Join restriction categories and the importance of being adhesive

J.R.B. Cockett and X. Guo

Department of Computer Science
University of Calgary
Alberta, Canada

robin@cpsc.ucalgary.ca

Category Theory 2007

Contents:

Restriction Categories

A category $\mathbb{C}$ is a **restriction category** if it has a restriction operator:

$$\frac{X \xrightarrow{f} Y}{X \xrightarrow[\overline{f}]{} X}$$

[R.1] $f\overline{f} = f$,
[R.2] $\overline{f}\,\overline{g} = \overline{g}\,\overline{f}$,
[R.3] $\overline{g\overline{f}} = \overline{g}\,\overline{f}$,
[R.4] $\overline{g}f = f\overline{gf}$.

The domain of definition of f is expressed by $\overline{f}$. Restriction categories are abstract categories of partial maps.

A map is **total** if $\overline{f} = 1$. The total maps form a subcategory.

More properties

- The restriction idempotents $e = \overline{e} : A \longrightarrow A$ form a semilattice written $\mathcal{O}(A)$ (in fact $\mathcal{O}$ is a contravariant functor to the category of semilattices with stable maps: a *corestriction* category). Think of these as the "open sets of A".
- Restriction categories are partial order enriched with $f \leq g \Leftrightarrow g\overline{f} = f$
- A map $f : A \longrightarrow B$ is a **partial isomorphism** in case there is an $f^{(-1)} : B \longrightarrow A$ such that $ff^{(-1)} = \overline{f^{(-1)}}$ and $f^{(-1)}f = \overline{f}$.
- A restriction category in which *all* maps are partial isomorphism is an inverse category. A one object inverse category is an inverse semigroup with a unit!

Inverse categories are to restriction categories what groupoids are to categories.

Compatibility

- Restriction categories are **compatibility** enriched with $f \smile g \Leftrightarrow g\overline{f} = f\overline{g}$. This relation is preserved by composition:
$$f \smile g \Rightarrow hfk \smile hgk.$$
- A set $S \subseteq \mathbb{C}(A, B)$ is **compatible** if for every $s, s' \in S$, $s \smile s'$.

It is reasonable to consider a join operation restricted to compatible maps ....

Join Restriction Categories

A restriction category $\mathbb{C}$ is a **join restriction category** if for each compatible subset $S \subseteq \mathbb{C}(A, B)$, the join $\bigvee_{s \in S} s \in \mathbb{C}(A, B)$ exists:

- $\bigvee_{s \in S} s$ is the join with respect to $\leq$ in $\mathbb{C}(A, B)$,
- The join is *stable* in the sense that: $(\bigvee_{s \in S} s)g = \bigvee_{s \in S}(sg)$.

Four consequences:

- The join is *universal* in the sense that $f(\bigvee_{s \in S} s) = \bigvee_{s \in S}(fs)$.
- The join commutes with the restriction $\overline{\bigvee_{s \in S} s} = \bigvee_{s \in S} \overline{s}$.
- Each $\mathcal{O}(A)$ is a *locale*. (In fact $\mathcal{O}$ is a covariant functor to the restriction category of locales with stable maps).
- Join restriction categories allow the manifold construction (Marco Grandis).

Free Join Restriction Categories

Given any restriction category $\mathbb{X}$, one may construct from it a free join restriction category $\mathbb{X} \longrightarrow \widehat{\mathbb{X}}$ (Marco Grandis) with

- **objects**: $X \in \mathbb{X}$;
- **maps**: $S : A \longrightarrow B$ where $S \subseteq \mathbb{X}(A, B)$ is a down-closed compatible set;
- **identities**: $1_A = \downarrow \{1_A\} = \{e | e = \overline{e} : A\} = \mathcal{O}(A)$;
- **composition**: for maps $S : A \longrightarrow B$ and $T : B \longrightarrow C$ $TS = \downarrow \{ts | s \in S,\ t \in T\}$;
- **restriction**: $\overline{S} = \{\overline{s} | s \in S\}$;
- **join**: $\bigvee_{i \in \Gamma} S_i = \bigcup_{i \in \Gamma} S_i$, where each S_i is a down closed compatible set and $\{S_i\}_{i \in \Gamma}$ are compatible sets.

