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MONOIDAL WIDTH

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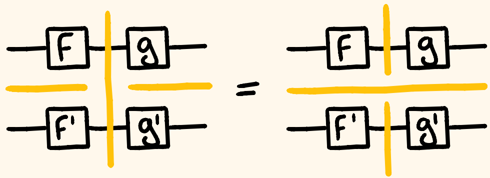
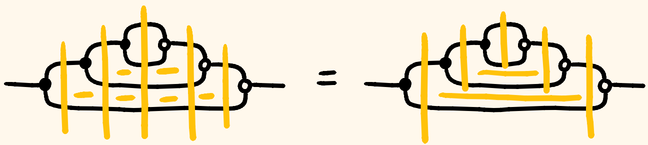
MOTIVATION (1)

- how efficient is to compute the semantics of morphisms in monoidal categories?





- we need an 'algebra of decompositions'

MOTIVATION (2)

- existing notions of complexity for graphs are based on decompositions : path width, tree width, branch width and rank width

- make explicit the algebra of decomposition that is hidden behind the definitions of these graph widths

MAIN RESULTS

- monoidal width as a measure of complexity for morphisms in monoidal categories
- monoidal decomposition as explicit decomposition algebra
- capture some known measures of complexity for graphs: path width, tree width, branch width and rank width

OUTLINE

- monoidal decompositions
- monoidal width for rank width

DECOMPOSITION SYSTEM

A decomposition system $(\mathcal{A}, \mathcal{O}, w)$ in a monoidal category $\mathcal{C}$ is given by

- $\mathcal{A}$: set of 'atomic' morphisms in $\mathcal{C}$
- $\mathcal{O} = \{\otimes, ;_X \text{ for } X \in obj(\mathcal{C})\}$: set of operations
- $w : \mathcal{A} \cup \mathcal{O} \to \mathbb{N}$: weight function
  such that:

$$\begin{cases} w(\otimes) = 0 \\ w(;_{X \otimes Y}) = w(;_X) + w(;_Y) \end{cases}$$

DECOMPOSITION SYSTEM - EXAMPLE

A decomposition system $(\mathcal{A}, \mathcal{O}, w)$ in $\mathcal{C}$ $\rightsquigarrow$ $\mathcal{F}in\mathcal{S}et$

- $\mathcal{A}$: set of 'atoms' $\rightsquigarrow \{ \supset\!-, \bullet\!-, \times, - \}$

- $\mathcal{O} = \{ \otimes, ;_X \text{ for } X \in obj(\mathcal{C}) \}$: set of operations

- $w : \mathcal{A} \cup \mathcal{O} \longrightarrow \mathbb{N}$: weight such that:

$$\begin{cases} w(\otimes) = 0 \\ w(;_{X \otimes Y}) = w(;_X) + w(;_Y) \end{cases}$$

$\rightsquigarrow$ $w(\supset\!-) = w(\times) = 2$
$w(\bullet\!-) = w(-) = 1$
$w(;_n) = n$

MONOIDAL DECOMPOSITION

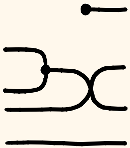
$f : X \to Y$ morphism in $\mathcal{C}$

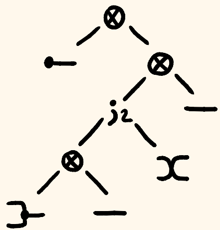
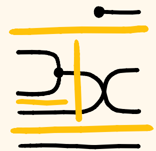
a monoidal decomposition $d \in \mathcal{D}_f$ of f is

$$
\begin{aligned}
d ::= \; & (f) && \text{if } f \in \mathcal{A} \\
\mid \; & d_1 \overset{;_c}{\wedge} d_2 && \text{if } f = f_1 ;_c f_2 \,,\; d_1 \in \mathcal{D}_{f_1} ,\; d_2 \in \mathcal{D}_{f_2} \\
\mid \; & d_1 \overset{\otimes}{\wedge} d_2 && \text{if } f = f_1 \otimes f_2 \,,\; d_1 \in \mathcal{D}_{f_1} ,\; d_2 \in \mathcal{D}_{f_2}
\end{aligned}
$$

