Computer Science 331
Classical Sorting Algorithms

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Lecture #22

Outline

The “Sorting Problem”

Precondition:

A: Array of length n, for some integer $n \geq 1$, storing objects of some ordered type

Postcondition:

A: Elements have been permuted (reordered) but not replaced, in such a way that

$$A[i] \leq A[i+1] \quad \text{for } 0 \leq i < n-1$$

Three Classical Algorithms

Discussed today: three “classical” sorting algorithms

- Reasonably simple
- Work well on small arrays
- Each can be used to sort an array of size n using $\Theta(n^2)$ operations (comparisons and exchanges of elements) in the worst case
- None is a very good choice to sort large arrays: asymptotically faster algorithms exist!

Reference: Textbook, Sections 3.1.2 and 8.2.2

Selection Sort

Idea:

- Repeatedly find "i^{th}-smallest" element and exchange it with the element in location $A[i]$
- Result: After i^{th} exchange,

  $$A[0], A[1], \ldots, A[i-1]$$

  are the i smallest elements in the entire array, in sorted order — and array elements have been reordered but are otherwise unchanged

Pseudocode

```
void Selection Sort(int[] A)
  for i from 0 to n − 2 do
    min = i
    for j from i + 1 to n − 1 do
      if A[j] < A[min] then
        min = j
      end if
    end for
    {Swap A[i] and A[min]}
    tmp = A[i]
    A[i] = A[min]
    A[min] = tmp
  end for
```


Example

A:

2	6	3	1	4

Idea: find smallest element in $A[i], \ldots, A[4]$ for each i from 0 to $n-1$

$i = 0$

-
-

A:

$i = 1$

-
-

A:

Example (cont.)

$i = 2$

-
-

A:

$i = 3$

-
-

A:

Finished! $A[0], \ldots, A[4]$ sorted

Inner Loop: Semantics

The inner loop is a **for** loop, which does the same thing as the following code (which includes a **while** loop):

$j = i + 1$
while $j < n$ **do**
  if $(A[j] < A[min])$ **then**
    $min = j$
  end if
  $j = j + 1$
end while

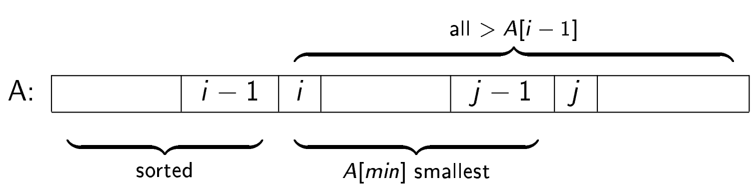
We will supply a "loop invariant" and "loop variant" for the above **while** loop in order to analyze the behaviour of the **for** loop we used to generate it

Inner Loop: Loop Invariant

Loop Invariant: At the beginning of each execution of the inner loop body

- $i, min \in \mathbb{N}$
- First subarray (with size i) is sorted with smallest elements:
  - $0 \leq i \leq n - 2$
  - $A[h] \leq A[h+1]$ for $0 \leq h \leq i - 2$
  - if $i > 0$ then $A[i-1] \leq A[h]$ for $i \leq h \leq n - 1$
- Searching for the next-smallest element:
  - $i + 1 \leq j < n$
  - $i \leq min < j$
  - $A[min] \leq A[h]$ for $i \leq h < j$
- Entries of A have been reordered; otherwise unchanged

Inner Loop: Interpretation of the Loop Invariant



Interpretation:

- $A[0] \leq A[1] \leq \cdots \leq A[i-1]$
- If $i > 0$ then $A[i-1] \leq A[\ell]$ for every integer ℓ such that $i \leq \ell \leq n$
- $i \leq min \leq j - 1$ and $A[min] \leq A[h]$ for every integer h such that $i \leq h \leq j - 1$
- entries of A have been reordered, otherwise unchanged

Application of the Loop Invariant

Loop invariant, final execution of the loop body, and *failure of the loop test* ensures that:

- $j = n$ immediately after the final execution of the inner loop body
- $i \leq min < n$ and $A[min] \leq A[\ell]$ **for all** ℓ such that $i \leq \ell < n$
- $A[min] \geq A[h]$ for all h such that $0 \leq h < i$

In other words, $A[min]$ is the value that should be moved into position $A[i]$

Inner Loop: Loop Variant and Application

Loop Variant: $f(n, i, j) = n - j$

- decreasing integer function
- when $f(n, i, j) = 0$ we have $j = n$ and the loop terminates

Application:

