines (like finite automata and Turing machines) that you already know about? How is it different?
   (b) Define an ***instantaneous description*** of a pushdown automaton.
   (c) Say, as precisely as you can, what it means for one instantaneous description of a pushdown automaton to ***yield*** another instantaneous description of the pushdown automaton.
   (d) Say, as precisely as you can, what it means for one instantaneous description of a pushdown automaton to ***derive*** another instantaneous description of the pushdown automaton.
   (e) Define the language, $T(M)$ accepted by M by final state, as precisely as you can.
   (f) Define the language, $N(M)$, accepted by M by empty store, as precisely as you can.
   (g) Define the language, $L(M)$, accepted by M by both final state and empty store, as precisely as you can.
   (h) How are the class of context-free languages, the languages accepted by pushdown automata by final state, the languages accepted by pushdown automata by empty

store, and the the languages accepted by pushdown automata by both final state and empty store, related (if they are related, at all)?

5. State the ***Pumping Lemma for Context-Free Languages***. Why is this useful?
6. State one or more ***closure properties*** for the class of context-free languages.
7. State one or more operations such that the class of ***regular languages*** is closed under each of these operations, but the class of ***context-free languages*** is not.
8. What is ***parse tree*** for a string in a context-free grammar?
9. What does it mean for a context-free grammar to be ***ambiguous***?
10. What does it mean for a context-free language to be ***inherently ambiguous***?
11. Describe one or more problems, concerning context-free grammars and their languages, that are probably ***solvable***.
12. Describe one or more problems, concerning context-free grammars and their languages, that are probably ***unsolvable***.

Problems To Be Solved

1. Consider a ***pushdown automaton*** $M = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, fF$ such that
   - $Q = \{q_0, q_1, q_2, q_3\}$
   - $\Sigma = \{\mathtt{a}, \mathtt{b}\}$
   - $\Gamma = \{\mathtt{a}, Z_0\}$
   - The function $\delta : Q \times \Sigma_\lambda \times \Gamma \to \mathcal{P}(Q \times \Gamma^\star)$ is as follows.
     - $\delta(q_0, \mathtt{a}, \lambda) = \{(q_1, \mathtt{a})\}$.
     - $\delta(q_0, \lambda, Z_0) = \{(q_3, \lambda)\}$.
     - $\delta(q_1, \mathtt{a}, \lambda) = \{(q_1, \mathtt{a})\}$.
     - $\delta(q_1, \mathtt{b}, \mathtt{a}) = \{(q_2, \lambda)\}$.
     - $\delta(q_2, \mathtt{b}, \mathtt{a}) = \{(q_2, \lambda)\}$.
     - $\delta(q_2, \lambda, Z_0) = \{(q_3, \lambda)\}$.
     - $\delta(q, \sigma, \tau) = \emptyset$ for all states $q \in Q$, symbols $\sigma \in \Sigma_\lambda$, and pushdown symbols $\tau \in \Gamma$ such that this part of the transition function is not given above.
   - $F = \{q_3\}$.

   (a) Prove that $\lambda \in L(M)$ (so that $\lambda \in T(M)$ and $\lambda \in N(M)$, as well).

   (b) Consider a string $\mathtt{b}\mu$, for $\mu \in \Sigma^\star$. Note that the only transition that can be applied, to the instantaneous description $(q_0, \mathtt{b}\mu, Z_0)$, is the transition (q_3, λ) — resulting in the instantaneous description

   $$(q_3, \mathtt{b}\mu, \lambda)$$

— and that no transitions can be used, after that. Note that it follows that $\mathtt{b}\mu \notin T(M)$, $\mathtt{b}\mu \notin N(M)$, and $\mathtt{b}\mu \notin L(M)$.

(c) Consider a string $\mathtt{a}^k\mu$ for a positive integer k and a string $\mu \in \Sigma^\star$. Prove that

$$(q_0, \mathtt{a}^k\mu, Z_0) \vdash^\star (q_1, \mu, Z_0\mathtt{a}^k).$$

(d) Once again, consider a string $\mathtt{a}^k\mu$ for a positive integer k and a string $\mu \in \Sigma^\star$. Prove that there are no other states $r \in Q$ and strings of pushdown symbols $\rho \in \Gamma^\star$ (besides the state $r = q_1$ and string $\rho = Z_0\mathtt{a}^k$) such that

$$(q_0, \mathtt{a}^k\mu, Z_0) \vdash^\star (r, \mu, \rho).$$

(e) Next consider integers k and ℓ such that $k \geq \ell \geq 1$, and let $\mu \in \Sigma^\star$. Prove that

$$(q_0, \mathtt{a}^k\mathtt{b}^\ell\mu, Z_0) \vdash^\star (q_2, \mu, Z_0\mathtt{a}^{k-\ell}).$$

(f) Prove that if $k > \ell \geq 1$, and $\mu \in \Sigma^\star$, then there are no other states $r \in Q$ and $\rho \in \Gamma^\star$ (besides $r = q_2$ and $\rho = Z_0\mathtt{a}^{k-\ell}$) such that

$$(q_0, \mathtt{a}^k\mathtt{b}^\ell\mu, Z_0) \vdash^\star (r, \mu, \rho).$$

(g) Use the above to prove that

$$T(M) = N(M) = L(M) = \{\mathtt{a}^n\mathtt{b}^n \mid n \in \mathbb{N}\}.$$

2. Let Σ and let $L_1, L_2 \subseteq \Sigma^\star$. Recall that if L_1 and L_2 are both context-free then there exist context-free grammars

$$G_1 = (V_1, \Sigma, \Pi_1, S_1) \quad \text{and} \quad G_2 = (V_2, \Sigma, \Pi_2, S_2)$$

such that $L_1 = L(G_1)$ and $L_2 = L(G_2)$. Renaming variables, if necessary, one can assume without loss of generality (in this case) that $V_1 \cap V_2 = \emptyset$ and that $S \notin V_1 \cup V_2$.

(a) Show that if L_1 and L_2 are both context-free then $L_1 \cup L_2$ is context-free as well.
Hint: Consider a context-free grammar with a set of variables $V = V_1 \cup V_2 \cup \{S\}$, start variable S, and whose productions include all the productions in Π_1, all the productions in Π_2, and another two productions (each with the new start variable, S, on the left-hand side).

(b) Show that if L_1 and L_2 are both context-free then $L_1 \circ L_2$ is context-free as well.

(c) Show that if L_1 is context-free then $L_1^\star$ is also context-free.

3. Let $L = \{\mathtt{a}, \mathtt{b}\}$. and consider the languages

$$L_1 = \{\mathtt{a}^n\mathtt{b}^n\mathtt{a}^m \mid n, m \in \mathbb{N}\}$$

and

$$L_2 = \{\mathtt{a}^n\mathtt{b}^m\mathtt{a}^m \mid n, m \in \mathbb{N}\}.$$

(a) Prove that L_1 is context-free.

(b) Prove that L_2 is context-free.

(c) Modify the proof in the lecture notes, that if $\widehat{\Sigma} = \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$ then the language

$$\{\mathtt{a}^n\mathtt{b}^n\mathtt{c}^n \mid n \in \mathbb{N}\} \subseteq \widehat{\Sigma}^\star$$

is *not* context-free, to show that $L_1 \cap L_2$ is not context-free either.
Then use this to prove part (a) of Claim #7 in the lecture notes.

(d) Use part (a) of Claim #7 to prove part (b) of that claim.
Hint: Consider De Morgan's laws.