$\rightsquigarrow$ it's a labelled binary tree

MONOIDAL DECOMPOSITION - EXAMPLE

 $: 4 \longrightarrow 4$ morphism in FinSet

$d =$  $\rightsquigarrow$ 

MONOIDAL WIDTH

$d \in \mathcal{D}_f$ monoidal decomposition of f

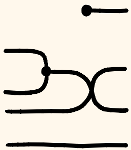
WIDTH OF d

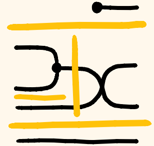
$$
\begin{aligned}
\mathrm{wd}(d) := \;& w(f) && \text{if } d = (f) \\
\mid\; & \max\{\mathrm{wd}(d_1),\, w(;_c),\, \mathrm{wd}(d_2)\} && \text{if } d = {}_{d_1}\diagup^{;_c}\diagdown_{d_2} \\
\mid\; & \max\{\mathrm{wd}(d_1),\, \mathrm{wd}(d_2)\} && \text{if } d = {}_{d_1}\diagup^{\otimes}\diagdown_{d_2}
\end{aligned}
$$

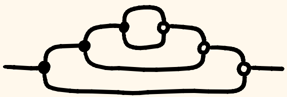
MONOIDAL WIDTH OF f

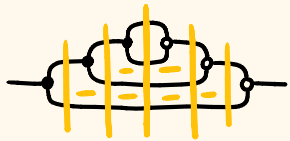
$$\mathrm{mwd}(f) := \min_{d \in \mathcal{D}_f} \mathrm{wd}(d)$$

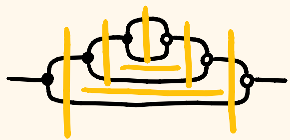
MONOIDAL WIDTH - EXAMPLE

$f =$ 

$\mathrm{wd}\Big($  $\Big) = 2$

$4 =$ 

$\mathrm{wd}\Big($  $\Big) = 4$

$\mathrm{wd}\Big($  $\Big) = 2$

OUTLINE

- monoidal decompositions

- monoidal width for rank width

RANK WIDTH [Oum & Seymour, 2006]

$G = (V, E, \mathrm{ends} : E \to \mathcal{P}_{\leq 2}(V))$ undirected graph

RANK DECOMPOSITION

(Y, r) where

- Y is a subcubic tree (= any node has at most 3 neighbours)
- $r : \mathrm{leaves}\ Y \xrightarrow{\simeq} V$ labelling bijection

WIDTH OF (Y, r)

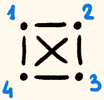
$$\mathrm{wd}(Y, r) := \max_{e \in \mathrm{edges}\, Y} \mathrm{rank}(X_e)$$

X_e adjacency matrix of the cut given by e through r

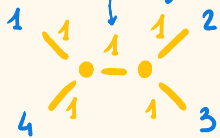
RANK WIDTH

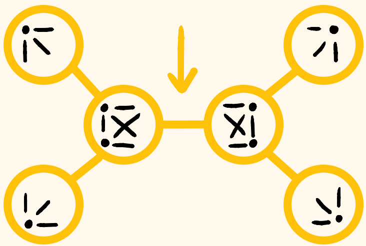
$$\mathrm{rwd}(G) := \min_{(Y, r)} \mathrm{wd}(Y, r)$$