- initial value is $j = i + 1$
- worst-case number of iterations is $f(n, i, i + 1) = n - (i + 1) = n - 1 - i$

Outer Loop: Semantics

The outer loop is a **for** loop whose index variable i has values from 0 to $n - 2$, inclusive

This does the same thing as a sequence of statements including

- an initialization statement, $i = 0$
- a **while** loop with test "$i \leq n - 2$" whose body consists of the body of the **for** loop, together with a final statement $i = i + 1$

We will provide a loop invariant and a loop variant for this **while** loop in order to analyze the given **for** loop

Outer Loop: Loop Invariant and Loop Variant

Loop Invariant: At the beginning of each execution of the outer loop body

- i is an integer such that $0 \leq i < n - 1$
- $A[h] \leq A[h + 1]$ for $0 \leq h < i$
- if $i > 0$, $A[i - 1] \leq A[\ell]$ for $i \leq \ell < n$
- Entries of A have been reordered; otherwise unchanged

Thus: $A[0], \ldots, A[i - 1]$ are sorted and are the i smallest elements in A

Loop Variant: $f(n, i) = n - 1 - i$

- decreasing integer function
- when $f(n, i) = 0$ we have $i = n - 1$ and the loop terminates
- worst-case number of iterations is $f(n, 0) = n - 1$

Analysis of Selection Sort

Worst-case: $\Theta(n^2)$ steps

- inner loop iterates $n - 1 - i$ times (constant steps per iteration)
- outer loop iterates $n - 1$ times
- total number of steps is at most

$$c_0 + \sum_{i=0}^{n-2} c_1(n - 1 - i) = c_0 + c_1 \frac{n(n-1)}{2}$$

Conclusion: Worst-case running time is in $O(n^2)$.

Analysis of Selection Sort, Concluded

Best-Case: Also in $\Omega(n^2)$:

- Both loops are **for** loops and a *positive* number of steps is used on each execution of the inner loop body
- Total number of steps is therefore *at least*

$$\widehat{c}_0 + \sum_{i=0}^{n-2} \widehat{c}_1(n-1-i) \in \Omega(n^2)$$

Conclusion: *Every* application of this algorithm to sort an array of length n uses $\Theta(n^2)$ steps

Insertion Sort

Idea:

- Sort progressively larger subarrays
- $n-1$ stages, for $i = 1, 2, \dots, n-1$
- At the end of the i^{th} stage
  - Entries originally in locations

    $$A[0], A[1], \dots, A[i]$$

    have been reordered and are now sorted
  - Entries in locations

    $$A[i+1], A[i+2], \dots, A[n-1]$$

    have not yet been examined or moved

Pseudocode

```
void Insertion Sort(int [] A)
  for i from 1 to n − 1 do
    j = i
    while ((j > 0) and (A[j] < A[j − 1])) do
      {Swap A[j − 1] and A[j]}
      tmp = A[j]
      A[j] = A[j − 1]
      A[j − 1] = tmp
      j = j − 1
    end while
  end for
```


Example

A:

2	6	3	1	4

Idea: insert $A[i]$ in the correct position in $A[0], \dots, A[i-1]$

- initially, $i = 0$ and $A[0] = 2$ is sorted

<u>$i = 1$</u>

-
-

A:

<u>$i = 2$</u>

-
-

A:

Example (cont.)

$i = 3$

-
-

A:

$i = 4$

-
-

A:

Finished! $A[0], \ldots, A[4]$ sorted

Inner Loop: Loop Invariant

Loop Invariant: at the beginning of each execution of the inner loop body

- $i, j \in \mathbb{N}$
- $1 \leq i < n$ and $0 < j \leq i$
- $A[h] \leq A[h+1]$ for $0 \leq h < j-1$ and $j \leq h < i$
- if $j > 0$ and $j < i$ then $A[j-1] \leq A[j+1]$
- Entries of A have been reordered; otherwise unchanged

Inner Loop: Interpretation of Loop Invariant

A:

	$j-1$	j	$j+1$		i	

sorted (under the cells up to $j-1$); sorted (under the cells from $j+1$ to i)

Can be used to establish that the following holds at the *end* of each execution of the inner loop body:

- i and j are integers such that $0 \leq j \leq i-1 \leq n-2$
- $A[0], \ldots, A[j-1]$ are sorted
- $A[j], \ldots, A[i]$ are sorted (so that $A[0], \ldots, A[i]$ are sorted if $j = 0$)
- if $j > 0$ and $j < i$, then $A[j-1] \leq A[j+1]$, so that $A[0], \ldots, A[i]$ are sorted if $A[j-1] \leq A[j]$

It follows that $A[0], \ldots, A[i]$ are sorted when this loop terminates.

