Computer Science 331
Merge Sort

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Lecture #23

Outline

Introduction

Merge Sort is is an asymptotically faster algorithm than the sorting algorithms we have seen so far.

- It can be used to sort an array of size n using $\Theta(n \log_2 n)$ operations in the worst case.

Presented here: A version that takes an input array A and produces another sorted array B (containing the entries of A, rearranged)

A solution to the "Merging Problem" (presented next) is a subroutine that is used to do much of the work.

Reference: Textbook, Section 11.1

The "Merging" Problem

Calling Sequence: `void merge(int [] ` A_1`, int [] `A_2`, int [] `B`)`

Precondition:

- A_1 is a sorted array of length n_1 (positive integer) such that

$$A_1[h] \leq A_1[h+1] \qquad \text{for } 0 \leq h \leq n_1 - 2$$

- A_2 is a sorted array of length n_2 (positive integer) such that

$$A_2[h] \leq A_2[h+1] \qquad \text{for } 0 \leq h \leq n_2 - 2$$

- Entries of A_1 and A_2 are integers (more generally, objects from the same ordered class)

The "Merging" Problem (cont.)

Postcondition:

- B is a sorted array of length $n_1 + n_2$, so that

$$B[h] \leq B[h+1] \qquad \text{for } 0 \leq h \leq n_1 + n_2 - 2$$

- Entries of B are the entries of A_1 together with the entries of A_2, reordered but otherwise unchanged
- A_1 and A_2 have not been modified

Idea for an Algorithm

Maintain indices into each array (each initially pointing to the leftmost element)

repeat

- Compare the current elements of each array
- Append the smaller entry onto the "end" of B, advancing the index for the array from which this entry was taken

until one of the input arrays has been exhausted

Append the rest of the other input array onto the end of B

Pseudocode

`void` **merge**(`int []` A_1, `int []` A_2, `int []` B)

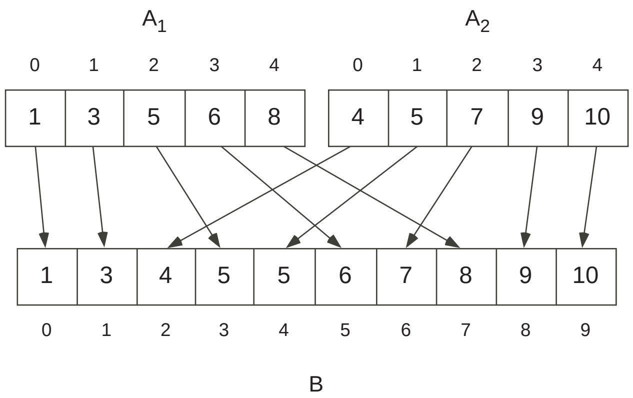
$n_1 = length(A_1)$; $n_2 = length(A_2)$
Declare B to be an array of length $n_1 + n_2$
$i_1 = 0$; $i_2 = 0$; $j = 0$
while $(i_1 < n_1)$ **and** $(i_2 < n_2)$ **do**
  if $A_1[i_1] \leq A_2[i_2]$ **then**
    $B[j] = A_1[i_1]$; $i_1 = i_1 + 1$
  else
    $B[j] = A_2[i_2]$; $i_2 = i_2 + 1$
  end if
  $j = j + 1$
end while

Pseudocode, Continued

{Copy remainder of A_1 (if any)}
while $i_1 < n_1$ **do**
  $B[j] = A_1[i_1]$; $i_1 = i_1 + 1$; $j = j + 1$
end while

{Otherwise copy remainder of A_2}
while $i_2 < n_2$ **do**
  $B[j] = A_2[i_2]$; $i_2 = i_2 + 1$; $j = j + 1$
end while

Example



Note: Running time is $\Theta(n_1 + n_2)$, where the input arrays have size n_1 and n_2

Merge Sort: Idea for an Algorithm

Suppose we:

1. Split an input array into two roughly equally-sized pieces.
2. *Recursively* sort each piece.
3. Merge the two sorted pieces.

This sorts the originally given array.

Note: this algorithm design strategy is known as *divide-and-conquer*:

- divide the original problem (sorting an array) into smaller subproblems (sorting smaller arrays)
- solve the smaller subproblems *recursively*
- combine the solutions to the smaller subproblems (the sorted subarrays) to obtain a solution to the original problem (merging the sorted arrays)

Pseudocode

`void` **mergeSort**(`int []` A, `int []` B)
  $n = A.length$
  if $n == 1$ **then**
    $B[0] = A[0]$
  else
    $n_1 = \lceil n/2 \rceil$
    $n_2 = n - n_1$ {so that $n_2 = \lfloor n/2 \rfloor$}
    Set A_1 to be $A[0], \ldots, A[n_1 - 1]$ {length n_1}
    Set A_2 to be $A[n_1], \ldots, A[n - 1]$ {length n_2}
    mergeSort(A_1, B_1)
    mergeSort(A_2, B_2)
    merge(B_1, B_2, B)
  end if

Example

	0	1	2	3	4	5	6	7
A :	7	3	9	6	5	2	1	8

1. Sort $A[0, \ldots, 3] = [7, 3, 9, 6]$ recursively:
   - Sort $A[0, 1] = [7, 3]$ recursively
     - Sort $A[0] = [7]$ recursively — base case
     - Sort $A[1] = [3]$ recursively — base case
     - Merge: result is $[3, 7]$
   - Sort $A[2, 3] = [9, 6]$ recursively. Result is $[6, 9]$
   - Merge: result is $[3, 6, 7, 9]$
2. Sort $A[4, \ldots, 7] = [5, 2, 1, 8]$ recursively. Result is $[1, 2, 5, 8]$
3. Merge: result is $[1, 2, 3, 5, 6, 7, 8, 9]$

Correctness of MergeSort

Theorem 1

If **mergeSort** *is run on an input array A of size* $n \geq 1$*, then the algorithm eventually halts, producing the desired sorted array as output.*

Prove by (strong) induction on n (assuming that **merge** is correct!):

Base Case: $n = 1$

- if $n = 1$, array consists of one element (array is sorted trivially)
- algorithm returns B containing a copy of the single element in the array (terminates with correct output)

Correctness, continued

Inductive hypothesis:

- assume the algorithm is correct for input arrays of size $k < n$

Prove that B is sorted under this assumption. Let A be an array of length $n \geq 2$:

- A_1 contains first n_1 elements of A sorted
- A_2 contains remaining n_2 elements of A
- $n_1 = \lceil n/2 \rceil < n$ and $n_2 = \lfloor n/2 \rfloor < n$, so inductive hypothesis implies that B_1 is A_1 sorted and B_2 is A_2 sorted
- **merge** computes B containing all elements of A sorted (assuming that **merge** is correct earlier)
- hence, algorithm is partially correct by induction. □

Termination and Efficiency

Let $T(n)$ be the number of steps used by this algorithm when given an input array of length n, in the worst case.

We can see the following by inspection of the code:

$$T(n) \leq \begin{cases} c_0 & \text{if } n = 1 \\ T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + c_1 n & \text{if } n \geq 2 \end{cases}$$

for some constants c_0 and c_1.

Special Case: If $n = 2^k$ is a power of two, we can rewrite this as

$$T(n) \leq \begin{cases} c_0 & \text{if } n = 1 \\ 2T(n/2) + c_1 n & \text{if } n \geq 2 \end{cases}$$

Termination and Efficiency

Theorem 2

If $n = 2^k$*, and* $c = \max(c_0, c_1)$*, then*

$$T(n) \leq cn \log_2(2n) = cn(k+1).$$

Prove by induction on k

- Base case ($k = 0$): for $k = 0$ we have $n = 2^0 = 1$, and

$$T(1) = c_0 \leq cn(k+1) = c$$

  because $c = \max(c_0, c_1)$.

Termination and Efficiency

Inductive hypothesis: Assume $k > 0$ and theorem holds for $k-1$:

Show that the theorem holds for k :

- By definition we have, for $n = 2^k$,

$$T(n) \leq 2T(n/2) + c_1 n$$

- by assumption $T(n/2) = T(2^{k-1}) \leq c(n/2)k$ and we obtain

$$\begin{aligned} T(n) &\leq 2(c(n/2)k) + c_1 n \\ &= cnk + c_1 n \\ &\leq cnk + cn \quad (c_1 \leq c = \max(c_0, c_1)) \\ &= cn(k+1) \end{aligned}$$

  as required.

Termination and Efficiency (General Case)

Consider the function $L(n) = \lceil \log_2 n \rceil$ for $n \geq 1$

Useful Property:

- $L(\lceil n/2 \rceil) = L(n) - 1$ and $L(\lfloor n/2 \rfloor) \leq L(n) - 1$ for every integer $n \geq 2$

Theorem 3

If $n \geq 1$ then $T(n) \leq cnL(2n) \leq cn(\log_2 n + 2)$.