RANK WIDTH - EXAMPLE

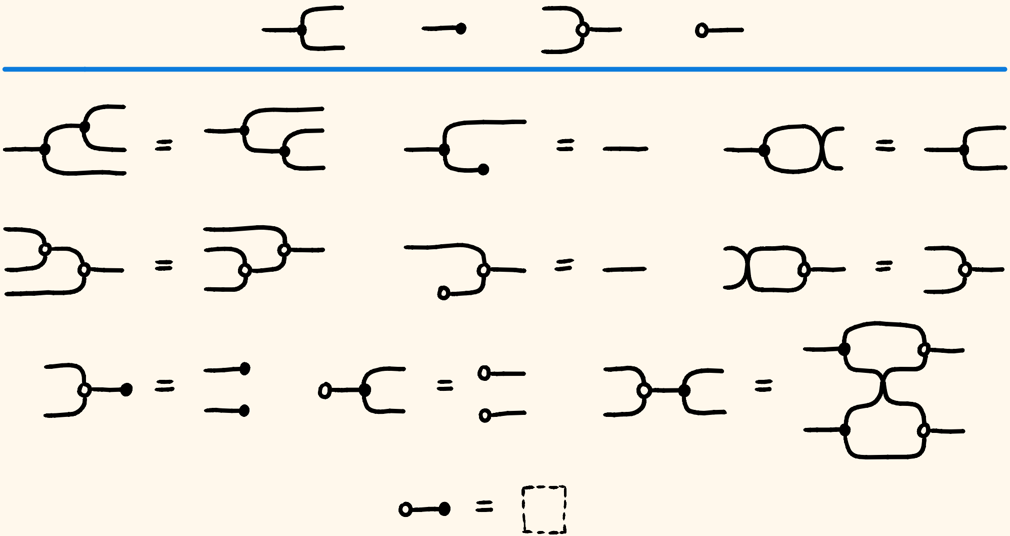
G = 

$$\begin{array}{c|cc} & 2 & 3 \\ \hline 1 & 1 & 1 \\ 4 & 1 & 1 \end{array} \qquad \begin{array}{c|ccc} & 1 & 3 & 4 \\ \hline 2 & 1 & 1 & 1 \end{array}$$



