Lecture #21: Chomsky Hierarchy II — Context-Free Languages

Lecture Presentation

Establishing the Language of a Context-Free Grammar

Let $G = (V, \Sigma, \Pi, S)$ where

- $V = \{S\}$.
- $\Sigma = \{\mathtt{a}, \mathtt{b}\}$.
- Π includes two productions, namely

  $$S \to \mathtt{a}S\mathtt{b}$$

  and

  $$S \to \lambda$$

- S is the start variable.

The goal of this part of the presentation is to prove that the language of this grammar is

$$L(G) = \{\mathtt{a}^n\mathtt{b}^n \mid n \geq 0\} \subseteq \Sigma^\star.$$

Lemma 1. *Let n be a non-negative integer. Then*

$$S \Rightarrow_G^\star \mathtt{a}^n\mathtt{b}^n,$$

so that $\mathtt{a}^n\mathtt{b}^n \in L(G)$.

How This Can Be Proved:

Proof of Lemma:

Lemma 2. *Let $\omega \in \Sigma^\star$ such that $S \Rightarrow_G^\star \omega$. Then $\omega = \mathtt{a}^n\mathtt{b}^n$ for some non-negative integer n.*

Note: Lemmas 1 and 2 imply that

$$L(G) = \{\mathtt{a}^n\mathtt{b}^n \mid n \in \mathbb{N}\} \subseteq \Sigma^\star,$$

as claimed in the lecture notes.

How This Second Lemma Can Be Proved:

Proof of Lemma:

Pumping Lemma for Context-Free Languages

Pumping Lemma for Context-Free Languages: Let $L \subseteq \Sigma^\star$. If L is a context-free language then there exists a positive integer p — called the ***pumping length*** for L — such that the following property is satisfied: For every string $s \in L$ such that $|s| \geq p$, there exist strings $u, v, w, x, y \in \Sigma^\star$ such that $s = uvwxy$ and the following properties are satisfied.

(i) $|vwx| \leq p$ — that is, this *middle* part of s is not very long.

(ii) $|vx| \geq 1$ — so that either $v \neq \lambda$ or $x \neq \lambda$ (or both).

(iii) $uv^iwx^iy \in L$ for every number $i \in \mathbb{N}$.

Sketch of Proof:

Application: Let $\Sigma = \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$ and let

$$L = \{\mathtt{a}^n\mathtt{b}^n\mathtt{c}^n \mid n \in \mathbb{N}\}.$$

Claim: L is not context-free.

Proof: