Categories of Kirchhoff Relations

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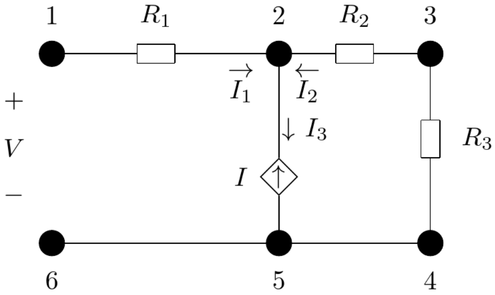
Introduction

- The goal of our work is to build a bridge between electrical circuits and quantum circuits.
- Comfort and Kissinger showed that $\mathsf{AffLagRel}_F$ is isomorphic to qudit stabilizer circuits.
- To make the connection to electrical circuits transparent, one defines subcategories of Lagrangian relations that satisfy Kirchhoff current law:

$$\mathsf{ResRel}_F \longhookrightarrow \mathsf{KirRel}_F \longhookrightarrow \mathsf{AffLagRel}_F = \begin{array}{c}\text{Stabilizer}\\ \text{QM}\\ \text{(odd-prime)}\end{array}$$

Basics of Electrical Circuits

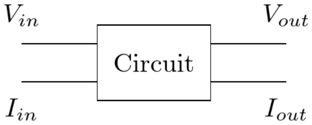


- Kirchhoff's current law (KCL) is just a restatement of conservation of charge, for example at node 2, KCL will give:

$$I_3 = I_1 + I_2$$

- Kirchhoff's voltage law (KVL) states that the voltage drop in a loop is zero.

Circuits as relations



- A circuit can be viewed as a relation between input currents and voltages and output currents and voltages.
- One can write a matrix to capture this relationship as follows:

$$Hx = 0$$

where H is a matrix which characterizes the circuit.

Categories of Interest

$$\begin{tikzcd}
\textbf{LosLagRel} \arrow[r, hook] \arrow[d, hook] & \textbf{LagRel} \arrow[r, hook] \arrow[d, hook] & \textbf{AffLagRel} \arrow[d, hook] \\
\begin{array}{c}\textbf{LosKirRel}\\ \text{(Spiders}\\ +\\ \text{Current Dividers)}\end{array} \arrow[r, hook] \arrow[d, hook] & \begin{array}{c}\textbf{KirRel}\\ \text{(LosKirRel + Resistors)}\end{array} \arrow[r, hook] \arrow[d, hook] & \begin{array}{c}\textbf{AffKirRel}\\ \text{(KirRel+Sources)}\end{array} \arrow[d, hook] \\
\begin{array}{c}\textbf{Spiders}\\ \text{(Junctions)}\end{array} \arrow[r, hook] & \begin{array}{c}\textbf{ResRel}\\ \text{(Resistor circuits)}\end{array} \arrow[r, hook] & \begin{array}{c}\textbf{AffResRel}\\ \text{(Resistors}\\ +\\ \text{Sources)}\end{array}
\end{tikzcd}$$

Category of Linear Relations

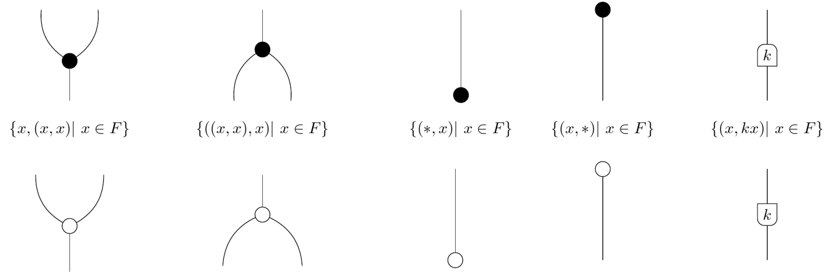
The category of linear relations LinRel_F over a field F is defined as:

- Objects: Natural Numbers m, n
- Maps: $m \to n$ linear relations $R \subseteq F^m \times F^n$
- Tensor: Direct Sum
- Composition: Relational Composition

Think of linear relations as linear subspaces!

Category of Linear Relations

- The category LinRel_F has a graphical calculus whose generators are:

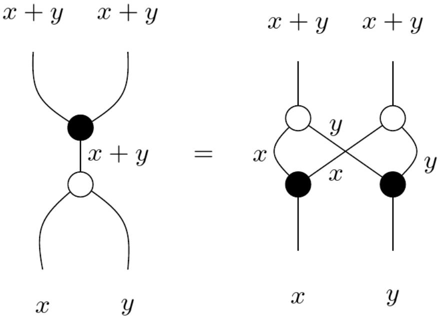


$\{x,(x,x)|\ x\in F\}$ $\{((x,x),x)|\ x\in F\}$ $\{(*,x)|\ x\in F\}$ $\{(x,*)|\ x\in F\}$ $\{(x,kx)|\ x\in F\}$

$\{(x,(x-y,y))|\ x,y\in F\}$ $\{((x,y),x+y|\ x,y\in F)\}$ $\{(*,0)\}$ $\{(0,*)\}$ $\{(x,k^{-1}x)|\ x\in F\}$

- These generators obey the laws of interacting Hopf algebras.

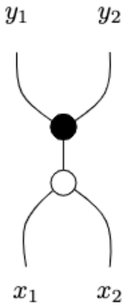
Example: Bialgebra law



Example Matrix

This diagram encodes the following relation:

$$\{(x_1, x_2), (y_1, y_2) \mid x_1 + x_2 = y_1 \ , \ y_1 = y_2\}$$



This relation can alternatively be written as follows:

$$\begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix} = 0$$

Lagrangian Relations

- A **symplectic form** is a bilinear, alternating and non-degenerate map $\langle \_, \_ \rangle : V \otimes V \to F$. A vector space with a symplectic form are called **symplectic vector space**.
- A symplectic vector space always has a **Darboux basis** that this is a basis $q_1, ..., q_n,\ p_1, ..., p_n$ for which $\langle q_i, q_j \rangle = \langle p_i, p_j \rangle = \delta_{ij}$ and $\langle q_i, p_j \rangle = \langle p_i, q_j \rangle = -\delta_{ij}$.
- In this basis, the symplectic form can be expressed using the $2n \times 2n$ block matrix J:

$$\langle x, y \rangle := \begin{pmatrix} q^T & p^T \end{pmatrix} J \begin{pmatrix} q \\ p \end{pmatrix} \quad \text{where} \quad J = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$$

- The **symplectic dual** of a linear subspace $U \subseteq V$ of a symplectic vector space V is the linear subspace $U' := \{u' \in V \mid \forall u \in U, \langle u', u \rangle = 0\} \subseteq V$.
- A **Lagrangian subspace** U of a symplectic vector space V is linear subspace which is its own symplectic dual, so that $U = U'$.

The Category of Lagrangian Relations

The category of Lagrangian category over a field F consists of:

- Objects $n \in \mathbb{N}$: correspond to the graded vector spaces $F^n \oplus F^n$ equipped with the cannonical symplectic form given by J.
- Maps $\mathcal{R} : n \to m$: are relations $\mathcal{R} \subseteq F^{n+m} \oplus F^{n+m}$ where the state is Lagrangian.
- Tensor: Given by addition on objects and the direct sum on maps.

The composition of Lagrangian relations is relational composition.

The Category of Kirchhoff Relations

A Lagrangian relation $\mathcal{R} : n \to m$ in LagRel$_F$ satisfies:

1. **Kirchhoff's Current Law** if, for all $((q, p), (q', p')) \in \mathcal{R}$, the following equality holds:

$$\sum_{j=1}^{n} p_j = \sum_{k=1}^{m} p'_k$$

2. **Translation invariance** if, whenever $\lambda \in F$ and $((q, p), (q', p')) \in \mathcal{R}$, then $((q + \vec{\lambda}_m, p), (q' + \vec{\lambda}_m, p'_n)) \in \mathcal{R}$, where $\vec{\lambda}_n$ is a vector of dimension n all of whose components are the same $\lambda \in F$.

Lemma

Kirchoff's current law implies translational invariance for LagRel$_F$

Parity-Check Matrices

Parity-check matrices H can be used to capture the constraints of a relation:

$$Hx = 0$$

Theorem

The parity-check matrix for Lagrangian relations has the following standard form:

$$H = \begin{pmatrix} Y & 0 & 1_{n_p} & A^T \\ -A & 1_{n_q} & 0 & 0 \end{pmatrix} \sigma_S \qquad (1)$$

where $n_p + n_q = n$ (the dimension of $\mathcal{R}$), σ_S is a symplectic permutation, A has dimensions $n_q \times n_p$, and $Y = Y^T$.

Corollary

The parity-check matrix for Kirchhoff relations has the same form with the following additional constraints:

$$Y\vec{1} = 0, \quad A\vec{1} = \vec{1}.$$

Here $\vec{1}$ is a column vector of all 1's. One $A\vec{1} = \vec{1}$ as A being **quasi-stochastic**.

Subcategories

- Lagrangian relations satisfying the Kirchhoff current law form a subcategory, $\mathsf{KirRel}_F \subseteq \mathsf{LagRel}_F$.
- Kirchhoff relations include resistor circuits and they also allow additional new components: namely **ideal current dividers**. This comes from the quasi-stochastic condition on A.
- A Kirchhoff relation is said to be **deterministic** if A is deterministic, in the sense that each row of A has only one non-zero entry. This corresponds to the subprop ResRel_F (resistor circuits) which was studied by Baez and Fong.
- Using the concept of **power-input** one can further isolate those subcategories that correspond to zero power, these are the lossless categories mentioned earlier.

Universality

Theorem

All maps in Kirchhoff relations KirRel_F *are generated by the current divider, resistors and junctions.*

Theorem

All maps in Affine Kirchhoff relations $\mathsf{AffKirRel}_F$ *are generated by the current divider, resistors, junctions and sources.*

Interrelation of Categories

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\end{tikzcd}$$

The Road ahead...

- Connections to qudit error correction...
- ZX calculus like graphical language for electrical circuits.
- Importing techniques of electrical network theory to quantum circuits.

References

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