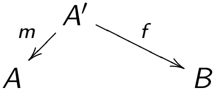
Partial Maps Categories

- A collection $\mathcal{M}$ of monics is **a stable system of monics** if it includes all isomorphisms, is closed under composition and is pullback stable.
- For any stable system of monics $\mathcal{M}$, if $mn \in \mathcal{M}$ and m is monic, then $n \in \mathcal{M}$.
- An $\mathcal{M}$-*category* is a pair $(\mathbb{C}, \mathcal{M})$, where $\mathbb{C}$ is a category and $\mathcal{M}$ is a stable system of monics in $\mathbb{C}$.
- Functors between $\mathcal{M}$-categories must preserve the selected monics *and* pullbacks of these monic. Natural transformations are "tight" (Manes) in the sense that they are cartesian over the selected monics.

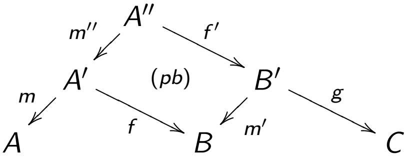
Partial Maps Categories

The category of partial maps $\mathrm{Par}(\mathbb{C}, \mathcal{M})$ is:

- **objects:** $A \in \mathbb{C}$;
- **maps:** $(m, f) : A \longrightarrow B$ (up to equivalence) with $m : A' \longrightarrow A$ is in $\mathcal{M}$ and $f : A' \longrightarrow B$ is a map in $\mathbb{C}$:



- **identities:** $(1_A, 1_A) : A \longrightarrow A$;
- **composition:** $(m', g)(m, f) = (mm'', gf')$:



- **restriction:** $\overline{(m, f)} = (m, m)$.

Completeness and representation

For a *split* restriction category, $\mathbb{X}$, the subcategory of total maps is an $\mathcal{M}$-category, where $m \in \mathcal{M}$ if and only if it is monic and a partial isomorphism. In that case $\mathsf{Par}(\mathsf{Total}(\mathbb{X}), \mathcal{M})$ is isomorphic to $\mathbb{X}$.

Theorem (Completeness: Cockett and Lack)

Every restriction category is a full subcategory of a partial map category.

There is also a representation theorem:

Theorem (Representation: Mulry)

Any restriction category $\mathbb{C}$ has a full and faithful restriction-preserving embedding into a partial map category of a presheaf category

$$\mathbb{C} \longrightarrow \mathsf{Par}(\mathbf{Set}^{\mathsf{Total}(\mathsf{split}_r(\mathbb{C}))^{\mathrm{op}}}, \widehat{\mathcal{M}})$$

Completeness and representation with joins

When does an $\mathcal{M}$-category have its partial map category a join restriction category?

The answer: $(\mathbb{X}, \mathcal{M})$ must be $\mathcal{M}$-adhesive ...

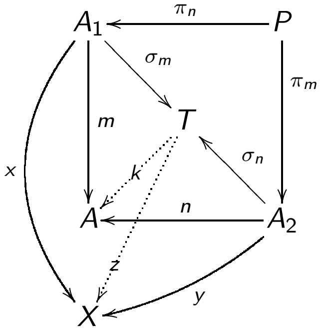
Theorem (Cockett and Guo)

Every join restriction category is a full subcategory of the partial map category of an adhesive $\mathcal{M}$-category whose gaps are in $\mathcal{M}$.

The rest of the talk is about the proof of this and a few consequences ...

First attempts ...