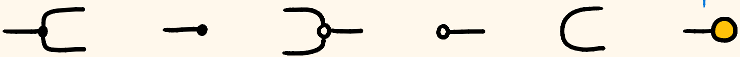


PROP OF MATRICES

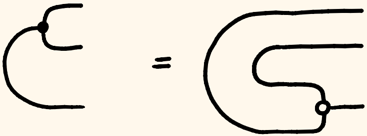


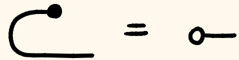
A PROP OF GRAPHS

vertex generator

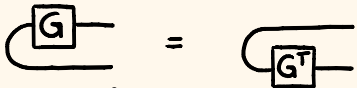


bialgebra equations +



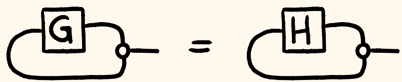


⇝ the cup transposes

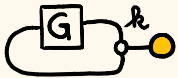


and captures equivalence of adjacency matrices

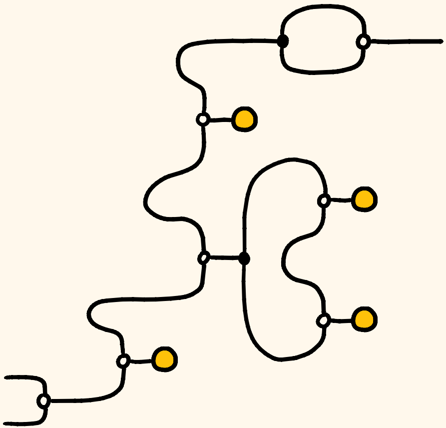
$[G] = [H] \iff$



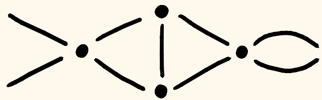
A PROP OF GRAPHS – EXAMPLE



⇝ graph on k vertices given by the adjacency matrix $[G]$

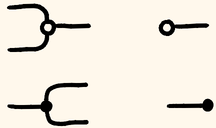


⇝



DECOMPOSITIONS IN THE PROP OF GRAPHS

Bialgebra structure



+ 'vertex' generator

ATOMS

$$\mathcal{A} = \{\text{all morphisms}\}$$

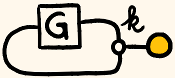
WEIGHT FUNCTION

$$w(g) := |\text{vertices } g|$$

$$w(j_n) := n$$

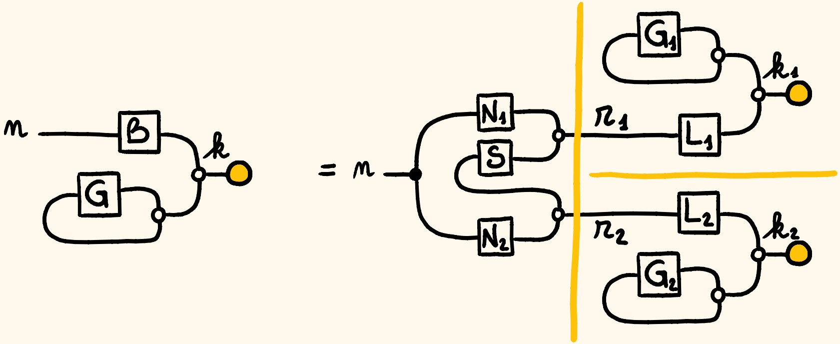
RANK WIDTH & MONOIDAL WIDTH

[G] undirected graph

$g =$  $: 0 \to 0$ in Grph

THEOREM

$$\frac{1}{2}\, \mathrm{rwd}(G) \leq \mathrm{mwd}(g) \leq 2\, \mathrm{rwd}(G)$$



SUMMARY OF RESULTS

MATRICES

$$\max_i \operatorname{rank} A_i \leq \operatorname{mwd} A \leq \max_i \operatorname{rank} A_i + 1$$

COSPANS OF GRAPHS

$$\operatorname{pwd}(G) = \operatorname{mpwd}(g)$$

$$\operatorname{twd}(G) \leq \operatorname{mtwd}(g) \leq 2 \cdot \operatorname{twd}(G)$$

$$\frac{1}{2} \operatorname{bwd}(G) \leq \operatorname{mwd}(g) \leq \operatorname{bwd}(G) + 1$$

PROP OF GRAPHS

$$\frac{1}{2} \operatorname{rwd}(G) \leq \operatorname{mwd}(g) \leq 2 \operatorname{rwd}(G)$$

FUTURE WORK

- obtain a result similar to Courcelle's theorem
- capture other widths (clique width, twin width, ... tree width for directed graphs and relational structures)

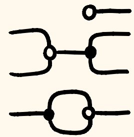
SOME REFERENCES

- Robertson & Seymour, Graph minors I-X, 1983-1991
- Oum & Seymour, Approximating clique width and branch width, 2006
- Arnborg, Courcelle, Proskurowski & Seese, An algebraic theory of graph reduction, 1993
- Courcelle & Olariu, Upper bounds to the clique width of graphs, 2000
- Berwanger, Dawar, Hunter, Kreutzer, Obdržálek, The DAG width of directed graphs, 2012
- Abramsky, Dawar & Pengming, The pebbling comonad in finite model theory, 2017
- Rosebrugh, Sabadini & Walters, Generic commutative separable algebras and cospans of graphs, 2005
- Chantawibul & Sobociński, Towards compositional graph theory, 2015
- Bonchi, Piedeleu, Sobociński & Zanasi, Graphical affine algebra, 2019
- Di Lavore, Hedges & Sobociński, Compositional modelling of network games, 2021

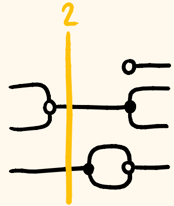
THIS WORK

- Di Lavore & Sobociński, Monoidal width: unifying tree width, path width and branch width, 2022
- Di Lavore & Sobociński, Monoidal width: capturing rank width, 2022

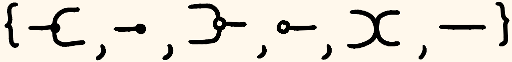
PROP OF MATRICES - EXAMPLE

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} =$$ 

FACT : the minimal vertical cut in a matrix is its rank : $\min\{k \in \mathbb{N} \mid A = B ;_k C\} = \operatorname{rank} A$

$\operatorname{rank} A = 2$ $\rightsquigarrow$ 

MONOIDAL WIDTH OF MATRICES

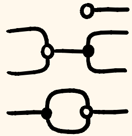
$\mathcal{A} =$ 

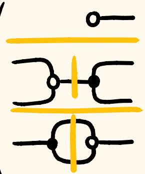
$$A = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & A_b \end{pmatrix} = A_1 \oplus A_2 \oplus \cdots \oplus A_b$$

THEOREM

$$\max_i \operatorname{rank} A_i \leq \operatorname{mwd} A \leq \max_i \operatorname{rank} A_i + 1$$

MONOIDAL WIDTH OF MATRICES – EXAMPLE

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} =$$ 

$$\mathrm{wd}\Bigg($$  $$\Bigg) = 2$$

$$= \max\left\{ \underset{\shortparallel \atop 0}{\mathrm{rank}(\,i\,)},\ \underset{\shortparallel \atop 1}{\mathrm{rank}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}},\ \underset{\shortparallel \atop 1}{\mathrm{rank}(2)} \right\} + 1$$

BRANCH WIDTH [Robertson & Seymour, 1991]

$G = (V, E, \text{ends} : E \to \mathcal{P}_{\leq 2}(V))$ undirected graph

BRANCH DECOMPOSITION

(Y, ℓ) where

- Y is a subcubic tree (= any node has at most 3 neighbours)
- ℓ : leaves $Y \xrightarrow{\simeq} E$ labelling bijection

WIDTH OF (Y, ℓ)

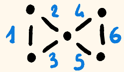
$$wd(Y, \ell) := \max_{e \in \text{edges}\, Y} |\text{ends}\, A_e \cap \text{ends}\, B_e|$$

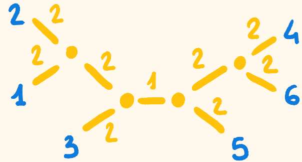
$\{A_e, B_e\}$ partition of E given by e through ℓ

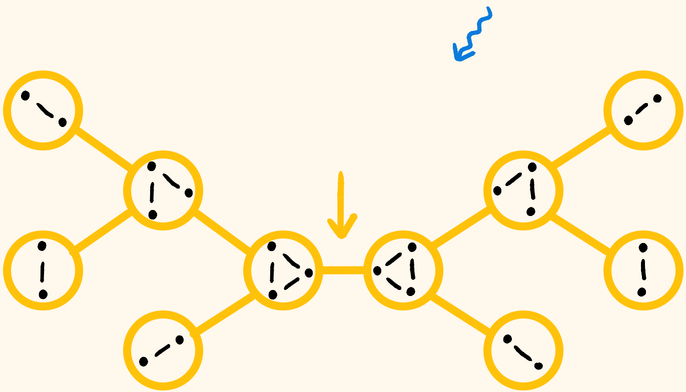
BRANCH WIDTH

$$bwd(G) := \min_{(Y, \ell)} wd(Y, \ell)$$

BRANCH WIDTH – EXAMPLE

$G =$ 

$(Y, \ell) =$ 



COSPANS OF GRAPHS

$\mathrm{Cospan}(\mathrm{UGraph})_\emptyset$

objects : sets $\rightsquigarrow$ discrete graphs

morphisms $X \to Y$: cospans
$$X \xrightarrow{\partial_X} G \xleftarrow{\partial_Y} Y$$
of graphs

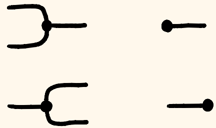
composition : by pushout $\rightsquigarrow$ glue along vertices

monoidal product : component-wise disjoint union

$\rightsquigarrow$ graphs with left and right sources

DECOMPOSITIONS IN COSPANS OF GRAPHS

Frobenius structure



+ 'edge' generator

ATOMS

$$\mathcal{A} = \{\text{all morphisms}\}$$

WEIGHT FUNCTION

$$w\left(X \rightarrow (V,E) \leftarrow Y \right) := |V|$$

$$w(j_X) := |X|$$

BRANCH WIDTH & MONOIDAL WIDTH

$G = (V, E)$ undirected graph

$g = \emptyset \xrightarrow{i} G \xleftarrow{i} \emptyset : \emptyset \to \emptyset$ in $\mathsf{Cospan}(\mathsf{UGraph})_\emptyset$

THEOREM

$$\frac{1}{2}\, \mathrm{bwd}(G) \leq \mathrm{mwd}(g) \leq \mathrm{bwd}(G) + 1$$