Inner Loop: Loop Variant and Application

Loop Variant: $f(n, i, j) = j$

- decreasing integer function
- when $f(n, i, j) = 0$ we have $j = 0$ and the loop terminates

Application:

- initial value is i
- worst-case number of iterations is i

Outer Loop: Semantics

Once again, the outer **for** loop can be rewritten as a **while** loop for analysis. Since the inner loop is already a **while** loop, the new outer **while** loop would be as follows.

$i = 1$
while $i \leq n - 1$ **do**
  $j = i$
  Inner loop of original program
  $i = i + 1$
end while

This program will be analyzed in order establish the correctness and efficiency of the original one.

Outer Loop

Loop Invariant: at the beginning of each execution of the outer loop body:

- $1 \leq i < n$
- $A[0], A[1], \ldots, A[i-1]$ are sorted
- Entries of A have been reordered; otherwise unchanged.

Thus, the loop invariant, final execution of the loop body, and failure of the loop test establish that

- $A[0], \ldots, A[i-1]$ are sorted,
- as $i = n$ when the loop terminates, A is sorted

Loop Variant: $f(n, i) = n - i$

- number of iterations is $f(n, 1) = n - 1$

Analysis of Insertion Sort

Worst-case: $\Theta(n^2)$ steps

- inner loop iterates i times (constant steps per iteration)
- outer loop iterates $n - 1$ times
- total number of steps is at most

$$c_0 + \sum_{i=1}^{n-1} c_1 i = c_0 + c_1 \frac{n(n-1)}{2}$$

Conclusion: Worst-case running time is in $O(n^2)$.

Analysis of Insertion Sort, Concluded

Worst-Case, Continued: For every integer $n \geq 1$ consider the operation on this algorithm on an input array A such that

- the length of A is n
- the entries of A are *distinct*
- A is sorted in **decreasing** order, instead of increasing order

It is possible to show that the algorithm uses $\Omega(n^2)$ steps on this input array.

Conclusion: The worst-case running time is in $\Theta(n^2)$.

Best-Case: $\Theta(n)$ steps are used in the best case.

- Proof: *Exercise.* Consider an array whose entries are already sorted as part of this.

Bubble Sort

Idea:

- Similar, in some ways, to "Selection Sort"
- Repeatedly sweep from right to left over the unsorted (rightmost) portion of the array, keeping the smallest element found and moving it to the left
- Result: After the i^{th} stage,

$$A[0], A[1], \ldots, A[i-1]$$

are the i smallest elements in the entire array, in sorted order

Pseudocode

void **Bubble Sort**(int [] A)
  for i from 0 **to** $n-1$ **do**
    for j **from** $n-2$ **down to** i **do**
      if $A[j] > A[j+1]$ **then**
        $\{$Swap $A[j]$ and $A[j+1]\}$
        $tmp = A[j]$
        $A[j] = A[j+1]$
        $A[j+1] = tmp$
      end if
    end for
  end for

Analysis of Inner Loop

Exercise!

- Rewrite the inner loop as an equivalent **while** loop (preceded by an initialization statement)
- Try to use your understanding of what the inner loop does to find a "loop invariant."
- This should include enough information so that it can be proved to hold (probably using mathematical induction) and so that it can be used to establish correctness of the outer loop.
- Try to find a "loop variant" for the inner loop as well.

Analysis of Outer Loop

Begin, as usual, by rewriting this loop as an equivalent **while** loop (preceded by an initialization statement)

- The loop invariant and loop variant given for the outer loop of the "Selection Sort" algorithm can be modified to include the fact that the *smallest* elements in the array are located in the part of the array that has currently been sorted.
- *Proving* this is different, since the details of the *inner* loops of these two algorithms are quite different.

The *application* of the loop invariant and loop variant to establish correctness are then much the same as for the "Selection Sort" algorithm.

Comparisons

All three algorithms have worst-case complexity $\Theta(n^2)$

- Selection sort only swaps $O(n)$ elements, even in the worst case. This is an advantage when exchanges are more expensive than comparisons.
- On the other hand, Insertion sort has the best "best case" complexity. It also performs well if the input as already partly sorted.
- Bubble sort is generally not used in practice.

Note: Asymptotically faster algorithms exist and will be presented next. These "asymptotically faster" algorithms are better choices when the input size is large and worst-case performance is critical.