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Operadic Tangent Categories

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What kinds of geometries can be encoded with tangent categories?

- Differential Geometry
- Synthetic Diff. Geometry
- Algebraic Geometry
- Nc Geometry?
- Any Other Geometry?

STRUCTURE

- Algebraic Geom. Construction
- Operads
- Algebras over Operads
- Monadic Interpretation
- Operadic Tangent Cats.
- Module over Alg. over op.
- Conclusions

ALGEBRAIC GEOMETRY CONSTRUCTION

$$T : cAlg^{op} \longrightarrow cAlg^{op}$$

$$A \longmapsto \underbrace{Sym_A}_{\text{Symmetric algebra}} \Omega^1_{Com} A$$

$cAlg_R$

Kähler Differential

Symmetric algebra

Kähler Differential

$$\mathrm{Der}(A,-): \mathrm{Mod}_A \longrightarrow \mathrm{Mod}_R$$
$$M \longmapsto \mathrm{Der}(A,M)$$

$\delta: A \to M$ derivation iff:

1. δ linear
2. $\delta(ab) = \underbrace{\delta(a)b}_{\uparrow\, x\cdot a := ax} + a\delta(b)$

$$\mathrm{Der}(A,M) \cong \mathrm{Mod}_A(\Omega^1_{\mathrm{com}}A, M)$$

Symmetric Algebra

$$\begin{tikzcd} \mathrm{Mod}_A \arrow[r, shift left] \arrow[r, phantom, "\perp"] & {}_{A/}\mathrm{cAlg} \arrow[l, shift left] \end{tikzcd} \qquad \mathrm{Sym}_A : \mathrm{Mod}_A \leftarrow {}_{A/}\mathrm{cAlg} : U_A$$

$$\left\{ \begin{array}{l} U_A(f: A \to B) \\ B \text{ with} \\ a \cdot x := f(a) x \end{array} \right.$$

Where does this construction come from?

Theorem

Let $(\mathbb{X}, T)$ tangent category.

If $S_n \dashv T_n \Rightarrow (\mathbb{X}^{op}, S)$ tangent category.

ALGEBRAIC GEOMETRY CONSTRUCTION

$$\mathrm{Sym}_A \Omega^1_{\mathrm{Com}} A \xrightarrow{f} B \qquad \frac{A}{\mathrm{cAlg}}$$

$$\Omega^1_{\mathrm{Com}} A \longrightarrow B \qquad \mathrm{Mod}_A$$

$$A \longrightarrow B \qquad \mathrm{Der}(A,B)$$

$$\frac{A}{\mathrm{cAlg}}$$

$$g : A \hookrightarrow IA \xrightarrow{f} B$$

ALGEBRAIC GEOMETRY CONSTRUCTION

$A \ltimes - : \mathrm{Mod}_A \to \mathrm{cAlg}$

$M \longmapsto A \ltimes M$

$A \oplus M$

$(a, m) \cdot (a', m') := (aa', am' + ma')$

$\delta : A \to M$ linear.

δ derivation $\iff$

$\tilde{\delta} : A \to A \ltimes M$

$a \mapsto (a, \delta(a))$

cAlg-morphism

ALGEBRAIC GEOMETRY CONSTRUCTION

$$\mathrm{Sym}_A \Omega^1_{Com} A \xrightarrow{f} B \qquad \frac{A}{\mathrm{CAlg}}$$

$$\Omega^1_{Com} A \longrightarrow B \qquad \mathrm{Mod}_A$$

$$A \longrightarrow B \qquad \mathrm{Der}(A,B)$$

$$A \longrightarrow A \ltimes B \longrightarrow B \ltimes B \qquad \frac{A}{\mathrm{CAlg}}$$

$$g : A \hookrightarrow TA \xrightarrow{f} B$$

$$\perp B := B \ltimes B$$

$$T \dashv \perp$$

ALGEBRAIC GEOMETRY CONSTRUCTION

Theorem. $(\mathsf{cAlg}, \mathcal{I})$ tangent category

Corollary. $(\mathsf{cAlg}^{op}, \mathcal{T})$ tangent category

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OPERADS

Def'n. (Symmetric Operads)

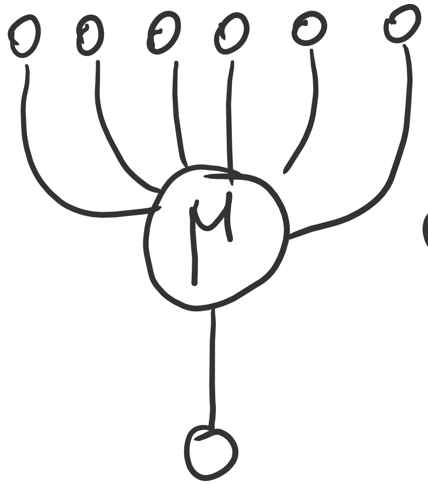
- $\{\mathcal{P}(n)\}$ sequence of vector spaces
- For $m, n, i \in \mathbb{N} \mid 1 \leq i \leq m$

$$\circ_i : \mathcal{P}(m) \otimes \mathcal{P}(n) \longrightarrow \mathcal{P}(m+n-1)$$

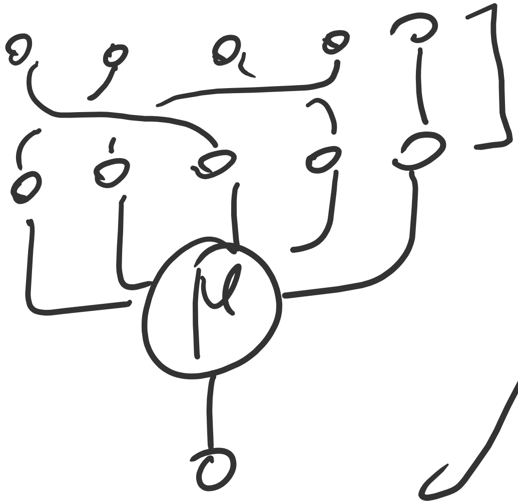
- $id_{\mathcal{P}} \in \mathcal{P}(1)$
- $\mathcal{P}(n) \otimes \mathbb{K}[S_n] \longrightarrow \mathcal{P}(n)$

It looks like a monad

OPERADS



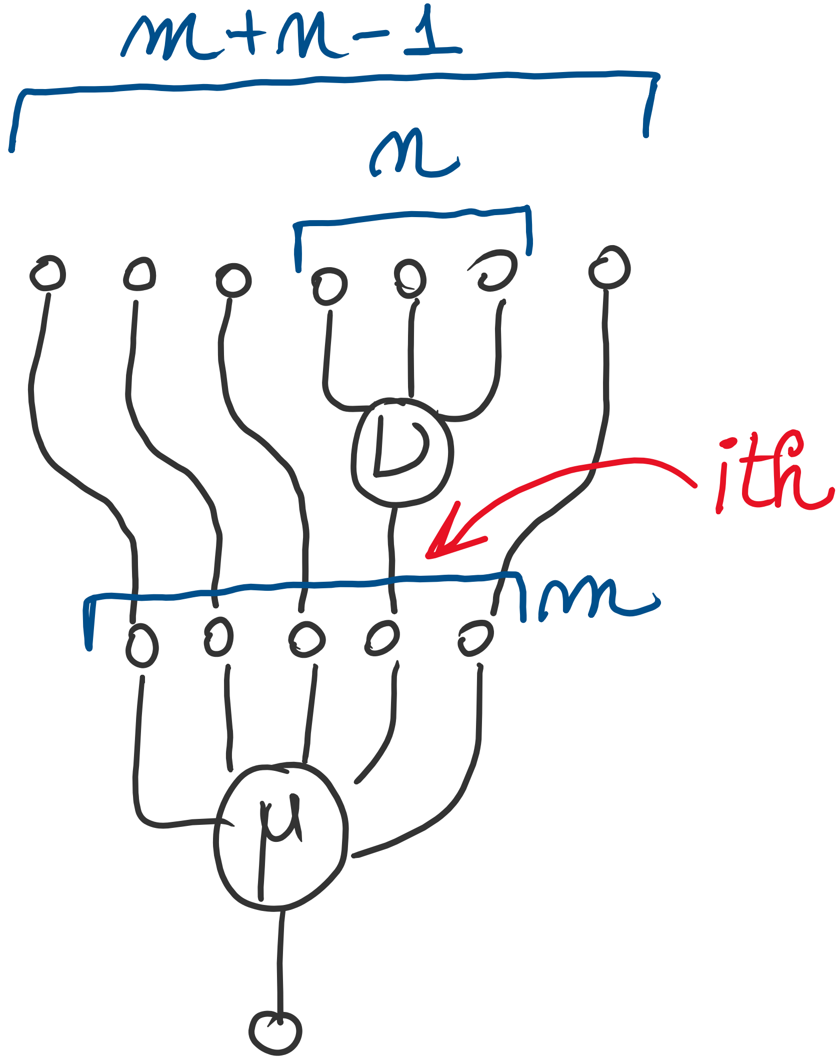
$\in \mathcal{P}(m)$



$\tau \in S_m$

$= \mu \cdot \tau$

$\mu \circ_i \nu =$



ALGEBRA OVER OPERADS

<u>Def'n</u> (Algebras over Operads).

$\mathcal{P}$ operad.

- A vector space
- For $n \in \mathbb{N}$

$$\theta_n : \mathcal{P}(n) \otimes A^{\otimes n} \longrightarrow A$$

This looks like an algebra over a monad $(A, SA \to A)$

ALGEBRA OVER OPERADS

$\mu \in \mathcal{P}(n)$, $a_1, \ldots, a_n \in A$

$\mu(a_1, \ldots, a_n) := \theta_n(\mu \otimes a_1 \otimes \ldots \otimes a_n)$

Examples:

$\mathcal{C}om \to$ Commutative alg.

$\mathcal{A}ss \to$ Associative alg.

$\mathcal{L}ie \to$ Lie alg.

$\mathcal{P}ois \to$ Poisson alg.

$\mathcal{L}eib \to$ Leibniz alg.

$\mathcal{Z}in \to$ Zinbiel alg.