Method of Proof: induction on n

Further Observations

It can be shown (by consideration of particular inputs) that the worst-case running time of this algorithm is also in $\Omega(n \log_2 n)$. It is therefore in $\Theta(n \log_2 n)$.

- This is preferable to the classical sorting algorithms, for sufficiently large inputs, if worst-case running time is critical.
- The classical algorithms are *faster* on sufficiently *small* inputs because they are simpler.

Alternative Approach: A "hybrid" algorithm:

- Use the recursive strategy given above when the input size is greater than or equal to some (carefully chosen) "threshold" value.
- Switch to a simpler, nonrecursive algorithm (that is faster on small inputs) as soon as the input size drops to below this "threshold" value.

Loop Invariant for Loop #1

At the beginning of each execution of the body of the first loop:

1. i_1, i_2 are integers such that $0 \leq i_1 < n_1$ and $0 \leq i_2 < n_2$
2.
   - $j = i_1 + i_2$;
   - $B[h] \leq B[h+1]$ for $0 \leq h \leq j-2$;
   - $B[0], B[1], \ldots, B[j-1]$ are the values

     $$A_1[0], A_1[1], \ldots, A_1[i_1 - 1] \quad \text{and} \quad A_2[0], A_2[1], \ldots, A_2[i_2 - 1],$$

     reordered but otherwise unchanged;
   - if $j \geq 1$ and $i_1 < n_1$ then $B[j-1] \leq A_1[i_1]$
   - if $j \geq 1$ and $i_2 < n_2$ then $B[j-1] \leq A_2[i_2]$
   - The arrays A_1 and A_2 have not been changed.

Analysis for Loop #1, Concluded

Application of Loop Invariant: At the end of every execution of the body of the first loop:

1. i_1, i_2 are integers such that $0 \leq i_1 \leq n_1$ and $0 \leq i_2 \leq n_2$
2. Condition 2 of the loop invariant is satisfied

Failure of the loop test ensures that these hold and either $i_1 = n_1$ or $i_2 = n_2$.

Loop Variant for Loop #1: $f(n_1, n_2, j) = n_1 + n_2 - j$

- loop invariant implies that $j = i_1 + i_2$, so that

$$f(n_1, n_2, j) = (n_1 - i_1) + (n_2 - i_2)$$

- Initial value is $n_1 + n_2$

Analysis for Loop #2

Loop Invariant for Loop #2: At the beginning of each execution of the body of the *second* loop

1. $i_2 = n_2$ and i_1 is an integer such that $0 \leq i_1 < n_1$
2. Part 2 of the loop invariant for the first loop is satisfied.

At the *end* of each execution of the body of this loop

1. $i_2 = n_2$ and i_1 is an integer such that $0 \leq i_1 \leq n_1$
2. Part 2 of the loop invariant for the first loop is satisfied.

Analysis for Loop #2, Continued

Failure of the loop test implies that, on termination of the *second* loop,

1. $i_1 = n_1$ and i_2 is an integer such that $0 \leq i_2 < n_2$
2. Part 2 of the loop invariant for the first loop is satisfied.

Note: we must consider the possibility that Loop #2 was skipped when considering the value of i_2 at this point in the code.

Loop Variant for Loop #2: Same as for Loop #1. Note that the value of this function has not been changed between the end of loop #1 and the beginning of loop #2.

Analysis for Loop #3

Loop Invariant for Loop #3: At the beginning of each execution of the body of the *third* loop

1. $i_1 = n_1$ and i_2 is an integer such that $0 \leq i_2 < n_2$
2. Part 2 of the loop invariant for the first loop is satisfied.

At the *end* of each execution of the body of this loop

1. $i_1 = n_1$ and i_2 is an integer such that $0 \leq i_2 \leq n_2$
2. Part 2 of the loop invariant for the first loop is satisfied.

Analysis for Loop #3, Continued

Failure of the loop test implies that, on termination of the *third* loop,

1. $i_1 = n_1$ and $i_2 = n_2$
2. Part 2 of the loop invariant for the first loop is satisfied.

These properties establish the postcondition of the merging problem.

Loop Variant for Loop #3: Same as for Loop #1. Note that the value of this function has not been changed between the end of loop #2 and the beginning of loop #3.

Analysis of the Merging Algorithm Concluded

Correctness:

- loop invariants can be used to prove partial correctness
- loop variant implies that the for loops (and hence the entire algorithm) terminate
- therefore, **merge** is correct

Efficiency:

- each loop body requires a constant number of steps
- total number of iterations is $n_1 + n_2$
- therefore, **merge** is $\Theta(n_1 + n_2)$