To form joins $(m, x) \vee (n, y)$ in $\mathrm{Par}(\mathbb{C}, \mathcal{M})$:

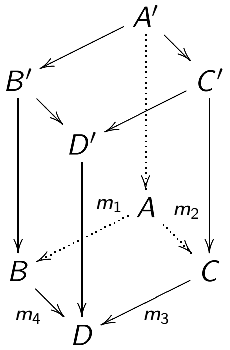


In order to have $(m, x) \vee (n, y) = (k, z)$, the **gap** k must in $\mathcal{M}$, the **pushout** (σ_m, σ_n) of (π_m, π_n) must be **stable** under pulling back.

.... also need stability under composition of spans: what on earth is this???!!! ...

van Kampen Squares

As in [4], a *van Kampen (VK) square* is a pushout (A, B, C, D) such that for each commutative cube:



whenever the back side faces are pullbacks, the front side faces are pullbacks iff the top face is a pushout.

Adhesive Categories

Definition (Adhesive category, [4])

A category $\mathbb{X}$ is said to be **adhesive** if

(*i*) $\mathbb{X}$ has pushouts along monics;

(*ii*) $\mathbb{X}$ has pullbacks;

(*iii*) pushouts along monics are van Kampen squares.

Set and elementary toposes are adhesive
but **Pos**, **Top**, **Grp**, and **Cat** are not [4].

We want to extend the notions of van Kampen squares and adhesive categories to **van Kampen colimits** and **adhesive** $\mathcal{M}$**-categories** ....

van Kampen colimits in general

A colimit $\alpha : D \Rightarrow C$, where $D : \mathbb{S} \longrightarrow \mathbb{C}$, is **van Kampen** if for any diagram $D' : \mathbb{S} \longrightarrow \mathbb{C}$, any cone $\alpha' : D' \Rightarrow X$ under D', and any commutative diagram

$$\begin{array}{ccc} D' & \xRightarrow{\alpha'} & X \\ {\scriptstyle\beta}\Downarrow & & \downarrow{\scriptstyle r} \\ D & \xRightarrow{\alpha} & C \end{array}$$

in which β is cartesian natural transformation, $\alpha' : D' \Rightarrow X$ is a colimit if and only if for each $s \in \mathbb{S}$

$$\begin{array}{ccc} D'(s) & \xrightarrow{\alpha'(s)} & X \\ {\scriptstyle\beta(s)}\downarrow & & \downarrow{\scriptstyle r} \\ D(s) & \xrightarrow{\alpha(s)} & C \end{array}$$

is a pullback diagram.

van Kampen colimits

Some properties:

- van Kampen colimits are pullback stable.
- Let D_i be diagrams on $\mathbb{S}_i$, $i = 1, 2$. If both $\alpha_1 : D_1 \Rightarrow X$ and $\alpha_2 : D_2 \Rightarrow X$ are van Kampen colimits, then so is $\alpha_1 \times_X \alpha_2 : D_1 \times_X D_2 \Rightarrow X$, where $D_1 \times_X D_2 : \mathbb{S}_1 \times \mathbb{S}_2 \longrightarrow \mathbf{C}$ is given by the following pullback diagram:

$$\begin{array}{ccc} (D_1 \times_X D_2)(s_1, s_2) & \xrightarrow{\beta(s_1,s_2)} & D_2(s_2) \\ {\scriptstyle \gamma(s_1,s_2)}\downarrow & & \downarrow{\scriptstyle \alpha_2(s_2)} \\ D_1(s_1) & \xrightarrow{\alpha_1(s_1)} & X \end{array}$$

  and $(\alpha_1 \times_X \alpha_2)(s_1, s_2) = \alpha_1(s_1)\gamma(s_1, s_2) = \alpha_2(s_2)\beta(s_1, s_2)$, for each $(s_1, s_2) \in \mathbb{S}_1 \times \mathbb{S}_2$.

van Kampen $\mathcal{M}$-amalgams

A **stable poset** is a poset with *binary* meets. When $\mathbb{S}$ is a stable poset and $D : \mathbb{S} \longrightarrow \mathcal{M}$ a diagram, an $\mathcal{M}$-cone $\alpha : D \Rightarrow X$ is an $\mathcal{M}$**-amalgam** in case for all $s_1, s_2 \in \mathbb{S}$ each

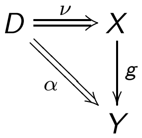
$$\begin{array}{ccc} D(s_1 \wedge s_2) & \xrightarrow{D(\leq)} & D(s_1) \\ {\scriptstyle D(\leq)}\downarrow & & \downarrow{\scriptstyle \alpha(s_1)} \\ D(s_2) & \xrightarrow{\alpha(s_2)} & X \end{array}$$

is a pullback diagram.

A stable poset $\mathcal{M}$-diagram $D : \mathbb{S} \longrightarrow \mathcal{M}$ is $\mathcal{M}$-**amalgamable** if there is an $\mathcal{M}$-amalgam under D.

$\mathcal{M}$-adhesive categories

- An $\mathcal{M}$-category $\mathbb{X}$ is an $\mathcal{M}$-**adhesive category** if each amalgamable $\mathcal{M}$-diagram D has a van Kampen colimit.
- A map $g : X \longrightarrow Y$ in an $\mathcal{M}$-adhesive category is an $\mathcal{M}$-**gap** if there is a van Kampen colimit $\nu : D \Rightarrow X$ such that each $g\nu(s) \in \mathcal{M}$ for each $s \in \mathbb{S}$:



Note: $\mathcal{M}$-gaps are necessarily monic so that these van Kampen colimits are $\mathcal{M}$-amalgams.

Mind the gap

What is the relation to van Kampen squares? When $\mathcal{M}$-gaps are $\mathcal{M}$ ...

Theorem

An $\mathcal{M}$-category is $\mathcal{M}$-adhesive with all $\mathcal{M}$-gaps in $\mathcal{M}$ if and only if all $\mathcal{M}$-amalgams which are pushouts have van Kampen colimits whose gaps are in $\mathcal{M}$.

The situation when the $\mathcal{M}$-gaps are *not* in $\mathcal{M}$ is of interest ...

$\mathcal{M}$-adhesive Categories

The class $\mathcal{M}_{\mathrm{gap}}$ of all $\mathcal{M}$-gaps in an $\mathcal{M}$-adhesive category **C** is a stable system of monics in **C** with $\mathcal{M} \subseteq \mathcal{M}_{\mathrm{gap}}$.

Theorem

If $\mathbb{X}$ is an $\mathcal{M}$-adhesive category, then

(*i*) $\mathbb{X}$ *is an $\mathcal{M}_{\mathrm{gap}}$-adhesive category;*

(*ii*) $(\mathcal{M}_{\mathrm{gap}})_{\mathrm{gap}} = \mathcal{M}_{\mathrm{gap}}$.

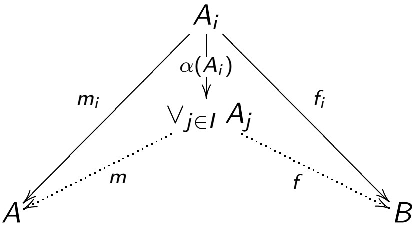
So one can always complete an $\mathcal{M}$-adhesive category to be closed to gaps.

Completeness for joins

Theorem

Let $\mathbb{X}$ be a category with a stable system of monics $\mathcal{M}$. Then $\mathrm{Par}(\mathbb{X},\mathcal{M})$ *is a join restriction category if and only if* $\mathbb{X}$ *is an* $\mathcal{M}$*-adhesive category and* $\mathcal{M}_{\mathrm{gap}} \subseteq \mathcal{M}$.

PROOF: ($\Leftarrow$) For any compatible set $\{(m_i, f_i) | i \in I\}$, $\nu : D \Rightarrow A$, given by $\nu(i) = m_i$, is a stable $\mathcal{M}$-cone on $\{A_i\}$, D has a VK colimit $(\vee_{j\in I} A_j, \alpha)$. $\exists ! m : \vee_{j\in I} A_j \longrightarrow A$ and $\exists f : \vee_{j \in I} A_j \longrightarrow B$:



$(m, f) = \vee\{(m_i, f_i) | i \in I\}$ and

$\mathrm{Par}(\mathbb{X},\mathcal{M})$ is a jrCat.

Completeness for joins

($\Rightarrow$) For all $\mathcal{M}$-diagrams $D : \mathbb{S} \longrightarrow \mathcal{M}$, and an $\mathcal{M}$-amalgam $\alpha : D \Rightarrow X$,

- The join $\vee_{s \in \mathbb{S}}(\alpha(s), \alpha(s)) = (m, m)$ exists, $m : C \longrightarrow X \in \mathcal{M}$;
- $(\alpha(s), \alpha(s)) \leq (m, f)$ implies there is an $\mathcal{M}$-map $\iota(s) : D(s) \longrightarrow C$ implies $\exists$ an amalgam $\mathcal{M}$-cone $\iota : D \Rightarrow C$.
- $\iota : D \Rightarrow C$ is a van Kampen colimit.

Free joins and $\mathcal{M}$-gaps

- Since any elementary topos is adhesive [4], $\mathbf{Set}^{\mathsf{Total}(\mathsf{split}_r(\mathbb{C}))^{\mathrm{op}}}$ is an adhesive category.
- Since $\mathcal{M} \subseteq \mathcal{M}_{\mathrm{gap}}$, there is a faithful embedding:

$$\begin{array}{c} \mathsf{Par}(\mathbf{Set}^{\mathsf{Total}(\mathsf{split}_r(\mathbb{C}))^{\mathrm{op}}}, \widehat{\mathcal{M}}) \\ \big\downarrow \mathcal{E} \\ \mathsf{Par}(\mathbf{Set}^{\mathsf{Total}(\mathsf{split}_r(\mathbb{C}))^{\mathrm{op}}}, \widehat{\mathcal{M}_{\mathrm{gap}}}) \end{array}$$

Free joins and $\mathcal{M}$-gaps

Hence there is a unique restriction functor $\mathcal{F} : \widehat{\mathbb{C}} \longrightarrow \mathsf{Par}(\mathbf{Set}^{\mathsf{Total}(\mathsf{split}_r(\mathbb{C}))^{\mathrm{op}}}, \widehat{\mathcal{M}_{\mathrm{gap}}})$ such that

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\mathsf{Par}(\mathcal{Y})J_{\mathbb{C}}} & \mathsf{Par}(\mathbf{Set}^{\mathsf{Total}(\mathsf{split}_r(\mathbb{C}))^{\mathrm{op}}}, \widehat{\mathcal{M}}) \\ {\scriptstyle \eta_{\mathbb{C}}}\downarrow & & \downarrow{\scriptstyle \mathcal{E}} \\ \widehat{\mathbb{C}} & \xrightarrow{\mathcal{F}} & \mathsf{Par}(\mathbf{Set}^{\mathsf{Total}(\mathsf{split}_r(\mathbb{C}))^{\mathrm{op}}}, \widehat{\mathcal{M}_{\mathrm{gap}}}) \end{array}$$

commutes

The functor $\mathcal{F}$ in the last commutative diagram is full and faithful.
So constructing joins in the Grothendieck category is the same as constructing joins directly ...

References

- J.R.B. Cockett and Stephen Lack, *Restriction categories I: Categories of partial maps*, Theoret. Comput. Science **270**(2002), 223-259.
- J.R.B. Cockett and Stephen Lack, *Restriction categories II: Partial map classification*, Theoret. Comput. Science **294**(2003), 61-102.
- H. Ehrig, M. Pfender, and H.J. Schneider, Graph-grammers: an algebraic approach, *IEEE Conf. on Automata and Switching Theory*, 167-180, 1973.
- S. Lack and P. Sobociński, *Adhesive and quasiadhesive categories*, Theoretical Informatics and Applications **39**(2005), 511-546.
- P.S. Mulry, *Partial map classifiers and partial cartesian closed categories*, Theoret. Comput. Sci. **99**(1992), 141-155.