MONADIC INTERPRETATION

$\mathcal{P}$ operad

Shur functor

$$S_{\mathcal{P}} : \mathrm{Vect} \longrightarrow \mathrm{Vect}$$
$$V \longmapsto S_{\mathcal{P}} V := \bigoplus_{n \geq 0} \mathcal{P}(n) \otimes_{\mathbb{S}_n} V^{\otimes n}$$

Theorem.
$\exists\, \gamma : S_{\mathcal{P}} \circ S_{\mathcal{P}} \Rightarrow S_{\mathcal{P}}$ and $\eta : \mathrm{id}_{\mathrm{Vect}} \Rightarrow S_{\mathcal{P}}$
s.t. $(S_{\mathcal{P}}, \gamma, \eta)$ monad and $\mathrm{Alg}_{S_{\mathcal{P}}} \cong \mathrm{Alg}_{\mathcal{P}}$.

MONADIC INTERPRETATION

Def'n (Tangent monads).

$(\mathbb{X}, T)$ tangent category.

- $(S : \mathbb{X} \longrightarrow \mathbb{X}, \gamma : S \circ S \Rightarrow S, \eta : \mathrm{id}_{\mathbb{X}} \Rightarrow S)$ monad
- $\alpha : S \circ T \Rightarrow T \circ S$

MONADIC INTERPRETATION

Theorem.
Let $(S, \gamma, \eta, \alpha)$ be a tangent monad over $(\mathbb{X}, \mathbb{T})$.
Then Alg_S has a tangent structure $\mathbb{T}_S$ s.t.

$$\mathrm{Alg}_S \xrightarrow{\text{Forget}} \mathbb{X}$$

is a tangent morphism.

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OPERADIC TANGENT CATEGORIES

Lemma. Vect has a tangent structure: $\mathbb{T}V := V \oplus V$

Theorem. $\mathcal{P}$ operad. $S_{\mathcal{P}}$ is a tangent monad.

Corollary. $\mathrm{Alg}_{\mathcal{P}} \cong \mathrm{Alg}_{S_{\mathcal{P}}}$ has a tangent structure $\mathbb{T}_{\mathcal{P}}$.

OPERADIC TANGENT CATEGORIES

$$T_{\mathcal{P}} : \mathsf{Alg}_{\mathcal{P}} \longrightarrow \mathsf{Alg}_{\mathcal{P}}$$

$$A \longmapsto \underbrace{\mathrm{Free}_A}_{\text{Free } A\text{-algebra}} \Omega^1_{\mathcal{P}} A$$

Kähler Differential

OPERADIC TANGENT CATEGORIES

Kähler Differential

$\delta : A \longrightarrow M$ derivation iff:

① linear

② $\delta(\mu(a_1, \ldots, a_n)) = \sum_{k=1}^{n} \mu(a_1, \ldots, \delta(a_k), \ldots, a_n)$

$$\mathrm{Der}^{(\mathcal{P})}(A, M) \cong \mathrm{Mod}_A^{(\mathcal{P})}(\Omega^1_{\mathcal{P}} A, M)$$

OPERADIC TANGENT CATEGORIES

Lemma.
$\mathrm{T}_{\mathcal{P}} \dashv \mathrm{L}_{\mathcal{P}}$.

Theorem.
For every operad $\mathcal{P}$
$(\mathrm{Alg}_{\mathcal{P}}^{op}, \mathrm{T}_{\mathcal{P}})$ tangent category.

$$(\mathrm{Alg}_{Com}^{op}, \mathrm{T}_{Com}) = (\mathrm{cAlg}^{op}, \mathrm{T})$$

$(\mathrm{Alg}_{Ass}^{op}, \mathrm{T}_{Ass})$ Ginzburg NC Geom.

MODULES OVER ALGEBRAS OVER OPERADS

Enveloping Operad

$\mathcal{P}$ operad, A algebra over $\mathcal{P}$.

$\exists\, \mathcal{P}_A$ operad s.t. $\mathrm{Alg}_{\mathcal{P}_A} \cong \dfrac{A}{\mathrm{Alg}_{\mathcal{P}}}$.

$\boxed{\text{Lemma. } \mathcal{P}_A(1) \text{ associative algebra.}}$

Universal Enveloping Algebra

MODULES OVER ALGEBRAS OVER OPERADS

Def'n (Modules).

$A \in \mathrm{Alg}_{\mathcal{P}}$.

$\mathrm{Mod}_A^{(\mathcal{P})} := \mathrm{Mod}_{\mathcal{P}_A(1)}$

Theorem.

$$\mathrm{DBun}_A^{(T_{\mathcal{P}})} \cong \mathrm{Mod}_A^{(\mathcal{P})\,op}$$

Differential Bundles

FUTURE DEVELOPMENTS

OPERADIC TANGENT CATEGORIES

- Deformation Theory
- Operad Theory
  - Functoriality
  - Distributive Laws
  - Tensor Products
- Tangent Category Theory
  - Connections
  - Differential Objs
  - Cohomology
  - Curve Objs
- Geometry
  - Serre–Swan Theorem


thanks

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