Lecture #23: Chomsky Hierarchy III — Context-Sensitive Languages

Lecture Presentation

Problem To Be Solved

Consider a phrase-structure grammar $G = (V, \Sigma, \Pi, S)$ such that

$$\Sigma = \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}, \quad V = \{S, B, C, T, W, Z\},$$

and Π includes the following productions.

$$\begin{array}{lll} S \to T & CB \to CZ & \mathtt{a}B \to \mathtt{ab} \\ S \to \lambda & CZ \to WZ & \mathtt{b}B \to \mathtt{bb} \\ T \to \mathtt{a}BC & WZ \to WC & \mathtt{b}C \to \mathtt{bc} \\ T \to \mathtt{a}TBC & WC \to BC & \mathtt{c}C \to \mathtt{cc} \end{array}$$

The goal of this presentation will be to prove that the language of this ***context-sensitive*** grammar is the language

$$L(G) = \{\mathtt{a}^n\mathtt{b}^n\mathtt{c}^n \mid n \in \mathbb{N}\},$$

so that this is a ***context-sensitive language***.

First Containment of Languages

Claim 1. *Let $G = (V, \Sigma, \Pi, S)$ be the context-free grammar that is given on shown on the first page. Then $S \Rightarrow^\star \mathtt{a}^n\mathtt{b}^n\mathtt{c}^n$, so that $\mathtt{a}^n\mathtt{b}^n\mathtt{c}^n \in L(G)$, for every non-negative integer n.*

How This Can Be Proved:

Proof:

Second Containment of Languages

A useful "second claim" is not as easy to state as one might think, but the following can be used.

Claim 2. *Let $G = (V, \Sigma, \Pi, S)$ be the context-sensitive grammar given on the first page. Let $\alpha \in (V \cup \Sigma)^\star$ be a non-empty string that satisfies the following properties.*

(a) Either $\alpha = \mathtt{a}^n T \nu$ or $\alpha = \mathtt{a}^n \nu$ for a non-negative integer n and a string $\nu \in \{B, C, \mathtt{b}, \mathtt{c}\}^\star$. Furthermore, ν begins with either "B" or "$\mathtt{b}$" — and ν begins with "B" if $\alpha = \mathtt{a}^n T \nu$.

(b) The sum of the number of copies of "B" in ν, and the number of copies of "$\mathtt{b}$" in ν, is equal to n.

(c) The sum of the number of copies of "C" in ν, and the number of copies of "$\mathtt{c}$" in ν, is equal to n.

(d) Neither "$\mathtt{Cb}$" nor "$\mathtt{cb}$" is a substring of ν.

If $\omega \in \Sigma^\star$ such that $\alpha \Rightarrow^\star \omega$, then $\omega = \mathtt{a}^m \mathtt{b}^m \mathtt{c}^m$ for some non-negative integer m.

How This Can Be Proved:

Proof:

Completing the Proof

Claim 3. *Let $G = (V, \Sigma, \Pi, S)$ be the context-sensitive grammar given on the first page. Then*

$$L(G) = \{\mathtt{a}^n\mathtt{b}^n\mathtt{c}^n \mid n \in \mathbb{N}\}.$$

Proof: