Min-Cost Multicast Networks in Euclidean Space

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Multicast Network Design

Problem

Design a network of minimum cost to support a unit multicast throughput among given terminals.

- Nodes are located in an Euclidean Space
- The cost of a link is defined as its capacity $\times$ its length
- Relay nodes may be added without extra cost
Related: The Euclidean Steiner Tree Problem

**Gauss, 1836:** how can a railway network of minimal length which connects the four German cities **Bremen**, **Hamburg**, **Hannover**, and **Braunschweig** be created?
Related: The Euclidean Steiner Tree Problem

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The Difference

**Information Flow $\neq$ Commodity Flow**

Due to Network Coding:

- Information flows can be both replicated and mixed during transmission.
- The min-cost network may not be a tree.
Example

(a) minimum information network

(b) minimum Steiner tree

Total Cost: $3\sqrt{3}m \times 0.5\text{bps} = 2.598 \text{ m}\cdot\text{bps}$

Total Cost: $\sqrt{7}m \times 1\text{bps} = 2.646 \text{ m}\cdot\text{bps}$
Another Example

Total Cost: $7.746m \times 0.5\text{bps} = 3.873 \text{ m\cdotbps}$

Total Cost: $3.947m \times 1\text{bps} = 3.947 \text{ m\cdotbps}$
Problem Formulation

- **Input**: the positions of source $s$ and receivers $R$.
- **Output**: a directed network $D(V, A)$ with link capacity $c$ and the position of inserted relay nodes.
- **Satisfying**: $D$ can support a multicast session of unit rate.
- **Minimizing**: the cost of $D$.

### Information Flow in Space

| minimize | $\sum_{uv \in A} c(\rightarrow uv) \ | \ uv \ |$ |
|----------|-------------------------------------|
| subject to | $\sum_{v \in V} [f^t(\rightarrow uv) - f^t(\rightarrow vu)] = \delta^t(u) \ \forall t \in R, \forall u \in V$ |
|           | $0 \leq f^t(\rightarrow uv) \leq c(\rightarrow uv) \ \forall t \in R, \forall u, v \in V$ |
Discrete System Model

- Assume link capacities take rational values.
- Scale each capacity with some integer $h$ to get an integral capacitated network.
- Use parallel links of unit capacity instead.

**Optimization Problem**

$$\min_{h \in \mathbb{N}_+} \frac{1}{h} \sum_{\vec{u}\vec{v} \in A} |\vec{u}\vec{v}|$$

subject to: $\lambda_D(s, t) \geq h, \quad \forall t \in R$

The Euclidean Steiner Tree Problem can be viewed as a special case with $h = 1$. 
The 1-to-2 Multicast Case

Theorem 1

If there are only 3 terminals, the minimum Steiner tree achieves the minimum cost, which can not be improved by network coding.

- For multicast in a network (instead of in a space), network coding starts to make a difference for three terminals.
- We have seen an example of 6 terminals where network coding makes a difference. The cases of 4 and 5 terminals are unknown.
Proof Sketch

- The minimum Steiner tree for 3 terminals has one relay node located at the Fermat point.
- It is sufficient to prove that the nodes of min-cost multicast network lie on the minimum Steiner tree.

Wedge Property [Gilbert and Pollak, 1968]

Let $W \subset \mathbb{R}^2$ be any open wedge-shaped region with an angle of at least $120^\circ$. If $W$ does not contain any terminal node and each relay node has degree 3 at most, $W$ contains no relay node.
Proof Sketch (continue)

- By the Wedge Property, it is sufficient to show that the relay nodes in the min-cost network have degree 3 at most.
- For relay node of degree larger than 3, we can split it without reducing the max flow to each receiver.

This is because we have 2 receiver, the flow on a link has 3 types: (0,1) (1,0) (1,1).
Bounding the Number of Relay Nodes

- There is a possibility the min cost can not be achieved by finite networks.
- If the number of relays are bounded, the problem can be formulated as a programming problem with finite variables.

**Theorem 2**

For an optimal multicast network with $h = 2$ (half integral capacities), there are $(2n - 3)(2n - 2) + n - 1$ relay nodes at most, where $n$ is the number of terminals.

**Theorem 3**

For a min-cost acyclic multicast network of max-flow $h$, there are $h^3(n - 1)^2 + n^h(n + h^3(n - 1)^2 - 2)$ relay nodes at most.
Conclusion & Future Work

What we have done:

- Min-cost multicast network $\neq$ minimum Steiner tree.
- Network coding is unnecessary for 3 terminals.
- The number of required relay nodes in an acyclic optimal network is upper-bounded.

For future work,

- Is the min cost achievable with a finite network?
- Computational complexity: P or NP-hard?
- How much difference can network coding make?

The case of Multiple Unicast is considered in another paper.
Thanks! & Questions?